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# Partially-massless higher spin algebras in four dimensions

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Abstract: We propose a realisation of partially-massless higher spin algebras in four dimensions in terms of bosonic *and* fermionic oscillators, using Howe duality between $sp(4,\mathbb{R}) \cong so(2,3)$ and $osp(1|2(\ell-1),\mathbb{R})$. More precisely, we show that the centraliser of $osp(1|2(\ell-1),\mathbb{R})$ in the Weyl-Clifford algebra generated by 4 bosonic and $8(\ell-1)$ fermionic symbols, modulo $osp(1|2(\ell-1),\mathbb{R})$ generators, is isomorphic to the higher spin algebra of the type-$A_\ell$ theory whose spectrum contains partially-massless fields of all spins and depths $t = 1, 3, \ldots, 2\ell-1$. We also discuss the possible existence of a deformation of this algebra, which would encode interaction for the type-$A_\ell$ theory.

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## Introduction

From a holographic perspective, theories with massless higher spin fields in anti-de Sitter (AdS) spacetime should be dual to free conformal field theories (CFT) [1–4]. In all dimensions, one can distinguish between either the free scalar or the free fermion theory, and in even dimensions, an additional possibility exists in the guise of the free $\frac{d-2}{2}$-form [5]. Giving up unitarity on the CFT side allows one to consider higher derivative versions of these theories,

$$S[\phi] = \frac{1}{2} \int_{\mathbb{R}^d} d^d x \, \phi^* \Box^\ell \phi \,, \qquad S[\psi] = i \int_{\mathbb{R}^d} d^d x \, \overline{\psi} \, \partial\!\!\!/^{2\ell-1} \psi \,,$$

where $\ell \geq 1$ is an integer, $\phi^*$ denotes the complex conjugate of the scalar field $\phi$ and $\overline{\psi}$ the Dirac adjoint of the spinor field $\psi$. Although non-unitary, these theories capture some non-trivial physics, namely they describe a special type of fixed point of the renormalisation group flow, called the multi-critical isotropic Lifshitz points [6, 7]. Such theories contain currents of arbitrary integer spin, which are both conserved and partially-conserved. More precisely, and focusing on the scalar case, the $2\ell$-derivative scalar — also known as order-$\ell$ singleton — has currents of the form

$$J^{(t)}_{a_1 \dots a_s} = \phi^* \partial_{a_1} \dots \partial_{a_s} \Box^{\ell - \frac{t+1}{2}} \phi + \dots \,,$$

for all integers $s \geq 1$ and $t = 1, 3, \dots, 2\ell - 1$, where the dots stand for additional terms which ensure that the above tensor is traceless and obey the partial-conservation law

$$\partial^{b_1} \dots \partial^{b_t} J^{(t)}_{a_1 \dots a_{s-t} b_1 \dots b_t} \approx 0 \,,$$

where $\approx$ signifies that this holds when the scalar field is on-shell, i.e. $\Box^\ell \phi \approx 0$ [6] (see also [8]). Such currents are dual to spin-$s$ partially-massless fields of depth-$t$ [9], which are fields propagating an intermediary number of degrees of freedom between those of a massless field and those of a massive one with the same spin [10–12] (see also [13–18]). These fields



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can be realised as totally symmetric rank-$s$ tensors in AdS, schematically $\varphi_{s,t}$, subject to $t$-derivative gauge symmetry

$$\delta_\xi \varphi_{s,t} \sim \nabla^t \xi_{s-t}\,,$$

where $\xi_{s-t}$ is any rank-$(s-t)$ symmetric tensor (see also [19] for their frame-like formulation, and [20, 21] for a recently proposed twistor-inspired formulation in four dimensions). The number of derivatives $t$ in the gauge transformations is called the 'depth' of the partially-massless, with $t=1$ corresponding to the massless case.

Based on the spectrum of partially-conserved currents of the order-$\ell$ singleton, the dual higher-spin gravity, usually referred to as the type-$\mathrm{A}_\ell$ theory, should contain partially-massless fields of any integer spins and *odd* depths from 1 to $2\ell-1$, i.e. schematically

$$\text{Type-}\mathrm{A}_\ell = \bigoplus_{t=1,3,\dots,2\ell-1} \bigoplus_{s=t}^{\infty} \varphi_{s,t} + (\cdots)\,,$$

where the dots stand for finitely many *massive* fields (i.e. with no gauge symmetry), and of spin lower than $2\ell-1$ (see e.g. [6, 22]). The higher spin algebra is the algebra of global symmetries of such theories, so that its spectrum is given directly by that of massless and partially-massless fields, namely it is given by gauge parameters which lead to trivial gauge symmetries. These are known to be generalised Killing tensors [23], which form finite-dimensional representations of $so(2,d)$ corresponding to Young diagrams with two rows, of length $s-1$ and $s-t$ respectively, so that the type-$\mathrm{A}_\ell$ higher spin algebra — denoted by $\mathfrak{hs}_\ell$ hereafter — admits the decomposition

$$\mathfrak{hs}_\ell \cong \bigoplus_{t=1,3,\dots,2\ell-1} \bigoplus_{s=t}^{\infty} \begin{array}{|l|}\hline \quad s-1 \quad\quad \\ \hline s-t \\ \hline \end{array}\,,$$

as an $so(2,d)$-module. Partially-massless higher spin algebras have been previously considered in [8, 24–28], and have been characterised both as a quotient of the universal enveloping algebra of $so(2,d)$ by a specific ideal, and in terms of the Howe duality between $so(2,d)$ and $sp(2,\mathbb{R})$ in the Weyl algebra generated by $2\times(d+2)$ variables [28]. Implementing these quotients may be cumbersome in practice, even in the case $\ell=1$, i.e. for the 'massless' higher spin algebra.

The latter does admit a simple realisation in *four dimensions*, namely as the even subalgebra of the Weyl algebra (in two pairs of oscillators). This is due to the existence of the low-dimensional isomorphism involving the $\mathrm{AdS}_4$ isometry algebra, $so(2,3)\cong sp(4,\mathbb{R})$, and the fact that the module of the $(\ell=1)$ singleton is realised simply in the Fock space associated to this Weyl algebra. In this paper, we argue that one can simply append a Clifford algebra (whose number of generators is related to $\ell$) to the Weyl algebra, and recover the type-$\mathrm{A}_\ell$ algebra as a 'simpler' quotient, using Howe duality between $sp(4,\mathbb{R})$ and $osp(1|2(\ell-1),\mathbb{R})$. Such a realisation, technically easier to work with, could be useful for the introduction of interactions and the derivation of a partially-massless higher spin gravity.

This paper is organised as follows: in section 1, we start by briefly reviewing the usual constructions of the type-$\mathrm{A}_\ell$ higher spin algebra mentioned previously, and then proceed



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to present our realisation based on extending the Weyl algebra with a Clifford algebra. In section 2, we propose a construction of the higher order singleton modules using the Fock space naturally associated with the Weyl-Clifford algebra used to realise the type-A$_\ell$ higher spin algebra. We discuss in section 3 the possible existence of formal partially-massless higher spin gravities, and exhibit a cycle of the type-A$_\ell$ algebra in arbitrary dimension as part of this analysis. We also explicitly verify that some of the Joseph ideal generators vanish in our oscillator realisation. We conclude the paper in section 4 with a discussion about the difficulties one may encounter in construction deformations of the type-A$_\ell$ higher spin algebra using our realisation. Appendix A contains additional details on the structure of the defining ideal for the type-A$_2$ algebra (as well as some comments for the arbitrary $\ell$ case) in arbitrary dimensions.

# 1 Partially-massless higher spin algebra

## 1.1 Lightning review in arbitrary dimensions

**Quotient of the universal enveloping algebra.** First, let us set some notation: we will denote by $M_{\mathsf{AB}}$ the generator of the Lie algebra $so(2,d)$, with indices $\mathsf{A}, \mathsf{B}, \ldots$ taking $d+2$ values, and by $\eta_{\mathsf{AB}}$ the (components of the diagonal) metric of signature $(-,-,+,\ldots,+)$. The Lie bracket of these $so(2,d)$ generators reads

$$[M_{\mathsf{AB}}, M_{\mathsf{CD}}] = \eta_{\mathsf{BC}}\, M_{\mathsf{AD}} - \eta_{\mathsf{AC}}\, M_{\mathsf{BD}} - \eta_{\mathsf{BD}}\, M_{\mathsf{AC}} + \eta_{\mathsf{AD}}\, M_{\mathsf{BC}}\,, \tag{1.1}$$

and we will simply denote the associative product in the universal enveloping algebra $\mathcal{U}(so(2,d))$ by juxtaposition, for instance we will write

$$\mathcal{C}_2 = -\frac{1}{2}\, M_{\mathsf{AB}}\, M^{\mathsf{AB}}\,, \tag{1.2}$$

for the quadratic Casimir operator of $so(2,d)$, where the indices have been raised with the inverse metric $\eta^{\mathsf{AB}}$. The higher spin algebra of type-A$_\ell$ is the quotient [25, 26]

$$\mathfrak{hs}_\ell = \mathcal{U}(so(2,d))/\mathcal{I}_\ell\,, \tag{1.3}$$

of the universal enveloping algebra of $so(2,d)$ by the (two-sided) ideal[1]

$$\mathcal{I}_\ell = \Big\langle V_{\mathsf{ABCD}} \oplus \Big(\mathcal{C}_2 + \frac{(d-2\ell)(d+2\ell)}{4}\,\mathbf{1}\Big) \oplus \mathcal{J}_{\mathsf{A}(2\ell)}\Big\rangle\,, \tag{1.4}$$

where

$$V_{\mathsf{ABCD}} := M_{[\mathsf{AB}}\, M_{\mathsf{CD}]}\,, \qquad \mathcal{J}_{\mathsf{A}(2\ell)} := M_{\mathsf{A}}{}^{\mathsf{B}_1}\, M_{\mathsf{AB}_1} \ldots M_{\mathsf{A}}{}^{\mathsf{B}_\ell}\, M_{\mathsf{AB}_\ell} - \text{traces}\,, \tag{1.5}$$

and where we used the convention (standard in the higher spin literature) that symmetrised indices are denoted by the same letter, with their number being indicated in parenthesis when necessary, e.g. $A(l) = (A_1 \ldots A_l)$.

[1]The notation $\big\langle (\cdots) \big\rangle$ means that the ideal is generated by the elements inside the brackets.

Recall that the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is isomorphic, as vector space[2], to the symmetric algebra $S(\mathfrak{g})$. This space is, by definition, the symmetrised tensor product of the adjoint representation $\ydiagram{1,1}$ of $so(2,d)$, and can be decomposed into a direct sum of finite-dimensional irreducible representations that we will denote by the corresponding Young diagram. In such terms, the subspace of elements quadratic in the Lie algebra generators reads

$$\ydiagram{1,1}^{\odot 2} \cong \ydiagram{2,2} \oplus \ydiagram{1,1,1,1} \oplus \ydiagram{2} \oplus \bullet\,, \tag{1.6}$$

where in particular

$$\ydiagram{1,1,1,1} \quad \longleftrightarrow \quad V_{\mathsf{ABCD}} \qquad \text{and} \qquad \bullet \quad \longleftrightarrow \quad \mathcal{C}_2\,. \tag{1.7}$$

When modding out the ideal $\mathcal{I}_\ell$, the totally antisymmetric diagram is removed, whereas the quadratic Casimir operator is related to a multiple of the identity. More precisely, the quadratic Casimir operator is set to take the value $-\frac{1}{4}(d-2\ell)(d+2\ell)$, which is the same value it takes when acting on the order-$\ell$ singleton module. Next we can look at the subspace of the universal enveloping algebra spanned by elements cubic in the Lie algebra,

$$\ydiagram{1,1}^{\odot 3} \cong \ydiagram{3,3} \oplus \ydiagram{2,2,1,1} \oplus \ydiagram{1,1,1,1} \oplus \ydiagram{2,1,1} \oplus \ydiagram{3,1} \oplus \ydiagram{1,1}\,, \tag{1.8}$$

and make the following observations.

(*i*) First, the three diagrams with more than two rows are contained in the product of the ideal generators $V_{\mathsf{ABCD}}$ and the Lie algebra generators $M_{\mathsf{AB}}$, and hence belong to the ideal $\mathcal{I}_\ell$,

$$\ydiagram{1,1,1,1} \oplus \ydiagram{2,2,1,1} \oplus \ydiagram{2,1,1} \subset \mathcal{I}_\ell\,, \tag{1.9}$$

so that they are removed once $\mathcal{I}_\ell$ is modded out from the universal enveloping algebra of $so(2,d)$. This is, in fact, a general pattern: Young diagrams with more than two rows appearing in the decomposition of $\mathcal{U}\big(so(2,d)\big)$ all belong to the ideal $\mathcal{I}_\ell$, and more specifically, to the ideal generated by $V_{\mathsf{ABCD}}$. As a consequence, the higher spin algebra $\mathfrak{hs}_\ell$ contains only Young diagrams with one or two rows.

