Local geometric Galois representations

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Context

We are interested in the study of the p-adic representations of

$$G_{\mathbb{Q}_p} = \mathsf{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p),$$

i.e. finite dimensional $\mathbb{Q}_p$-vector spaces with linear continuous $G_{\mathbb{Q}_p}$-action. Or equivalently, continuous group homomorphisms

$$G_{\mathbb{Q}_p} \longrightarrow \mathrm{GL}_n(\mathbb{Q}_p).$$

Particularly the ones that come from geometry.

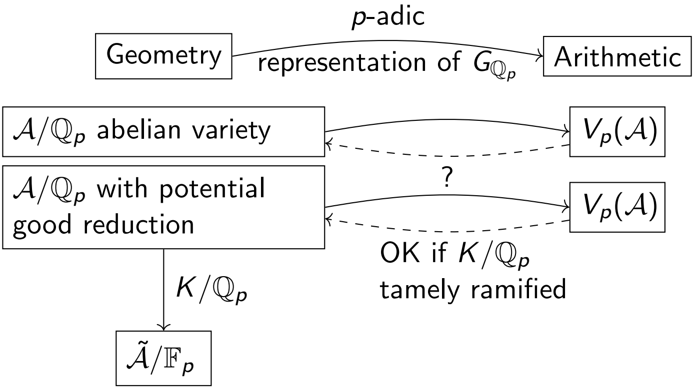
Arising from geometry

Let $\mathcal{A}/\mathbb{Q}_p$ be an abelian variety. We denote by $\mathcal{A}[p^n]$ the set of p^n-torsion points in $\mathcal{A}(\overline{\mathbb{Q}}_p)$.

- $\mathcal{A}[p^n]$: $2\dim\mathcal{A}$ free $\mathbb{Z}/p^n\mathbb{Z}$-module, linear continuous $G_{\mathbb{Q}_p}$-action
- $[p] : \mathcal{A}[p^{n+1}] \to \mathcal{A}[p^n] : x \mapsto px$
- $V_p(\mathcal{A}) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n \mathcal{A}[p^n]$

We obtain a p-adic representation of $G_{\mathbb{Q}_p}$ of dimension $2\dim\mathcal{A}$, called the p-adic Tate module of $\mathcal{A}$.

An inverse problem



Question

What if $K/\mathbb{Q}_p$ is wildly ramified? First goal: elliptic curves over $\mathbb{Q}_3$.

The one-dimensional case: elliptic curves

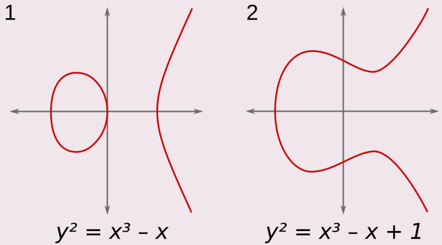
Definition

An elliptic curve $E/\mathbb{Q}_p$ is a projective curve given by an equation of the form

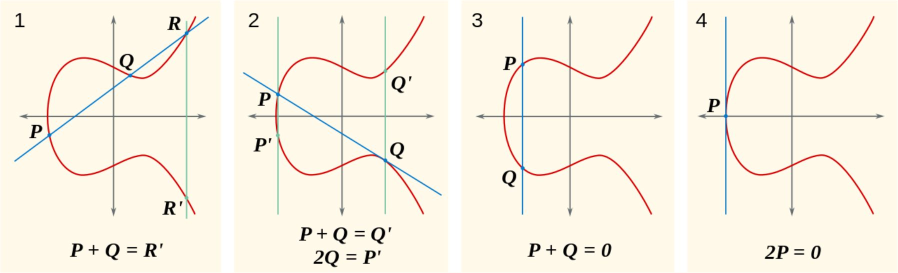
$$y^2 = x^3 + Ax + B$$

where $A, B \in \mathbb{Q}_p$ with $\Delta = -16(4A^3 + 27B^2) \neq 0$.

Example



Group law



The group law is given by rational functions with coefficients in $\mathbb{Q}_p$. The identity element is the point "at infinity".

Good reduction

Let

$$E : y^2 = x^3 + Ax + B$$

we can assume up to a minimal change of coordinates that $A, B \in \mathbb{Z}_p$.
It then makes sense to consider the reduction $\bmod p$:

$$\tilde{E} : y^2 = x^3 + \tilde{A}x + \tilde{B}$$

which is an algebraic curve over $\mathbb{F}_p$.
If $\tilde{E}/\mathbb{F}_p$ is an elliptic curve, we say that E has good reduction.

Field of good reduction

Let $E/\mathbb{Q}_p$ with potential good reduction. There exists $K/\mathbb{Q}_p$ with minimal ramification index e, such that E acquires good reduction over K. Such a field is called a field of good reduction.

A link between arithmetic and geometry

Theorem

An elliptic curve $E/\mathbb{Q}_p$ has good reduction if and only if $V_p(E)$ is crystalline.

In this case, $V = V_p(E)$ is completely determined by its associated filtered φ-module

$$(D, L) = \mathbf{D}^*_{\mathrm{cris},\mathbb{Q}_p}(V) = \mathbf{Hom}_{\mathbb{Q}_p[G_{\mathbb{Q}_p}]}(V, \mathbf{B}_{\mathrm{cris}}),$$

where

- D is a 2-dimensional $\mathbb{Q}_p$-vector space
- $\varphi : D \xrightarrow{\sim} D$ is $\mathbb{Q}_p$-linear
- L is a line in D

First results

A full classification of the 3-adic representations arising from elliptic curves over $\mathbb{Q}_3$ with potential good reduction.

e	Reduction type	K	Frobenius	#Classes
3	Supersingular	$L^{\mathrm{na}}(\zeta_4)$	$a_3 = 0$	$\mathbb{P}^1(\mathbb{Q}_3)$
		$L^{\mathrm{a}} = \mathbb{Q}_3(\pi)$	$a_3 = -3$	2
			$a_3 = 0$	2
			$a_3 = 3$	2
12	Supersingular	K_1	$a_3 = 0$	$\mathbb{P}^1(\mathbb{Q}_3)$
		K_2	$a_3 = 0$	$\mathbb{P}^1(\mathbb{Q}_3)$
		K_3	$a_3 = 0$	$\mathbb{P}^1(\mathbb{Q}_3)$
		K_4	$a_3 = 0$	$\mathbb{P}^1(\mathbb{Q}_3)$
		K_5	$a_3 = 0$	$\mathbb{P}^1(\mathbb{Q}_3)$