(*ii*) Second, the diagram $\ydiagram{1,1}$ is obtained as the product of the quadratic Casimir operator $\mathcal{C}_2$ with the Lie algebra generators $M_{\mathsf{AB}}$. Since, after modding out the ideal $\mathcal{I}_\ell$, the value of $\mathcal{C}_2$ is fixed, the adjoint representation only appears with multiplicity one in $\mathfrak{hs}_\ell$.

One can immediately extract from the previous item the following lesson: in order for the quotient algebra $\mathfrak{hs}_\ell$ to admit a *multiplicity-free* decomposition under $so(2,d)$, i.e. that all irreducible representations (irreps) appear only once in the decomposition of $\mathfrak{hs}_\ell$ under the adjoint action of $so(2,d)$, the center of the universal enveloping algebra has to be fixed. In

[2]Actually as a $\mathfrak{g}$-module, and as a (co-commutative) coalgebra.

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other words, modding out the ideal $\mathcal{I}_\ell$ should fix the values of all Casimir operators (quadratic and higher), as the latter form a basis of the center of $\mathcal{U}(so(2,d))$.

Finally, since $so(2,d)$-module appearing in the decomposition of the universal enveloping algebra $\mathcal{U}(so(2,d))$ are contained, by definition, in tensor product of its adjoint representation, all these irreps are characterised by Young diagrams with an *even* number of boxes. In view of the previous discussion, this means that they are necessarily of the form , where difference between the number of boxes in the first and in the second row is *even*. This difference equals $t-1$ where, as before, $t$ is the depth of the partially massless field. Modding out by the symmetric diagram $\boxed{\;2\ell\;}$ effectively removes all diagrams with $t > 2\ell-1$, as they would belong to the product of the former with another diagram in the spectrum.

**Howe duality in the Weyl algebra.** Now consider the Weyl algebra $\mathcal{A}_{2(d+2)}$ generated by $\{Y_i^{\mathsf{A}}\}$ where $i=\pm$, and with the Moyal-Weyl star-product

$$f \star g = f\, \exp\left(\frac{\overleftarrow{\partial}}{\partial Y_i^{\mathsf{A}}}\, \eta^{\mathsf{AB}} \epsilon_{ij}\, \frac{\overrightarrow{\partial}}{\partial Y_j^{\mathsf{B}}}\right) g\,, \tag{1.10}$$

where $\epsilon_{ij}$ are the components of the canonical $2\times 2$ symplectic matrix, as associative product. Quadratic monomials in $Y_i^{\mathsf{A}}$, i.e. linear combinations of the generators

$$K_{ij}^{\mathsf{AB}} := \frac{1}{2}\, Y_i^{\mathsf{A}} Y_j^{\mathsf{B}}\,, \tag{1.11}$$

span a Lie subalgebra isomorphic to $sp(2(d+2),\mathbb{R})$,

$$[K_{ij}^{\mathsf{AB}}, K_{kl}^{\mathsf{CD}}]_\star = \eta^{\mathsf{BC}} \epsilon_{jk}\, K_{il}^{\mathsf{AD}} + \eta^{\mathsf{AC}} \epsilon_{ik}\, K_{jl}^{\mathsf{BD}} + \eta^{\mathsf{BD}} \epsilon_{jl}\, K_{ik}^{\mathsf{AC}} + \eta^{\mathsf{AD}} \epsilon_{il}\, K_{jk}^{\mathsf{BC}}\,, \tag{1.12}$$

where $[-,-]_\star$ denotes the commutator with respect to the star-product. The index structure on display here allows one to easily identify two mutually commuting Lie subalgebras,

$$so(2,d) \oplus sp(2,\mathbb{R}) \subset sp(2(d+2),\mathbb{R})\,, \tag{1.13}$$

respectively generated by

$$M_{\mathsf{AB}} := \frac{1}{2}\, \epsilon^{ij}\, Y_i^{\mathsf{A}} Y_j^{\mathsf{B}}\,, \qquad \text{and} \qquad \tau_{ij} := \frac{1}{2}\, \eta_{\mathsf{AB}}\, Y_i^{\mathsf{A}} Y_j^{\mathsf{B}}\,. \tag{1.14}$$

Such pairs of algebras are usually called reductive dual pairs, or Howe dual pairs [29–31], and can be used to construct a realisation of the type-A$_\ell$ higher spin algebra in the Weyl algebra (as used in the case $\ell = 1$ to propose nonlinear equations of motions for interacting massless higher spin fields [32]).

To do so, we will first need to identify the centraliser $\mathcal{Z}_{\mathcal{A}_{2(d+2)}}(sp(2,\mathbb{R}))$ of $sp(2,\mathbb{R})$ in the Weyl algebra $\mathcal{A}_{2(d+2)}$, which is the space of elements annihilated by

$$[\tau_{ij}, -]_\star = Y_{(i}^{\mathsf{A}}\, \epsilon_{j)k}\, \frac{\partial}{\partial Y_k^{\mathsf{A}}}\,, \tag{1.15}$$

or equivalently by the three operators

$$Y_+^{\mathsf{A}}\, \frac{\partial}{\partial Y_+^{\mathsf{A}}} - Y_-^{\mathsf{A}}\, \frac{\partial}{\partial Y_-^{\mathsf{A}}}\,, \qquad Y_+^{\mathsf{A}}\, \frac{\partial}{\partial Y_-^{\mathsf{A}}}\,, \qquad Y_-^{\mathsf{A}}\, \frac{\partial}{\partial Y_+^{\mathsf{A}}}\,. \tag{1.16}$$



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The first operator imposes that elements in the centraliser of $sp(2,\mathbb{R})$ be of the same degree in $Y_+^{\mathsf{A}}$ and $Y_-^{\mathsf{A}}$, while the other two operators both impose that the coefficients of monomials in $Y_\pm^{\mathsf{A}}$ have the symmetry of a rectangular Young diagram in the $so(2,d)$ indices. In other words,

$$f(Y) \in \mathcal{Z}_{\mathcal{A}_{2(d+2)}}(sp(2,\mathbb{R})) \qquad \Longleftrightarrow \qquad f(Y) = \sum_{s=1}^{\infty} f_{\mathsf{A}(s-1),\mathsf{B}(s-1)}\, Y_+^{\mathsf{A}(s-1)}\, Y_-^{\mathsf{B}(s-1)}\,, \tag{1.17}$$

with $f_{\mathsf{A}(s-1),\mathsf{AB}(s-2)} = 0$. Note however that these tensors are still traceful, and hence are *reducible* representations of $so(2,d)$. Decomposing them into irreducible representations, one would find all possible finite-dimensional irreps of $so(2,d)$ labelled by Young diagrams of the form

$$\begin{array}{|c|c}\hline \quad s-1 \quad & \qquad \\ \hline s-t & \\ \hline \end{array} \qquad \text{with} \qquad s \geq 1\,, \quad t \in 2\,\mathbb{N}+1\,, \tag{1.18}$$

i.e. all Young diagrams with two rows whose lengths differ by an even number of boxes. Comparing to the universal enveloping algebra construction reviewed previously, we found ourselves with the same content as we do after modding out $\mathcal{U}(so(2,d))$ by the ideal generated by $V_{\mathsf{ABCD}}$. We also face the same multiplicity problem: recall that we need to fix the center of the universal enveloping algebra in order to obtain a multiplicity-free spectrum. Here, the source of multiplicities is not only the center of the universal enveloping algebra of $so(2,d)$, but also that of $sp(2,\mathbb{R})$ which is, by definition, also contained in the centraliser of $sp(2,\mathbb{R})$ in $\mathcal{A}_{2(d+2)}$. Fortunately, both problems can be solved at once thanks to the fact that the quadratic Casimir operators of $so(2,d)$ and $sp(2,\mathbb{R})$ are related via

$$\mathcal{C}_2[so(2,d)] + \mathcal{C}_2[sp(2,\mathbb{R})] = -\frac{1}{4}\,(d-2)(d+2)\,, \tag{1.19}$$

and similarly for higher order Casimir operator (see e.g. [33–36] and [37, section 9] for more details). Since $sp(2,\mathbb{R})$ only has one independent Casimir operator, it is sufficient to fix its value to also fix the values of all Casimir operators of $so(2,d)$. In particular, imposing

$$\mathcal{C}_2[sp(2,\mathbb{R})] = -(\ell-1)(\ell+1)\,, \tag{1.20}$$

sets the quadratic Casimir operator of $so(2,d)$ to

$$\mathcal{C}_2[so(2,d)] = -\frac{1}{4}\,(d-2\ell)(d+2\ell)\,, \tag{1.21}$$

as it should in the type-$\mathrm{A}_\ell$ algebra $\mathfrak{hs}_\ell$. Finally, notice that the diagrams of shape $(s-1,s-t)$ with $t=2k+1$ and $k \geq 0$ appear as the $k$th trace of rectangular diagrams, and that these traces are proportional to $k$ times the $sp(2,\mathbb{R})$ generators. As a consequence, one can recover the partially-massless higher spin algebra as the quotient[3]

$$\mathfrak{hs}_\ell \cong \mathcal{Z}_{\mathcal{A}_{2(d+2)}}(sp(2,\mathbb{R}))\Big/\big\langle \tau_{(i_1 j_1} \dots \tau_{i_\ell j_\ell)} \oplus \mathcal{C}_2[sp(2,\mathbb{R})] - \frac{1}{2}\,(\ell-1)(\ell+1)\mathbf{1}\big\rangle\,, \tag{1.22}$$

as modding out the ideal generated by the elements $\tau_{(i_1 j_1} \dots \tau_{i_\ell j_\ell)}$ guarantees that only $so(2,d)$ Young diagrams corresponding to partially-massless fields of depth $t = 1,3,\dots,2\ell-1$ remain. See e.g. [28, 38–42] for more details on the construction of higher spin algebras from the perspective of Howe duality.

[3]Note that $\tau_{(i_1 j_1} \dots \tau_{i_\ell j_\ell)}$ generate the annihilator of the finite-dimensional $sp(2,\mathbb{R})$-irrep of highest weight $\ell-1$, which is a reflection of the fact that the order-$\ell$ singleton is Howe dual to this $\ell$-dimensional irrep of $sp(2,\mathbb{R})$ [27, 28].



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### 1.2 Four-dimensional specificities

**Dual pairs and the Weyl-Clifford algebra.** Consider a set of bosonic $(Y^A)$ and fermionic $(\phi_i^A)$ where the capital indices $A, B, \ldots$ take $2n$ values and the lower case indices $i, j, \ldots$ take $2p$ values. These oscillators are subject to the commutation and anticommutation relations

$$[\hat{Y}^A, \hat{Y}^B] = 2\, C^{AB}\, \mathbf{1}\,, \tag{1.23a}$$

$$\{\hat{\phi}_i^A, \hat{\phi}_j^B\} = 2\, C^{AB} \epsilon_{ij} \mathbf{1}\,, \tag{1.23b}$$

where $C^{AB} = -C^{BA}$ and $\epsilon_{ij} = -\epsilon_{ji}$ are two antisymmetric, non-degenerate matrices, with inverses given by

$$C^{AC}\, C_{BC} = \delta_B^A\,, \qquad \epsilon^{ik}\, \epsilon_{jk} = \delta_j^i\,. \tag{1.24}$$

We can therefore use these matrices to raise and lower indices, which we will do using the convention

$$C^{AB}\, X_B = X^A\,, \qquad X^A\, C_{AB} = X_B\,, \tag{1.25}$$

and similar convention for $\epsilon_{ij}$. The associative algebra generated by these oscillators modulo the above anti/commutation relations forms the Weyl-Clifford algebra $\mathcal{A}_{2n|4np}$, which is simply the tensor product of the Weyl algebra generated by the bosonic oscillators, and the Clifford algebra generated by the fermionic ones.

The elements quadratic in these oscillators (modulo the previous anti/commutation relations),

$$K^{AB} := \frac{1}{4}\{\hat{Y}^A, \hat{Y}^B\}\,, \qquad M_{ij}^{AB} := \frac{1}{4}\,[\hat{\phi}_i^A, \hat{\phi}_j^B]\,, \qquad Q_i^{A|B} := \frac{1}{2}\,\hat{Y}^A\, \hat{\phi}_i^B\,, \tag{1.26}$$

form a subalgebra isomorphic to $osp(4np|2n, \mathbb{R})$, whose bosonic subalgebra $o(4np) \oplus sp(2n, \mathbb{R})$ is generated by $M_{ij}^{AB}$ and $K^{AB}$, and the odd/fermionic generators — the supercharges — correspond to $Q_i^{A|B}$. Their anti/commutation relations read

$$[K^{AB}, K^{CD}] = C^{BC}\, K^{AD} + C^{AC}\, K^{BD} + C^{BD}\, K^{AC} + C^{AD}\, K^{BC}\,, \tag{1.27a}$$

$$[M_{ij}^{AB}, M_{kl}^{CD}] = C^{BC} \epsilon_{jk}\, M_{il}^{AD} - C^{AC} \epsilon_{ik}\, M_{jl}^{BD} - C^{BD} \epsilon_{jl}\, M_{ik}^{AC} + C^{AD} \epsilon_{il}\, M_{jk}^{BC}\,, \tag{1.27b}$$

$$[K^{AB}, Q_i^{C|D}] = C^{BC}\, Q_i^{A|D} + C^{AC}\, Q_i^{B|D}\,, \tag{1.27c}$$

$$[M_{ij}^{AB}, Q_k^{C|D}] = C^{BD} \epsilon_{jk}\, Q_i^{C|A} - C^{AD} \epsilon_{ik}\, Q_j^{C|B}\,, \tag{1.27d}$$

$$\{Q_i^{A|C}, Q^{B|D_j}\} = C^{AB}\, M_{ij}^{CD} + C^{CD} \epsilon_{ij}\, K^{AB}\,. \tag{1.27e}$$

Note that the orthogonal algebra is presented in a slightly unconventional basis here: one should think of the pair of indices $(A, i)$ on the generators $M_{ij}^{AB}$ and $Q_i^{B|A}$ as a single index for the fundamental representation of $o(4np)$. This is accordance with the fact that only the first capital index of the fermionic generators $Q_i^{A|B}$ (the index '$A$' here) is rotated by the $sp(2n, \mathbb{R})$ generators $K^{AB}$, whereas the second capital index is rotated, along with the lower case index (the indices '$B$' and '$i$' here) are rotated together by the $o(4np)$ generators $M_{ij}^{AB}$.

This unusual structure of indices for the $o(4np)$ generators, which stems from the choice of indices carried out by the fermionic oscillators $\phi_i^A$, is motivated by the fact that we are interested in singling out the pair of subalgebras

$$sp(2n,\mathbb{R}) \oplus sp(2p,\mathbb{R}) \subset o(4np)\,, \tag{1.28}$$

generated by

$$J^{AB} := \epsilon^{ij}\, M_{ij}^{AB}\,, \qquad \text{and} \qquad \tau_{ij} := C_{AB}\, M_{ij}^{AB}\,, \tag{1.29}$$

i.e. the generators obtained by contracting those of $o(4np)$ with the invariant tensors $\epsilon^{ij}$ of $sp(2p,\mathbb{R})$, and $C_{AB}$ of $sp(2n,\mathbb{R})$, respectively. As is clear from the index structure of these generators, these two subalgebras commute with one another, i.e. they are contained in each other's centraliser in $o(4np)$, and in fact they are exactly their respective centralisers.

**An interlude on Howe duality.** Such pair of subalgebras are usually called 'dual pairs' and have particularly interesting applications in representation theory and physics. The most famous examples come from dual pairs in a symplectic group $\mathrm{Sp}(2N,\mathbb{R})$, which are the central object of study of Howe duality [29, 30]. In this case, one can show that the oscillator representation of $\mathrm{Sp}(2N,\mathbb{R})$, i.e. the Fock space generated by $N$ pairs of bosonic creation-annihilation operators, admits a decomposition into direct sum of the tensor product of a representation of each group of the dual pair.

Another variation on the same theme consists in considering dual pairs in an *orthogonal* group, say $O(2N)$. This is precisely the case we are presented with above, with the pair $(sp(2n,\mathbb{R}), sp(2p,\mathbb{R})) \subset o(4np)$. For such dual pairs, the natural representation of the orthogonal group is the Fock space generated by *fermionic* pairs of creation-annihilation operators. Indeed, bilinears in these operators define a representation of the orthogonal group (or the double cover thereof) on the fermionic Fock space, which can then be decomposed into irreducible representations of the dual pair of interest. See e.g. [43, 44] for more details on this 'skew-Howe' duality.

Since we have both bosonic and fermionic oscillators at hand, we can consider dual pairs in the orthosymplectic group [45–47]. In our case, the relevant pair is composed of $sp(2n,\mathbb{R})$, generated by

$$T^{AB} := K^{AB} - \epsilon^{ij}\, M_{ij}^{AB} = \frac{1}{4}\,\{\hat{Y}^A, \hat{Y}^B\} - \frac{1}{4}\,[\hat{\phi}_i^A, \hat{\phi}^{Bi}]\,, \tag{1.30}$$

and satisfying the commutation relations

$$[T^{AB}, \hat{Y}^C] = C^{AC}\, \hat{Y}^B + C^{BC}\, \hat{Y}^A\,, \tag{1.31a}$$
$$[T^{AB}, \hat{\phi}_i^C] = C^{AC}\, \hat{\phi}_i^B + C^{BC}\, \hat{\phi}_i^A\,, \tag{1.31b}$$
$$[T^{AB}, T^{CD}] = C^{AC}\, T^{BD} + C^{AD}\, T^{BC} + C^{BD}\, T^{AC} + C^{BC}\, T^{AD}\,, \tag{1.31c}$$

and $osp(1|2p,\mathbb{R})$, generated by

$$Q_i = \frac{1}{2}\, C_{AB}\, \hat{Y}^A\, \hat{\phi}_i^B\,, \qquad \text{and} \qquad \tau_{ij} \equiv \{Q_i, Q_j\} = \frac{1}{4}\, C_{AB}\, [\hat{\phi}_i^A, \hat{\phi}_j^B]\,, \tag{1.32}$$

obeying,

$$[\tau_{ij}, Q_k] = \epsilon_{kj}\, Q_i + \epsilon_{ki}\, Q_j\,, \tag{1.33a}$$

$$[\tau_{ij}, \tau_{kl}] = \epsilon_{ki}\tau_{jl} + \epsilon_{kj}\tau_{il} + \epsilon_{li}\tau_{jk} + \epsilon_{lj}\tau_{ik}\,. \tag{1.33b}$$

**Casimir operators.** The quadratic Casimir operators for $sp(2n,\mathbb{R})$ and $osp(1|2p,\mathbb{R})$ are respectively given by,

$$\mathcal{C}_2\big[sp(2n,\mathbb{R})\big] = -\frac{1}{4}\, T_{AB}\, T^{AB}\,, \qquad \mathcal{C}_2\big[osp(1|2p,\mathbb{R})\big] = -\frac{1}{2}\, Q_i Q^i - \frac{1}{4}\, \tau_{ij}\tau^{ij}\,, \tag{1.34}$$

and a direct computation shows that, in the previously described oscillator realisation, these Casimir operators are related to one another via

$$\mathcal{C}_2\big[sp(2n,\mathbb{R})\big] = \frac{n}{8}\,(2p-1)(2p+2n+1) - \mathcal{C}_2\big[osp(1|2p,\mathbb{R})\big]\,. \tag{1.35}$$

In particular, for $n = 2$ and $p = \ell - 1$, one finds

$$\mathcal{C}_2\big[sp(4,\mathbb{R})\big] + \mathcal{C}_2\big[osp(1|2(\ell-1),\mathbb{R})\big] = -\frac{1}{4}\,(3-2\ell)(3+2\ell)\,, \tag{1.36}$$

this last number being the values of the quadratic Casimir operator of $so(2,3) \cong sp(4,\mathbb{R})$ on the module of the order-$\ell$ scalar singleton. This is a first hint that one may recover the type-A$_\ell$ higher spin algebra as the centraliser of $osp(1|2(\ell-1),\mathbb{R})$ in the Weyl-Clifford algebra, modulo $osp(1|2(\ell-1),\mathbb{R})$ generators, as we shall prove in the next paragraphs.

**Partially-massless higher spin algebra.** In order to identify the type-A$_\ell$ higher spin algebra, let us first give an equivalent presentation of the Weyl-Clifford algebra in terms of symbols of the previous oscillators, that we will denote by $Y^A$ and $\phi_i^A$ and which are commuting and anticommuting respectively. Their product is the graded version of the previously discussed Moyal-Weyl product[4]

$$f \star g = f\, \exp\left(\frac{\overleftarrow{\partial}}{\partial Y^A}\, C^{AB}\, \frac{\overrightarrow{\partial}}{\partial Y^B} + \frac{\overleftarrow{\partial}}{\partial \phi_i^A}\, C^{AB}\, \epsilon_{ij}\, \frac{\overrightarrow{\partial}}{\partial \phi_j^B}\right) g\,, \tag{1.37}$$

where $f$ and $g$ are arbitrary polynomials in $Y^A$ and $\phi_i^A$. The symbols of the $sp(2n,\mathbb{R})$ and $osp(1|2p,\mathbb{R})$ generators are simply

$$T^{AB} = \frac{1}{2}\, Y^A Y^B - \frac{1}{2}\, \epsilon^{ij}\, \phi_i^A \phi_j^B\,, \qquad Q_i = \frac{1}{2}\, C_{AB}\, Y^A \phi_i^B\,, \qquad \tau_{ij} = \frac{1}{2}\, C_{AB}\, \phi_i^A \phi_j^B\,, \tag{1.38}$$

respectively.

Now let us characterise the centraliser of $osp(1|2p,\mathbb{R})$ in $\mathcal{A}_{2n|4np}$, that is the space of elements annihilated by

$$[Q_i, -]_\star = \phi_i^A\, \frac{\partial}{\partial Y^A} - \epsilon_{ij}\, Y^A\, \frac{\partial}{\partial \phi_j^A}\,, \tag{1.39}$$

[4] Note that, for a homogeneous element $f \in \mathcal{A}_{2n|4np}$ of degree $|f|$, the left and right derivatives with respect to $Y^A$ and $\phi_i^A$ are related by $f\,\frac{\overleftarrow{\partial}}{\partial Y^A} = \frac{\partial}{\partial Y^A} f$ and $f\,\frac{\overleftarrow{\partial}}{\partial \phi_i^A} = -(-1)^{|f|}\,\frac{\partial}{\partial \phi_i^A} f$.

where $[-,-]_\star$ should be understood as the graded commutator (i.e. $[f,g]_\star = f\star g - (-1)^{|f||g|}\, g\star f$ for homogeneous elements of the Weyl-Clifford algebra $f$ and $g$). This condition is solved by considering any function of the symbol of the $sp(2n,\mathbb{R})$ generators $T^{AB}$,

$$f(Y^A,\phi_i^B) \in \mathcal{Z}_{\mathcal{A}_{2n|4np}}(osp(1|2p,\mathbb{R})) \qquad \Leftrightarrow \qquad f(Y^A,\phi_i^B) = f(T^{AB})\,, \tag{1.40}$$

since the symbols $T^{AB}$ are characteristics of the first order partial differential equations $[Q_i,f]_\star = 0$. Due to the fact that the $sp(2n,\mathbb{R})$ contain a piece quadratic in the anticommuting variables $\phi_i^A$, the only possible diagram that can appear when decomposing the centraliser of $osp(1|2p,\mathbb{R})$ are those whose second row (and by extension, all rows except the first one) are of length smaller than $2p$. Indeed, upon splitting the $sp(2p,\mathbb{R})$ indices as $i=(+\alpha,-\alpha)$ with $\alpha = 1,\dots,p$, we have

$$\frac{1}{2}\,\epsilon^{ij}\,\phi_i^A\,\phi_j^B = \sum_{\alpha=1}^{p} \varphi_\alpha^{AB}\,, \qquad \varphi_\alpha^{AB} := \phi_{+\alpha}^{(A}\,\phi_{-\alpha}^{B)}\,, \tag{1.41}$$

where

$$\varphi_\alpha^{(AB}\,\varphi_\alpha^{CD)} = 0\,, \tag{1.42}$$

by virtue of the fact that $\phi_i^A$ are anticommuting. Note also that since the 'building blocks' of the centraliser of $osp(1|2p,\mathbb{R})$ are rank-2 symmetric tensors of $sp(2n,\mathbb{R})$, all diagrams appearing will have an even number of boxes, and in particular, each row will be of even length. This means that, for $n=2$, diagrams appearing in the centraliser of $osp(1|2p,\mathbb{R})$ will be of the form

$$\begin{array}{|l|}\hline \quad 2s-t-1 \quad \\ \hline t-1 \\ \cline{1-1} \end{array}\,, \tag{1.43}$$

with $s \geq 1$ and $t = 1,3,\dots,2p+1$, so that upon setting $p = \ell - 1$ as before, we recover exactly the spectrum of diagrams expected to appear in the higher spin algebra $\mathfrak{hs}_\ell$ (in the $sp(4,\mathbb{R})$ basis). However, these diagrams are not traceless in $sp(4,\mathbb{R})$ sense a priori: consider for instance the product of two $sp(4,\mathbb{R})$ generators, which can be projected onto a totally symmetric part,

$$T^{ABCD} := T^{(AB}\,T^{CD)} \qquad \longleftrightarrow \qquad \begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \end{array}\,, \tag{1.44}$$

and a piece with the symmetry of a 'window-shaped' diagram,

$$T^{AB,CD} := T^{AB}\,T^{CD} - T^{A(C}\,T^{D)B} \qquad \longleftrightarrow \qquad \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline \end{array}\,. \tag{1.45}$$

While the first one, the totally symmetric part, is trivially traceless, the second one is not since

$$C_{BC}\,T^{AB,CD} \;\propto\; 2\,\epsilon^{ij}\,\phi_i^{[A}Y^{D]}\,Q_j - \epsilon^{ij}\epsilon^{kl}\,\phi_i^A\phi_k^D\,\tau_{jl}\,, \tag{1.46}$$

does not vanish identically, but is proportional to the $osp(1|2(\ell-1),\mathbb{R})$ generators. This is in fact a general feature, namely all traces are proportional to these generators. Indeed, taking a trace in the $sp(4,\mathbb{R})$ sense means contracting the capital latin indices $A,B,\dots$ with the invariant tensor $C_{AB}$, which thereby produces the generators $Q_i$ and $\tau_{ij}$. Consequently, we can remove traces by modding out the centraliser of $osp(1|2(\ell-1),\mathbb{R})$ by the ideal generated by $Q_i$ and $\tau_{ij}$, and thereby obtain the type-A$_\ell$ higher spin algebra in four dimensions as the quotient[5]

$$\mathfrak{hs}_\ell \cong \mathcal{Z}_{\mathcal{A}_{4|8(\ell-1)}}\big(osp(1|2(\ell-1),\mathbb{R})\big)\Big/\langle Q_i \oplus \tau_{ij}\rangle \,. \tag{1.47}$$

The main difference compared to the realisation reviewed in the previous section is that here, the 'order' $\ell$ of the theory is no longer controlled by choosing different ideal to mod out from the centraliser of the Howe dual algebra, but by the choice of the Howe dual algebra itself. This allows us to slightly simplify the identification of the type-A$_\ell$ higher spin algebra in four dimensions with respect to the arbitrary dimension construction.

Given that $\mathfrak{hs}_\ell$ is the symmetry algebra of the order-$\ell$ scalar singleton, and having found a realisation of it within the Weyl-Clifford algebra, it is natural to seek a realisation of the order-$\ell$ singleton in the Fock space generated by the bosonic and fermionic oscillators used above, which we will do in the next section.

## 2 Higher order singleton module

The Weyl-Clifford algebra generated by the oscillators $\hat{Y}^A$ and $\hat{\phi}_i^A$ introduced in section 1 naturally acts on the Fock space generated by $n$ pairs of bosonic creation-annihilation operators,

$$[\mathsf{a}_a, \bar{\mathsf{a}}^b] = \delta_a^b\,\mathbf{1}\,, \qquad \bar{\mathsf{a}}^a := (\mathsf{a}_a)^\dagger\,, \qquad a,b,\dots = 1,\dots,n\,, \tag{2.1}$$

and $2np$ fermionic ones,

$$\{\mathsf{c}_a^i, \bar{\mathsf{c}}_j^b\} = \delta_j^i\,\delta_a^b\,\mathbf{1}\,, \qquad \bar{\mathsf{c}}_i^a := (\mathsf{c}_a^i)^\dagger\,, \qquad i,j,\dots = 1,\dots,2p\,. \tag{2.2}$$

In fact, the Weyl-Clifford algebra is the algebra of endomorphisms of this Fock space. Bilinears in these creation-annihilation operators form a Lie subalgebra isomorphic to $osp(2n|4np,\mathbb{R})$, which contain the dual pair $sp(2n,\mathbb{R}) \oplus osp(1|2p,\mathbb{R})$ discussed previously. Introducing the notation,

$$v \cdot w := \epsilon^{ij}\,v_i\,w_j = \epsilon_{ij}\,v^i\,w^j\,, \tag{2.3}$$

for the contraction of the $sp(2p,\mathbb{R})$ indices, the generators of $sp(2n,\mathbb{R})$ are given by

$$T^{ab} := \bar{\mathsf{a}}^a\,\bar{\mathsf{a}}^b - \bar{\mathsf{c}}^a \cdot \bar{\mathsf{c}}^b\,, \qquad\qquad T_{ab} := \mathsf{a}_a\,\mathsf{a}_b - \mathsf{c}_a \cdot \mathsf{c}_b\,, \tag{2.4a}$$

$$T^a{}_b := \bar{\mathsf{a}}^a\,\mathsf{a}_b + \bar{\mathsf{c}}^a \cdot \mathsf{c}_b + \frac{1-2p}{2}\,\delta_b^a\,\mathbf{1}\,, \tag{2.4b}$$

while the generators of $osp(1|2p,\mathbb{R})$ read

$$Q_i := \frac{1}{2}\left(\bar{\mathsf{c}}_i^a\,\mathsf{a}_a - \bar{\mathsf{a}}^a\,\epsilon_{ij}\,\mathsf{c}_a^j\right), \qquad \tau_{ij} := \epsilon_{k(i}\,\bar{\mathsf{c}}_{j)}^a\,\mathsf{c}_a^k\,. \tag{2.5}$$

[5]Note that this definition also works for $\ell = 1$, even though this case may seem degenerate at first glance. Indeed, in this case the Howe dual algebra becomes trivial, which is simply a consequence of the fact the relevant Howe dual *group* is the finite group $\mathbb{Z}_2$. This group acts on the Weyl algebra by reflections $Y^A \to -Y^A$, so that its centraliser is nothing but the *even* subalgebra, the subalgebra of polynomials in an even number of $Y^A$'s.



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Now let us isolate the $sp(2n,\mathbb{R})$ representation dual to the trivial irrep of $osp(1|2p,\mathbb{R})$. Doing so amounts to finding states in the Fock space which are annihilated by the action of the $osp(1|2p,\mathbb{R})$ supercharges, i.e.

$$Q_i\, f(\mathsf{\bar a},\mathsf{\bar c})|0\rangle = 0\,, \tag{2.6}$$

which is solved by

$$f(\mathsf{\bar a},\mathsf{\bar c}) = f(T^{ab})\,, \tag{2.7}$$

that is, any function of the $sp(2n,\mathbb{R})$ raising operators $T^{ab}$. Since the vacuum $|0\rangle$ of the Fock space is $osp(1|2p,\mathbb{R})$-invariant, it defines a lowest weight vector for the dual $sp(2n,\mathbb{R})$-module, with weight

$$\Big(\underbrace{\frac{1-2p}{2},\dots,\frac{1-2p}{2}}_{n\,\text{times}}\Big)\,, \tag{2.8}$$

with respect to the Cartan subalgebra spanned by the generators $T^a{}_a$ (no summation implied). The subspace of homogeneous polynomials of degree $k$ in $T^{ab}$ is preserved by the action of the $u(n)$ subalgebra generated by $T^a{}_b$ (since the latter preserve the number of creation/annihilation operators). The decomposition of these subspaces into irreducible representations of $u(n)$ consists of all Young diagram with $2k$ boxes, whose rows are all of even length and such that the second row is of length at most $2p$. In particular, for $n = 2$, the lowest weight $sp(4,\mathbb{R})$-module dual to the trivial $osp(1|2p,\mathbb{R})$-representation admits the decomposition

$$\mathcal{D}_{sp(4,\mathbb{R})}\Big(\frac{1-2p}{2},\frac{1-2p}{2}\Big) \cong \bigoplus_{s=0}^{\infty}\bigoplus_{k=0}^{p}\Big[\frac{1-2p}{2}+2s+2k,\frac{1-2p}{2}+2k\Big]_{u(2)}\,, \tag{2.9}$$

under the maximal compact subalgebra $u(2)\subset sp(4,\mathbb{R})$. Taking into account the isomorphism

$$[\lambda_1,\lambda_2]_{u(2)} \cong \Big[\frac{\lambda_1+\lambda_2}{2},\frac{\lambda_1-\lambda_2}{2}\Big]_{so(2)\oplus so(3)}\,, \tag{2.10}$$

between finite-dimensional irreps of $u(2)$ and $so(2)\oplus so(3)$ and setting $p=\ell-1$, this decomposition matches the one of the order-$\ell$ singleton module

$$\mathcal{D}_{so(2,3)}\Big(\frac{3-2\ell}{2},0\Big) \cong \bigoplus_{s=0}^{\infty}\bigoplus_{t=1,3,\dots}^{2\ell-1}\Big[\frac{3-2\ell}{2}+t-1+s,s\Big]_{so(2)\oplus so(3)} \tag{2.11}$$

in three dimensions (see e.g. [48, section 3.4.2]).

**A word about non-unitarity.** Let us conclude this section by commenting on the non-unitarity of these modules. Recall that higher-order singletons are lowest weight irreps of $so(2,d)$, which can therefore be described by its lowest weight vector $|\phi\rangle$, obeying

$$(D-\Delta_\phi)|\phi\rangle = 0\,,\qquad J_{ab}|\phi\rangle = 0\,,\qquad K_a|\phi\rangle = 0\,, \tag{2.12}$$

where $D$, $J_{ab}$ and $K_a$ are the dilation, Lorentz, and special conformal transformation generators. All states of the modules are obtained by repeated application of the translation generators $P_a$ on $|\phi\rangle$. Using the relations

$$[K_a, P_b] = \eta_{ab}\, D - M_{ab}\,, \qquad [D, P_a] = P_a\,, \qquad [M_{ab}, P_c] = 2\,\eta_{c[b}\, P_{a]}\,, \tag{2.13}$$

one finds

$$K_a\, P^2\, |\phi\rangle = 2\left(\Delta_\phi - \frac{d-2}{2}\right) P_a|\phi\rangle\,, \tag{2.14}$$

and with $P_a^\dagger = K_a$, this implies

$$\|P^2\, |\phi\rangle\|^2 = 2d\, \Delta_\phi \left(\Delta_\phi - \frac{d-2}{2}\right) \langle\phi\,|\,\phi\rangle\,. \tag{2.15}$$

The above identities tells us that $P^2\,|\phi\rangle$ is singular and null for $\Delta_\phi = \frac{d-2}{2}$, while for $\Delta_\phi < \frac{d-2}{2}$ it is not singular but acquires a negative norm. For the order-$\ell$ singleton, $\Delta_\phi = \frac{d-2\ell}{2}$ and hence the presence of $P^2|\phi\rangle$ is one of the first indications that the module is non-unitary.

Now coming back to our construction, it may be surprising that such a non-unitary module can be realised in a Fock space, which is usually itself a unitary module (for the Heisenberg algebra, or its supersymmetric version relevant here). A first consistency check is that this negative norm state $P^2|\phi\rangle$ is indeed present, since we recover the correct $u(2)$ decomposition. More importantly, the Hermitian conjugation *does not preserve* the $osp(1|2p,\mathbb{R})$ generators in this realisation, which is why we have a non-unitary module in a Fock space.

**Higher order spinor singleton?** Note that one could look for other representation of $osp(1|2p,\mathbb{R})$ than the trivial one. For instance, the 'next-to-simplest' representation is of dimension $2p+1 \equiv 2\ell-1$ and splits into a direct sum of $sp(2p,\mathbb{R})$ irreps, the trivial and the vector (or fundamental) one. It can be realised in the Fock space considered here as the subspace with basis

$$\bar{\mathsf{a}}^a|0\rangle \qquad \text{and} \qquad \bar{\mathsf{c}}_i^a|0\rangle\,, \tag{2.16}$$

which are indeed, for $osp(1|2p,\mathbb{R})$, a scalar and a vector respectively. This subspace is preserved by the action of $osp(1|2p,\mathbb{R})$ since

$$Q_i\, \bar{\mathsf{a}}^a|0\rangle = \frac{1}{2}\, \bar{\mathsf{c}}_i^a|0\rangle\,, \qquad Q_i\, \bar{\mathsf{c}}_j^a|0\rangle = -\frac{1}{2}\, \epsilon_{ij}\, \bar{\mathsf{a}}^a|0\rangle\,, \tag{2.17}$$

while the $sp(2p,\mathbb{R})$ generators $\tau_{ij}$ merely rotate this states, as expected. As usual in the context of Howe duality, this representation appears with a *multiplicity*, as indicated by the fact that the above basis vectors also carry an $sp(2n,\mathbb{R})$ index. In fact, as in the case of the trivial representation, any state obtained from the above basis vectors by the action of $osp(1|2p,\mathbb{R})$-invariant operators, which are generated by the Howe dual algebra $sp(2n,\mathbb{R})$, will not change the representation. In other words, this finite-dimensional representation of $osp(1|2p,\mathbb{R})$ appears with *infinite multiplicity* in the Fock space, but this feature is merely the reflection of the fact that it is Howe dual to a lowest weight module of $sp(2n,\mathbb{R})$, which is infinite-dimensional.



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The lowest weight $sp(2n,\mathbb{R})$-module in question is induced by the lowest $u(n)$-irrep spanned by the state $\bar{\mathsf{a}}^a|0\rangle$ and $\bar{\mathsf{c}}^a_i|0\rangle$, and generated by the action of the raising operators $T^{ab}$. The lowest weight reads

$$\left(\frac{3-2p}{2}, \frac{1-2p}{2}, \dots, \frac{1-2p}{2}\right), \tag{2.18}$$

which, in the case of $sp(4,\mathbb{R}) \cong so(2,3)$, corresponds to the lowest weight

$$\left[\frac{3-2p}{2}, \frac{1-2p}{2}\right]_{u(2)} = \left[2-\ell, \frac{1}{2}\right]_{so(2)\oplus so(3)}, \tag{2.19}$$

whose components are respectively the conformal weight and spin of the spinor singleton of order $\ell$ (i.e. a free spinor $\psi$ subject to the higher order Dirac equation $\partial\!\!\!/^{2\ell-1}\psi \approx 0$ as recalled in the Introduction). In other words, we find that the higher order spinor singleton

$$\mathcal{D}_{sp(4,\mathbb{R})}\left(\frac{3-2p}{2}, \frac{1-2p}{2}\right) \cong \mathcal{D}_{so(2,3)}\left(2-\ell, \frac{1}{2}\right), \tag{2.20}$$

is Howe dual to the finite-dimensional $osp(1|2p,\mathbb{R})$ representation made out of the trivial and vector $sp(2p,\mathbb{R})$-irreps.

This therefore begs the question: can we find the type-$\mathrm{B}_\ell$ higher spin algebra in our construction? To do so, one would need to quotient the centraliser of $osp(1|2p,\mathbb{R})$ in the Weyl-Clifford algebra by a different ideal than the one generated by $osp(1|2p,\mathbb{R})$. Indeed, we saw previously that the scalar singleton is Howe dual to the *trivial* representation of $osp(1|2p,\mathbb{R})$, and hence the full algebra is the annihilator of this representation. The quotient by $osp(1|2p,\mathbb{R})$ should be understood as the quotient by the annihilator of this trivial representation — as recalled above when we discussed the definition of the type-$\mathrm{A}_\ell$ algebra in arbitrary dimensions. Having this framework in mind, we should quotient the centraliser of $osp(1|2p,\mathbb{R})$ by the annihilator of its $(2\ell-1)$-dimensional irrep in order to obtain the type-$\mathrm{B}_\ell$ higher spin algebra. Schematically, this means modding out by higher powers of the $osp(1|2p,\mathbb{R})$ generators, which in turn amounts to keeping some of the traces in the diagrams (1.43), as may be expected to reproduce the spectrum of the type-$\mathrm{B}_\ell$ algebra. Such an analysis is however beyond the scope of this paper, and we leave it for potential future work.

## 3 Formal partially-massless higher spin gravity

Having build an oscillator realisation of the higher spin algebra $\mathfrak{hs}_\ell$ in four dimensions, we will now use it to try and construct an interacting theory of partially-massless higher spin fields.

The most common way of constructing formal higher spin gravities is to consider a gauge connection $\omega$ of the relevant higher spin algebra $\mathfrak{hs}$, together with a zero-form $C$ taking value in a module of this algebra (see e.g. [27, 32, 39, 41, 49–54]). This data is associated with the coordinates on a $Q$-manifold, which we denote by the same symbols, and whose (co)homological vector field $Q$ encodes the interactions. More precisely, one is then charged with constructing equations of motion

$$d\omega = \mathcal{V}(\omega,\omega) + \mathcal{V}(\omega,\omega,C) + \dots\,, \tag{3.1}$$

$$dC = \mathcal{U}(\omega,C) + \mathcal{U}(\omega,C,C) + \dots\,, \tag{3.2}$$

where $\mathcal{V}$ and $\mathcal{U}$ are the component of $Q$, and the initial data for the deformation problem reads

$$\mathcal{V}(a,b) = a \star b\,, \qquad\qquad \mathcal{U}(a,u) = a \star u - u \star \pi(a)\,, \tag{3.3}$$

where $\pi$ is an anti-involution of the higher spin algebra. At this point it is convenient to define ${}^{\mathbb{Z}_2}\mathfrak{hs} = \mathfrak{hs} \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{1, \pi\}$. In practice, one adds an element $k$ such that $k^2 = 1$ and $k \star a \star k = \pi(a)$.

Under some fairly general assumptions, one can show that the problem of constructing the $A_\infty$-algebra underlying the $Q$-manifold reduces to a much simpler problem of deforming ${}^{\mathbb{Z}_2}\mathfrak{hs}$ as an associative algebra [42, 54–56]. Moreover, often times it is easy to see that ${}^{\mathbb{Z}_2}\mathfrak{hs}$ can be deformed and even construct such a deformation, which we call $\mathbf{A}_u$, explicitly. Once $\mathbf{A}_u$ is available, there is an explicit procedure to construct all vertices. For example,

$$\mathcal{V}(a,b,u) = \phi_1(a,b) \star \pi(u)\,, \tag{3.4}$$

where $\phi_1$ is a (Hochschild) 2-cocycle that determines the first order deformation of ${}^{\mathbb{Z}_2}\mathfrak{hs}$ to $\mathbf{A}_u$:

$$a \circ b = a \star b + u\,\phi_1(a,b)k + \mathcal{O}(u^2)\,. \tag{3.5}$$

It has to be noted that the above form of the vertices is non-minimal: the equations, in general, 'mix' different spins even at the free level. One therefore needs to find a suitable field redefinition to bring the vertex in its 'minimal' form wherein such mixing are absent.

The deformed algebra $\mathbf{A}_u$ is defined from $\mathfrak{hs}$, the latter being usually obtained via either one of the following constructions:

($a$) A quotient of the universal enveloping algebra $\mathcal{U}(so(d,2))$ by a two-sided ideal $\mathcal{I}$ (in most cases called the Joseph ideal), which corresponds to the annihilator of a given irreducible $so(2,d)$-module, e.g. [28, 32, 57–60];

($b$) Using an oscillator realisation, wherein one embeds $so(2,d)$ and its enveloping algebra in a Weyl(–Clifford) algebra and typically obtain $\mathfrak{hs}$ as the quotient of the centraliser of a Howe dual algebra, as discussed above for the type-$\mathrm{A}_\ell$ algebra, as well as in [38, 39, 41, 59, 61, 62] and references therein;

($c$) Via the *quasi-conformal* realisation, which consists in explicitly solving the defining relations of the (Joseph) ideal mentioned previously, see e.g. [59, 63–65].

The first order deformation defined by the 2-cocycle $\phi_1$ makes its presence felt already at the free level. Indeed, linearizing the above equations around an (A)dS$_{d+1}$ background,

$$\omega_0 = e^a\,P_a + \frac{1}{2}\,\varpi^{a,b}\,L_{ab}\,, \qquad C_0 = 0\,, \qquad d\omega_0 + \frac{1}{2}\,[\omega_0, \omega_0] = 0\,, \tag{3.6}$$

their first order in the field fluctuations should reproduce the free field equations for partially-massless in the frame-like formalism [19], whose schematic form reads

$$R[\omega_1]^{a(s-1),b(s-t)} = e_c \wedge e_d\, C_1^{a(s-1)c,b(s-t)d}\,, \qquad R[\omega_1]^{a(s-1-m),b(s-t-n)} = 0\,, \tag{3.7}$$

where $R[\omega_1]^{a(s-1-m),b(s-t-n)} = \nabla\omega_1^{a(s-1-m),b(s-t-n)} + \dots$, with $\omega_1$ the first order fluctuations of a 1-form valued in $\mathfrak{hs}_\ell$. More specifically, the components of 1-form $\omega_1$ takes values in the finite-dimensional representations of $so(2,d)$ labelled by the two-row Young diagrams of the form

$$\begin{array}{|l|}\hline s-1 \\ \hline s-t \\ \hline \end{array} \qquad t=1,3,\dots,2\ell-1\,, \quad s=t,t+1,\dots, \tag{3.8}$$

which corresponds to generators of the form

$$M_{\mathsf{A}(s-1),\mathsf{B}(s-t)} = \underbrace{M_{\mathsf{AB}}\cdots M_{\mathsf{AB}}}_{s-t}\,\underbrace{M_{\mathsf{A}}{}^{\mathsf{C}}M_{\mathsf{AC}}\cdots M_{\mathsf{A}}{}^{\mathsf{C}}M_{\mathsf{AC}}}_{\frac{t-1}{2}}+\dots\,, \tag{3.9}$$

where the dots denote terms ensuring that the right hand side has the symmetry of the above Young diagram, and is traceless. The first order fluctuation of the zero-form takes values in a representation of $\mathfrak{hs}_\ell$, usually called the 'twisted-adjoint representation'.[6] This module of the type-A$_\ell$ algebra is defined on the same vector space as $\mathfrak{hs}_\ell$, but where the latter acts via a 'twisted commutator'

$$\mathcal{U}(\omega_0,C_1)=\omega_0\star C_1-C_1\star\pi(\omega_0)\,, \tag{3.10}$$

hence the name of this representation. The zero-forms can therefore be expanded in a basis of generators of the (partially-massless) higher spin algebra [32, 48]. Typically, the Weyl tensor $C^{a(s),b(s-t+1)}$, for the spin-$s$ and depth-$t$ partially-massless field is the component of $C_1$ along the generator of $\mathfrak{hs}_\ell$ which schematically reads,

$$\underbrace{L_{ab}\dots L_{ab}}_{s-t+1}\underbrace{P_a\dots P_a}_{t-1}P^{2\ell-t-1}+\dots\,, \tag{3.11}$$

where we separated generators of $so(2,d)$ into those of the Lorentz subalgebra $so(1,d)$, denoted by $L_{ab}$, and the transvection (or AdS-translation) generators denoted by $P_a$.

The low spin ($s=1$ and 2) components of these fluctuations are given by

$$\omega_1=A\cdot\mathbf{1}+h^a\,P_a+\frac{1}{2}\,\omega^{ab}\,L_{ab}+\dots\,,\qquad C_1=\frac{1}{2}\,F^{a,b}\,\mathcal{M}_{a,b}+\dots\,, \tag{3.12}$$

where, to keep this discussion fairly general, we denoted by $\mathcal{M}_{a,b}$ the generator of $\mathfrak{hs}$ along which one finds the Maxwell tensor, independently of the higher spin algebra of interest. In the type-A case, it would simply be $\mathcal{M}_{a,b}=L_{ab}$, while in the type-A$_\ell$ case, it would be of the form $L_{ab}P^{2(\ell-1)}+(\dots)$ instead. When comparing (3.1) to the previous free equations of motion, e.g. in the spin-1 sector,

$$dA=e_a\wedge e_b\,F^{a,b}+\dots\,, \tag{3.13}$$

we can deduce that $\mathcal{V}$ yields

$$\mathcal{V}(P_a,P_b;\mathcal{M}_{c,d})=2\,\eta_{a[c}\,\eta_{d]b}\,\mathbf{1}+\dots\,, \tag{3.14}$$

when evaluated on $P_a\otimes P_b\otimes\mathcal{M}_{c,d}$. From (3.4), we know that the dots in the previous equation originate from the expression

$$\phi_1(P_a,P_b)\star\mathcal{M}_{a,b}=2\,\eta_{a[c}\,\eta_{d]b}\,\mathbf{1}+\dots\,, \tag{3.15}$$

[6] Although it may be more relevant to think of it as a coadjoint module [66, appendix B].

modulo the field-redefinition bringing the vertex in its 'minimal' form. In other words, the product of $\phi_1(P_a, P_b)$ and the generator that corresponds to the Maxwell tensor must contain the unit of $\mathfrak{hs}$. Let us note that $\phi_1(P_a, P_b)$ is also the simplest term to probe the deformation since $\phi_1(\mathbf{1}, \bullet) = 0$ and $\phi_1(L_{ab}, \bullet) = 0$. The first condition means that the unit is not deformed and the second one protects Lorentz symmetry.

Recalling that $\mathfrak{hs}$ has an invariant trace tr (defined as the projection onto the unit), the above condition can also be rewritten as

$$\mathrm{tr}(\phi_1(P_a, P_b) \star \mathcal{M}_{a,b}) \neq 0\,. \tag{3.16}$$

Since the basis of any higher spin algebra $\mathfrak{hs}$ can be decomposed into finite-dimensional $so(d,2)$-modules, and that the trace respects $so(d,2)$, different generators are orthogonal to each other. As a result, we have to have

$$\phi_1(P_a, P_b) \propto \eta_{ab} + T_{ab} + \dots \qquad\qquad \mathrm{tr}(T_{ab} \star \mathcal{M}_{a,b}) \neq 0\,, \tag{3.17}$$

where $T_{ab}$ is a generator of $\mathbf{A}_u$ that *deforms* the commutator $[P_a, P_b]$. In other words, the generator $T_{ab}$ of this deformation is a multiple of the *dual* of the Maxwell tensor generator $\mathcal{M}_{a,b}$. In order to define $\mathbf{A}_u$, one needs to define $[P_a, P_b] = L_{ab} + uk\, T_{ab}$ and deform the Joseph ideal accordingly.

The Maxwell tensor (and the whole decomposition) can be found by decomposing the twisted-adjoint action $\{P_a, \bullet\}$ of translations on $C$. The adjoint of the Lorentz algebra may appear with multiplicity greater than 1 (this happens for instance in the type-B or Type-A$_\ell$, $\ell > 1$, cases). The Maxwell equations should have the form

$$\nabla F^{a,b} = h_c F^{ac,b} \tag{3.18}$$

$$\nabla F^{ab,c} \propto h^{(a} F^{b),c} - \frac{1}{d} h_\times \left(\eta^{ab} F^{c,\times} - \eta^{c(a} F^{b),\times}\right) + \dots\,, \tag{3.19}$$

where the first line comes from the anticommutator $\{P_m, \mathcal{M}_{ac,b}\}$, where $\mathcal{M}_{ab,c}$ is a traceless and hook-symmetric generator of the form $\mathcal{M}_{a,b}P_c + (\dots)$, and in the second line from $\{P_m, \mathcal{M}_{a,b}\}$. Most importantly, $F^{a,b}$ must not contribute anywhere else. The second equation means that, at the algebra level, one finds

$$\{P_{(a}, \mathcal{M}_{b),c}\} = \mathcal{M}_{ab,c}\,. \tag{3.20}$$

which implies that $\{P^a, \mathcal{M}_{a,b}\} = 0$, and hence this anticommutator must be a part of the two-sided ideal defining $\mathfrak{hs}$. This is indeed the case for the Type-A algebra, whose Joseph ideal contains $\{P^m, L_{mb}\} = 0$. For the type-A$_\ell$ case, $\mathcal{M}_{a,b}$ must be in the adjoint representation that sits inside the subspace of monomials of order $\ell - 1$ in the $so(2,d)$ generator (i.e. one degree less than the generator $\mathcal{J}_{\mathsf{A}(2\ell)}$ of the Joseph ideal).

**Probing deformation through cycles.** Cocycles are more complicated than cycles to derive since cocycles are defined on the whole algebra (must be assigned some value for all possible arguments), while cycles involve few specific elements of the algebra. Nontrivial cocycles can be evaluated on nontrivial cycles, the result being nonzero. In the type-A case, the Maxwell equation probes the cycle [42, appendix B]

$$c_{(1)} = L_{ab} \otimes P^a \otimes P^b + \frac{1}{4}\,(\mathbf{1} \otimes L_{ab} \otimes L^{ab}) - \frac{1}{4}\,C_L(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1})\,, \qquad C_L = -\frac{d(d-2)}{4}\,, \tag{3.21}$$

which is closed by virtue of the fact that $\{L_{ab}, P^b\} \sim 0$ and $-\frac{1}{2}\,L_{ab}L^{ab} \sim C_L\,\mathbf{1}$ due to the quotient by the Joseph ideal. As it turns out, one can find a counterpart of this cycle in $\mathfrak{hs}_\ell$, namely

$$c_{(\ell)} = \mathcal{M}_{a,b} \otimes P^a \otimes P^b + \frac{1}{2}\,(\mathbf{1} \otimes \mathcal{M}_{a,b} \otimes L^{ab} - P^a \otimes P^b \otimes \mathcal{M}_{a,b} - P^a \otimes \mathcal{M}_{a,b} \otimes P^b) + \dots\,, \tag{3.22}$$

which is closed as a consequence of the fact that $\{\mathcal{M}_{a,b}, P^b\} \sim 0$, *up to a term* $\mathbf{1} \otimes \mathcal{M}_{a,b}L^{ab}$ (hence the dots). To verify that this is indeed a cycle, first note that

$$\partial(\mathcal{M}_{a,b} \otimes P^a \otimes P^b) = \{\mathcal{M}_{a,b}, P^a\} \otimes P^b - \frac{1}{2}\mathcal{M}_{a,b} \otimes L^{ab} \sim -\frac{1}{2}\mathcal{M}_{a,b} \otimes L^{ab}\,, \tag{3.23}$$

as we have previously argued that $\{\mathcal{M}_{a,b}, P^a\}$ belongs to the defining ideal of $\mathfrak{hs}_\ell$ (in fact, any higher spin algebra containing a massless spin-1 fields in its spectrum), and hence is modded out. The remaining term is compensated thanks to

$$\partial(\mathbf{1} \otimes \mathcal{M}_{a,b} \otimes L^{ab}) = \mathcal{M}_{a,b} \otimes L^{ab} - \mathbf{1} \otimes \mathcal{M}_{a,b}\,L^{ab} + L^{ab} \otimes \mathcal{M}_{a,b}\,, \tag{3.24}$$

which however brings in two other terms. The last one can be cancelled using

$$\partial(P^a \otimes P^b \otimes \mathcal{M}_{a,b} + P^a \otimes \mathcal{M}_{a,b} \otimes P^b) \sim L^{ab} \otimes \mathcal{M}_{a,b}\,, \tag{3.25}$$

where we made use of $\{\mathcal{M}_{a,b}, P^b\} \sim 0$ again, which leaves us with the final task of eliminating the term proportional to $\mathbf{1} \otimes \mathcal{M}_{a,b}\,L^{ab}$. In the type-A example, we could take advantage of the fact that the term $L_{ab}L^{ab}$ is proportional to the identity in $\mathfrak{hs}$. This is a simple consequence of quotienting $\mathcal{U}(so(2,d))$ by the Joseph ideal.[7] We can expect that a similar property also holds for type-A$_\ell$ algebras, by inspecting its spectrum: since $\mathcal{M}_{a,b} = L_{ab}P^{2(\ell-1)} + (\dots)$ belongs to the $so(2,d)$-irrep $(2\ell-1,1)$, the contraction $\mathcal{M}_{a,b}\,L^{ab}$ belongs to $(2\ell)$,

$$\mathcal{M}_{a,b} \in \begin{array}{|c|l}\hline \multicolumn{2}{|c|}{\qquad 2\ell-1 \qquad} \\ \hline \;\; & \\ \cline{1-1}\end{array} \qquad \Longrightarrow \qquad \mathcal{M}_{a,b}\,L^{ab} \in \begin{array}{|c|}\hline \qquad\quad 2\ell \qquad\quad \\ \hline\end{array} \subset \mathcal{I}_\ell\,, \tag{3.26}$$

since the latter is the only $so(2,d)$ diagram susceptible to contain a Lorentz scalar. In other words, $\mathcal{M}_{a,b}\,L^{ab}$ is related to the scalar part of the generator $\mathcal{J}_{\mathsf{A}(2\ell)}$, whose structure is discussed in appendix A. We can expect that $\mathcal{M}_{a,b}\,L^{ab}$ is proportional to $P^{2(\ell-1)}$, or a polynomial in $P^2$ of degree $\ell-1$ more generally.

This is indeed the case for the type-A$_2$ algebra, where $\mathcal{M}_{a,b}\,L^{ab} \sim \#\,P^2$, as we show in appendix A. We can therefore use this identity and compensate the term $\mathbf{1}\otimes\mathcal{M}_{a,b}\,L^{ab}$ by adding

$$\mathbf{1} \otimes P_a \otimes P^a \qquad \Longrightarrow \qquad \partial(\mathbf{1} \otimes P_a \otimes P^a) = -\mathbf{1} \otimes P^2\,, \tag{3.27}$$

which, when added with the proper coefficient to $c_{(2)}$ above, defines a cycle of $\mathfrak{hs}_2$.

[7] More precisely, the scalar component of the generator $\mathcal{J}_{AB}$, when decomposed under the Lorentz algebra, relates $P^2$ to the quadratic Casimir operator of $so(2,d)$ which is itself proportional to the identity. Since $\mathcal{C}_2 = -\frac{1}{2}\,L_{ab}L^{ab} + P^2$, one therefore concludes that $-\frac{1}{2}\,L_{ab}L^{ab}$ is also proportional to the identity.



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**Oscillator realisation for type-A$_2$.** Let us compute the Maxwell generator in our oscillator realisation. To do so, first recall that the Lorentz generators are embedded in $sp(4,\mathbb{R})$ as

$$L^{\alpha\beta} = \frac{1}{4}\,\{\hat{y}^\alpha,\hat{y}^\beta\} - \frac{1}{4}\,\epsilon^{ij}\,[\hat{\phi}_i^\alpha,\hat{\phi}_j^\beta]\,, \qquad L^{\alpha'\beta'} = \frac{1}{4}\,\{\hat{y}^{\alpha'},\hat{y}^{\beta'}\} - \frac{1}{4}\,\epsilon^{ij}\,[\hat{\phi}_i^{\alpha'},\hat{\phi}_j^{\beta'}]\,, \tag{3.28}$$

where we split the oscillators (1.23a) as $\hat{Y}^A = (\hat{y}^\alpha,\hat{y}^{\alpha'})$ with $\alpha,\alpha' \in \{1,2\}$ indices for two-components spinors. Similarly, the transvection generators read

$$P^{\alpha\alpha'} = \frac{1}{4}\,\{\hat{y}^\alpha,\hat{y}^{\alpha'}\} - \frac{1}{4}\,\epsilon^{ij}\,[\hat{\phi}_i^\alpha,\hat{\phi}_j^{\alpha'}]\,. \tag{3.29}$$

Let us also introduce the notation

$$q_i = \frac{1}{2}\,\hat{y}_\alpha\,\hat{\phi}_i^\alpha\,, \qquad \overline{q}_i = \frac{1}{2}\,\hat{y}_{\alpha'}\,\hat{\phi}_i^{\alpha'}\,, \qquad t_{ij} = \frac{1}{4}\,\epsilon_{\alpha\beta}\,[\hat{\phi}_i^\alpha,\hat{\phi}_j^\beta]\,, \qquad \overline{t}_{ij} = \frac{1}{4}\,\epsilon_{\alpha'\beta'}\,[\hat{\phi}_i^{\alpha'},\hat{\phi}_j^{\beta'}]\,, \tag{3.30}$$

in terms of which the $osp(1|2p,\mathbb{R})$ generators read

$$Q_i = q_i + \overline{q}_i\,, \qquad \tau_{ij} = t_{ij} + \overline{t}_{ij}\,. \tag{3.31}$$

Note that $q_i$ and $t_{ij}$ form an $osp(1|2p,\mathbb{R})$ algebra, and $\overline{q}_i$ and $\overline{t}_{ij}$ as well. The square of the translation generators can be written as

$$P^2 = -\frac{1}{2}\,P_{\alpha\alpha'}\,P^{\alpha\alpha'} = \frac{2p-1}{2} + q_i\,\overline{q}^i + \frac{1}{2}\,t_{ij}\,\overline{t}^{ij}\,, \tag{3.32}$$

where the factor $-\frac{1}{2}$ comes from the $\gamma$-matrices used to convert vector indices into spinor ones.[8]

The Maxwell generator for $p=1 \Leftrightarrow \ell=2$ therefore becomes

$$\mathcal{M}_{\alpha\beta} = L_{\alpha\beta}\left(q_i\,\overline{q}^i + \frac{1}{2}\,\overline{t}_{ij}\,t^{ij} + \frac{1}{2}\right), \tag{3.33}$$

and similarly for $\mathcal{M}_{\alpha'\beta'}$, upon exchanging $L_{\alpha\beta}$ with $L_{\alpha'\beta'}$. A direct computation leads to

$$\{\mathcal{M}_{\alpha\beta}, P^\beta{}_{\alpha'}\} \sim 0\,, \tag{3.34}$$

upon using the identities

$$\hat{\phi}_i^\alpha\,t^2 = -2\,\hat{\phi}_j^\alpha\,t_i{}^j\,, \qquad t_{ik}\,t_j{}^k = -2\,t_{ij} + \frac{1}{2}\,\epsilon_{ij}\,t^2\,, \qquad \text{and} \qquad (t_{ij}+2\epsilon_{ij})\,t^2 = 6\,t_{ij}\,, \tag{3.35}$$

with $t^2 \equiv t_{ij}\,t^{ij}$, which can be proved thanks to Fierz identities.

Let us conclude this section by pointing a subtlety in the computation of the ideal generators in our oscillator realisation. Introducing $\hbar$ in the canonical anti/commutation relations as

$$[\hat{Y}^A,\hat{Y}^B] = 2\hbar\,C^{AB}\,, \qquad \{\hat{\phi}_i^A,\hat{\phi}_j^B\} = 2\hbar\,C^{AB}\,\epsilon_{ij}\,, \tag{3.36}$$

the $osp(1|2p,\mathbb{R})$ anti/commutation relations read

$$\{Q_i,Q_j\} = \hbar\,\tau_{ij}\,, \qquad [\tau_{ij},Q_k] = 2\hbar\,\epsilon_{k(i}\,Q_{j)}\,, \qquad [\tau_{ij},\tau_{kl}] = \hbar\,(\epsilon_{kj}\,\tau_{il} + \dots)\,, \tag{3.37}$$

[8]This can also be check by comparing $[L_{ab},P^b]$ and $[P_a,P^2]$ with their spinor counterparts, which shows that one should use $L_a \to \frac{1}{2}\left(\epsilon_{\alpha\beta}L_{\alpha'\beta'} + \epsilon_{\alpha'\beta'}L_{\alpha\beta}\right)$ and $P_a \to \frac{i}{\sqrt{2}}\,P_{\alpha\alpha'}$.

i.e. the right hand side of any anti/commutator is proportional to $\hbar$. Contracting the second relation with $\epsilon^{jk}$ yields

$$\hbar\, q_i = -\frac{1}{2p+1}\,[t_{ij}, Q^j] \qquad \Longrightarrow \qquad \hbar^2\, t_{ij} = -\frac{1}{2p+1}\,\{[t_{ij}, Q^j], q_j\}\,, \tag{3.38}$$

which could, in the absence of $\hbar$, lead one to conclude that $q_i$ and $t_{ij}$ can be set to zero (and similarly for $\bar{q}_i$ and $\bar{t}_{ij}$), when taking the quotient by $osp(1|2p,\mathbb{R})$. This would however be incorrect since it would amount to quotienting by $osp(1|2p,\mathbb{R}) \oplus osp(1|2p,\mathbb{R})$, one copy generated by $q_i$ and $t_{ij}$, and another copy by $\bar{q}_i$ and $\bar{t}_{ij}$. This direct sum is Howe dual to $sp(2,\mathbb{R}) \oplus sp(2,\mathbb{R})$, and not to $sp(4,\mathbb{R})$, as each copy of $osp(1|2p,\mathbb{R})$ does not commute with the transvection generators $P_{\alpha\alpha'}$. Consequently, it would be inconsistent to mod out $q_i$ and $\bar{q}_i$ separately (and similarly for $t_{ij}$ and $\bar{t}_{ij}$) in the centraliser of $sp(4,\mathbb{R})$ — in the sense that the resulting algebra would not be related to the type-A$_\ell$ higher spin algebra.

The introduction of $\hbar$ in computation also proves useful when it comes to checking that the scalar generator of the ideal also vanishes: the expression (3.32) of $P^2$ can be re-written as

$$-\frac{1}{2}\,P_{\alpha\alpha'}\,P^{\alpha\alpha'} = \hbar^2\,\frac{2p-1}{2} + q_i\,\bar{q}^i + \frac{1}{2}\,t_{ij}\,\bar{t}^{ij} = \hbar^2\,\frac{2p-1}{2} + q_i\,(Q^i - q^i) + \frac{1}{2}\,t_{ij}\,(\tau^{ij} - t^{ij})\,, \tag{3.39}$$

Evaluating $\mathscr{J}_{\bullet}^{(2)}$, which involves the previous equation for $p=1$ and modulo $Q_i$ and $\tau_{ij}$, yields

$$\mathscr{J}_{\bullet}^{(2)} = \left(P^2 - \frac{\hbar^2}{2}\right)\left(P^2 - \frac{5\hbar^2}{2}\right) \sim \left(q_i\,q^i + \frac{1}{2}\,t_{ij}\,t^{ij}\right)\left(q_k\,q^k + \frac{1}{2}\,t_{kl}\,t^{kl} + 2\,\hbar^2\right) \sim \left(q_i\,q^i + \frac{1}{2}\,t_{ij}\,t^{ij}\right)^2, \tag{3.40}$$

upon using $\hbar\, q_i \sim 0 \sim \hbar^2\, t_{ij}$. Using again Fierz identity and (3.35), one can show that

$$\left(q_i\,q^i + \frac{1}{2}\,t_{ij}\,t^{ij}\right)^2 \sim 0 \qquad \text{i.e.} \qquad \mathscr{J}_{\bullet}^{(2)} \sim 0\,, \tag{3.41}$$

modulo $Q_i$ and $\tau_{ij}$, as required.

## 4 Discussion

In this paper, we proposed a new realisation of the type-A$_\ell$ higher spin algebra in four dimensions, based on extending the Weyl algebra with a Clifford algebra. This allows for an arguably simpler realisation of $\mathfrak{hs}_\ell$, wherein the limit of the range of values of the depth of the partially-massless fields is constrained by the dimension of the Clifford algebra. We also exhibited a Hochschild 3-cycle of $\mathfrak{hs}_\ell$, which suggests that there should exist non-trivial deformations of the partially-massless higher spin algebras.

Unfortunately, the usual technique used to construct deformation of higher spin algebra that consists in using deformed oscillators [67], i.e. trading $\hat{Y}^A$ for $\hat{q}^A$ which satisfy

$$[\hat{q}^A, \hat{q}^B] = 2\,C^{AB}\,(\mathbf{1} + k\,\nu)\,, \tag{4.1}$$

where $k$ is the generator of the $\mathbb{Z}_2$ action on the Weyl algebra, discussed in the previous section, does *not* seem to work: we were unable to use this deformation and preserve a

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realisation of the $osp(1|2p,\mathbb{R})$ algebra *undeformed*. Indeed, note first that, assuming that the deformed oscillators $\hat{q}^A$ still *commute* with the fermionic oscillators implies

$$0 = [\phi_i^A, [\hat{q}^B, \hat{q}^C]] = 2\nu\, C^{BC}\, [\phi_i^A, k] \qquad \Longrightarrow \qquad [\phi_i^A, k] = 0\,, \tag{4.2}$$

i.e. the fermionic oscillators should also *commute* with the generator of the $\mathbb{Z}_2$ action — also called the Klein operator. A direct computation yields

$$\{Q_i, Q_j\} = (\mathbf{1} + k\,\nu)\,\tau_{ij}\,, \qquad \text{with} \qquad Q_i \equiv \frac{1}{2}\, C_{AB}\, \hat{q}^A\, \hat{\phi}_i^B\,, \qquad \tau_{ij} = \frac{1}{4}\, C_{AB}\, [\hat{\phi}_i^A, \hat{\phi}_j^B]\,, \tag{4.3}$$

which deforms the $osp(1|2p,\mathbb{R})$ algebra. One could think of modifying the odd generators as

$$Q_i^{\text{new}} = f(k,\nu)\, Q_i\,, \qquad f(k,\nu) = a(\nu)\,\mathbf{1} + b(\nu)\,k\,, \qquad a(0) = 1\,, \quad b(0) = 0\,, \tag{4.4}$$

however, this leads to

$$\{Q_i^{\text{new}}, Q_j^{\text{new}}\} = f(k,\nu) f(-k,\nu)\,(\mathbf{1} + k\,\nu)\,\tau_{ij}\,, \tag{4.5}$$

which does not allow us to remove the factor $\mathbf{1} + k\nu$ by suitably choosing $f(k,\nu)$, since the combination that appears, $f(k,\nu)f(-k,\nu) = \big(a(\nu)^2 - b(\nu)^2\big)\mathbf{1}$, is only proportional to the identity. If one could require that the $\mathbb{Z}_2$ generator anticommute with the fermionic oscillators, $\{k, \phi_i^A\} = 0$, the right hand side of the above equation could be fixed to be the (undeformed) generators of the $sp(2p,\mathbb{R})$ subalgebra $\tau_{ij}$ by suitably choosing $f(k,\nu)$, thereby providing us with a realisation of $osp(1|2p,\mathbb{R})$ in the deformed oscillator algebra. Unfortunately, we saw that requiring the Klein operator and the fermionic oscillators to anticommute is inconsistent.

This situation seems surprising since in the case of the type-B algebra, whose realisation is also based on a quotient of the Weyl-Clifford algebra [38], and are known to admit deformations of this type [53, 54]. The deformed oscillator algebra, which first appeared in a paper of Wigner [68], is one of the simplest example of a *symplectic reflection algebra* (originally introduced by Etingof and Ginzburg [69], see also [70–72] for more recent reviews). The algebras are deformations of the smash product of the Weyl algebra with a finite group (acting on it by automorphisms). The latter naturally contains reductive dual pairs $(\mathfrak{g}, \mathfrak{g}')$ of bosonic type, which can — at least in some cases [73–75] — be deformed by finding a realisation of one of the algebra of the pair, say $\mathfrak{g}$, in a symplectic reflection algebra. Typically, the other algebra $\mathfrak{g}'$ is deformed to an *associative* (not Lie) algebra. In any case, both algebras are mutual centralisers of one another, and hence one again finds a bijection between their representations (appearing in the appropriate Fock space). Recently, some examples of dual pairs of Lie *superalgebras* have been deformed [76, 77] using symplectic reflection algebras. The difference with respect to the pair $\big(sp(2n,\mathbb{R}), osp(1|2p,\mathbb{R})\big)$ of interest for us is that the superalgebra $osp(1|2p,\mathbb{R})$ that we would like to preserve when using deformed oscillator has its bosonic subalgebra $sp(2p,\mathbb{R})$ realised using only fermionic oscillators which are not deformed (since they generate a Clifford algebra which is finite-dimensional, it does not admit a non-trivial deformation). This seems to be one of the reason why preserving $osp(1|2p,\mathbb{R})$ appears impossible, at least if we simply replace the bosonic oscillators $\hat{Y}^A$ by deformed ones $\hat{q}^A$ in our realisation. We hope to come back to this issue in the near future.



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## A More on the type-$\mathrm{A}_2$ algebra

Any higher spin algebra whose spectrum consists of totally symmetric fields *only*, and defined as a quotient of $\mathcal{U}(so(2,d))$ by an ideal $\mathcal{I}$, will necessarily contain the antisymmetric generator[9]

$$V_{\mathsf{ABCD}} = 4\, M_{[\mathsf{AB}}\, M_{\mathsf{CD}]} = M_{[\mathsf{AB}}\, M_{\mathsf{C}]\mathsf{D}} - M_{[\mathsf{AB}}\, \eta_{\mathsf{C}]\mathsf{D}}\,, \tag{A.1}$$

in its defining ideal $\mathcal{I}$. We will therefore start this appendix by reviewing how factoring out $V_{\mathsf{ABDC}}$ relates all Casimir operators to the quadratic one (see also [48, section 2.1] and [78]). Let us illustrate this mechanism in the case of the quartic Casimir operator, defined as[10]

$$\mathcal{C}_4 = \frac{1}{2}\, M_{\mathsf{A}}{}^{\mathsf{B}}\, M_{\mathsf{B}}{}^{\mathsf{C}}\, M_{\mathsf{C}}{}^{\mathsf{D}}\, M_{\mathsf{D}}{}^{\mathsf{A}}\,, \qquad \mathcal{C}_2 = -\frac{1}{2}\, M_{\mathsf{AB}}\, M^{\mathsf{AB}}\,. \tag{A.2}$$

A direct computation yields[11]

$$V_{\mathsf{ABC}}{}^{\bullet}\, M_{\mathsf{D}\bullet} = M_{\mathsf{AB}}\, M_{\mathsf{C}}{}^{\bullet}\, M_{\mathsf{D}\bullet} + 2\, M_{\mathsf{C}[\mathsf{A}}\, M_{\mathsf{D}}{}^{\bullet}\, M_{\mathsf{B}]\bullet} + M_{\mathsf{AB}}\, M_{\mathsf{CD}} - 2\,(d-1)\, M_{\mathsf{C}[\mathsf{A}}\, M_{\mathsf{B}]\mathsf{D}}\,, \tag{A.3}$$

which, upon taking a trace in $\mathsf{CD}$ and contracting with $M_{\mathsf{AB}}$, gives

$$V_{\mathsf{ABCD}} \sim 0 \qquad \Longrightarrow \qquad \mathcal{C}_4 \sim \mathcal{C}_2\left(\mathcal{C}_2 + \frac{d(d-1)}{2}\right), \tag{A.4}$$

in agreement with [48, section 2.1] in the special case of the singleton, and with [79] in general. Similarly, taking the Lorentz components $V_{abcd}$, and contracting them with $L^{ab}$ (on the left) and $L^{cd}$ (on the right), one finds

$$V_{abcd} \sim 0 \qquad \Longrightarrow \qquad \mathcal{C}_4^L \sim (\mathcal{C}_2 - P^2)\left(\mathcal{C}_2 - P^2 + \frac{1}{2}\,(d-1)(d-2)\right), \tag{A.5}$$

[9] The factor 4 in $V_{\mathsf{ABCD}}$ has been added for simplicity.

[10] More generally, we follow the convention that the Casimir operator of $so(2,d)$ of order $2n$ is given by $\mathcal{C}_{2n} := \frac{1}{2}\, M_{\mathsf{A}_1}{}^{\mathsf{A}_2}\, M_{\mathsf{A}_2}{}^{\mathsf{A}_3} \dots M_{\mathsf{A}_{2n}}{}^{\mathsf{A}_1}$.

[11] For all computations in this appendix, one needs to use a few identities that are specific to orthogonal algebra, which we will list here. For $so_N$, with generators $R_{IJ} = -R_{JI}$ obeying $[R_{IJ}, R_{KL}] = \eta_{JK}\, R_{IL} + (\dots)$ with $\eta$ of *arbitrary signature*, one has

$$[R_I{}^{\bullet}, R_{J\bullet}] = -(N-2)\, R_{IJ}\,, \qquad [V^{\bullet}, R_{I\bullet}] = -(N-1)\, V_I\,, \qquad R_I{}^J\, R_J{}^K\, R_K{}^I = -\frac{N-2}{2}\, R_{IJ}\, R^{IJ}\,,$$

where $V_I$ is any vector of $so_N$.

where $\mathcal{C}_4^L = \frac{1}{2}\, L_a{}^b\, L_b{}^c\, L_c{}^d\, L_d{}^a$ is the quartic Casimir operator of the Lorentz subalgebra. This is the same type of relation as (A.4) with $d \to d-1$, upon using the fact the quadratic Casimir operator $\mathcal{C}_2^L$ of $so(1,d)$ is given in terms of that of $so(2,d)$ by $\mathcal{C}_2^L = \mathcal{C}_2 - P^2$. Contracting $V_{\mathsf{ABCD}}$ with more generators produces similar identities, relating Casimir operators of order $2n$ to lower order ones, and ultimately to $\mathcal{C}_2$.

When decomposing the generator $V_{\mathsf{ABCD}}$ under $so(1,d)$, one finds an additional anti-symmetric generator of rank 3, namely $V_{abc0'}$. Contracting it with $L^{ab}$ (on the left) and $P^c$ (on the right) yields the identity

$$V_{abc0'} \sim 0 \qquad \Longrightarrow \qquad L_a{}^{\bullet}\, L_{b\bullet}\, \{P^a, P^b\} \sim -2\, (\mathcal{C}_2 - P^2) \left(P^2 + \frac{d-1}{2}\right), \tag{A.6}$$

which will be useful for us later on.

**Type-$\mathbf{A_2}$.** Let us define

$$W_{\mathsf{AB}} := M_{(\mathsf{A}}{}^{\mathsf{C}}\, M_{\mathsf{B})\mathsf{C}}\,, \tag{A.7}$$

and consider the symmetric generator for the ideal defining the partially-massless higher spin algebra $A_2$, which is the traceless part of $W^{(\mathsf{AB}}\, W^{\mathsf{CD})}$, given by,

$$\begin{aligned} \mathcal{J}_{\mathsf{ABCD}} := {} & W_{(\mathsf{AB}}\, W_{\mathsf{CD})} - \frac{4}{d+6}\, \eta_{(\mathsf{AB}}\, \big(W_{\mathsf{C}}{}^{\mathsf{M}}\, W_{\mathsf{D})\mathsf{M}} - \mathcal{C}_2\, W_{\mathsf{CD})}\big) \\ & + \frac{4}{(d+4)(d+6)}\, \eta_{(\mathsf{AB}}\, \eta_{\mathsf{CD})} \left(\mathcal{C}_4 + \mathcal{C}_2 \left[\mathcal{C}_2 - \left(\frac{d}{2}\right)^2\right]\right), \end{aligned} \tag{A.8}$$

where we used the relation

$$\frac{1}{2}\, W_{\mathsf{AB}}\, W^{\mathsf{AB}} = \mathcal{C}_4 - \left(\frac{d}{2}\right)^2 \mathcal{C}_2\,, \tag{A.9}$$

relating the contraction of the generator $W_{AB}$ with itself and the quadratic and quartic Casimir operators. Note that we can also express this generator of the ideal $\mathcal{I}_2$ as

$$\mathcal{J}_{\mathsf{ABCD}} = \mathcal{J}_{(\mathsf{AB}}\, \mathcal{J}_{\mathsf{CD})} - \frac{4}{d+6}\, \eta_{(\mathsf{AB}}\, \mathcal{J}_{\mathsf{C}}{}^{\bullet}\, \mathcal{J}_{\mathsf{D})\bullet} + \frac{4}{(d+4)(d+6)}\, \eta_{(\mathsf{AB}}\eta_{\mathsf{CD})} \left(\mathcal{C}_4 - \frac{2}{d+2}\, \mathcal{C}_2^2 - \left(\frac{d}{2}\right)^2 \mathcal{C}_2\right), \tag{A.10}$$

where

$$\mathcal{J}_{\mathsf{AB}} := M_{(\mathsf{A}}{}^{\mathsf{C}}\, M_{\mathsf{B})\mathsf{C}} + \frac{2}{d+2}\, \eta_{\mathsf{AB}}\, \mathcal{C}_2\,, \tag{A.11}$$

is the traceless part of $W_{\mathsf{AB}}$, which is also one of the generator of the Joseph ideal of the type-A algebra, and where we used

$$\frac{1}{2}\, \mathcal{J}_{\mathsf{AB}}\, \mathcal{J}^{\mathsf{AB}} = \mathcal{C}_4 - \frac{2}{d+2}\, \mathcal{C}_2^2 - \left(\frac{d}{2}\right)^2 \mathcal{C}_2\,. \tag{A.12}$$

This generator can be decomposed under the Lorentz subalgebra, and in particular contains a scalar piece,

$$\begin{aligned} \mathcal{J}_{\bullet}^{(2)} = {} & \frac{d+2}{d+6}\, P^4 + \frac{4}{(d+6)} \left(\frac{1}{4}\, \{L_{ab}, P^b\}\, \{L^{ac}, P_c\} - \mathcal{C}_2\, P^2\right) \\ & + \frac{4}{(d+4)(d+6)} \left(\mathcal{C}_4 + \mathcal{C}_2 \left[\mathcal{C}_2 - \left(\frac{d}{2}\right)^2\right]\right), \end{aligned} \tag{A.13}$$

which can be re-written in terms of $\mathcal{C}_4$, $\mathcal{C}_2$, $P^4$ and $P^2$, using some previously discussed results. To do so, notice first that

$$\frac{1}{4}\{L_{ab},P^b\}\{L^{ac},P_c\} = \frac{1}{2}\,L_a{}^{\bullet}\,L_{b\bullet}\,\{P^a,P^b\} + \frac{d+1}{2}\,(\mathcal{C}_2 - P^2) - \frac{d^2}{4}\,P^2\,, \tag{A.14}$$

The first term on the right hand side can be eliminated using (A.6), and using the relation (A.4) between $\mathcal{C}_4$ and $\mathcal{C}_2$, as well as imposing $\mathcal{C}_2 \sim -\frac{1}{4}(d-4)(d+4)$, we end up with

$$\mathcal{J}^{(2)}_{\bullet} = \left(P^2 + \frac{d-4}{2}\right)\left(P^2 + \frac{d-8}{2}\right). \tag{A.15}$$

The symmetric generator $\mathcal{J}_{\mathsf{A}(2\ell)}$ of the defining ideal for the type-A$_\ell$ higher spin algebra verifies

$$[M_{\mathsf{AB}},\mathcal{J}_{\mathsf{C}(2\ell)}] = 4\,\ell\,\eta_{\mathsf{C}[\mathsf{B}}\,\mathcal{J}_{\mathsf{A}]\mathsf{C}(2\ell-1)}\,, \tag{A.16}$$

by definition. Decomposing this identity under the Lorentz subalgebra yields

$$[P_a,\mathcal{J}^{(\ell)}_{b(2\ell-k)}] = (2\ell-k)\,\eta_{ab}\,\mathcal{J}^{(\ell)}_{b(2\ell-k-1)} + k\,\mathcal{J}^{(\ell)}_{ab(2\ell-k)}\,, \tag{A.17}$$

for $k=0,\ldots,2\ell$. We can use the above equation to express the various Lorentz generators, obtained by decomposing $\mathcal{J}_{\mathsf{A}(2\ell)}$, in terms of the scalar one

$$\mathcal{J}^{(\ell)}_{\bullet} = \sum_{k=0}^{\ell} \nu_{2k}(\mathcal{C}_{2n})\,P^{2k}\,, \tag{A.18}$$

where $\nu_k$ are polynomials in the Casimir operators of $so(2,d)$, and $\nu_{2\ell}=1$. Indeed, for $k=2\ell$ the equality (A.17) yields

$$\mathcal{J}^{(\ell)}_{a} = \frac{1}{2\ell}\,[P_a,\mathcal{J}^{(\ell)}_{\bullet}]\,, \tag{A.19}$$

while for $k=2\ell-1$ it gives,

$$\mathcal{J}^{(\ell)}_{ab} = \frac{1}{2\ell\,(2\ell-1)}\,[P_a,[P_b,\mathcal{J}^{(\ell)}_{\bullet}]] + \frac{1}{(2\ell-1)}\,\eta_{ab}\,\mathcal{J}^{(\ell)}_{\bullet}\,, \tag{A.20}$$

which can then be used to obtain, recursively, expressions for all generators $\mathcal{J}^{(\ell)}_{a(2\ell-k)}$ given by various linear combinations of nested commutators of $P_a$ and $\mathcal{J}^{(\ell)}$. Schematically,

$$V^{(\ell)}_{a(k)} = \sum_{j=0}^{[k/2]} \#\,\underbrace{\eta_{aa}\ldots\eta_{aa}}_{j\,\text{times}}\,\underbrace{[P_a,\ldots,[P_a,}_{k-2j\,\text{times}}\mathcal{J}^{(\ell)}_{\bullet}]\ldots]\,, \tag{A.21}$$

where $\#$ generically denotes combinatorial coefficients that can be obtained by recursion. For instance, for the $\ell=1$ case, i.e. the usual type-A higher spin algebra, the symmetric generator

$$\mathcal{J}_{\mathsf{AB}} = M_{(\mathsf{A}}{}^{\mathsf{C}}\,M_{\mathsf{B})\mathsf{C}} + \frac{2}{d+2}\,\eta_{\mathsf{AB}}\,\mathcal{C}_2\,, \tag{A.22}$$

decomposes into three generators,

$$\mathcal{J}^{(1)}_{ab} = L_{(a}{}^{c}\,L_{b)c} - P_{(a}\,P_{b)} + \frac{2}{d+2}\,\eta_{ab}\,\mathcal{C}_2\,,\quad \mathcal{J}^{(1)}_{a} = \frac{1}{2}\,\{L_{ab},P^b\}\,,\quad \mathcal{J}^{(1)}_{\bullet} = P^2 - \frac{2}{d+2}\,\mathcal{C}_2\,, \tag{A.23}$$

and one can check that the rank-2 symmetric and the vector generators can be re-written as

$$\mathcal{J}^{(1)}_{ab} = \frac{1}{2}\Big[P_a,[P_b,P^2]\Big] + \eta_{ab}\left(P^2 - \frac{2}{d+2}\,\mathcal{C}_2\right)\,, \qquad \mathcal{J}^{(1)}_{a} = \frac{1}{2}\,[P_a,P^2]\,. \tag{A.24}$$

For $\ell = 2$, one finds

$$\mathcal{J}^{(2)}_{a} = \frac{1}{2}\Big\{L_{ab}\,(P^2+d-3), P^b\Big\}\,, \tag{A.25}$$

which is similar to the $\ell = 1$ case, in that it is given by the anticommutator of $P^b$ with a monomial of order $2\ell - 1 = 3$ in generators, which is an antisymmetric Lorentz tensor. In light of the discussion in section 3, the generator $\frac{1}{2}\,(L_{ab}P^2 + d - 3)$ can be identified as the Maxwell generator in type-$A_2$. In fact, this pattern holds for arbitrary values of $\ell$: a simple recursion leads to

$$[P_a, P^{2k}] = \sum_{j=1}^{k}(2-\delta_{j,k})\,d^{j-1}\,\{L_{ab}\,P^{2(k-j)}, P^b\}\,, \qquad k \geq 1\,, \tag{A.26}$$

which yields

$$\mathcal{J}^{(\ell)}_{a} = \frac{1}{2\ell}\sum_{k=0}^{\ell-1} a_{2k}\,\{L_{ab}\,P^{2k}, P^b\}\,, \qquad \text{with} \qquad a_{2k} = (2-\delta_{k,0})\sum_{j=k+1}^{\ell} d^{j-k-1}\,\nu_{2j}\,, \tag{A.27}$$

where $\nu_{2j}$ denote the coefficients in the expression of $\mathcal{J}^{(\ell)}_{\bullet}$ as a polynomial in $P^2$ (A.18).

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**Code Availability Statement.** This article has no associated code or the code will not be deposited.



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