UNIQUENESS, NON-DEGENERACY, AND EXACT MULTIPLICITY OF POSITIVE SOLUTIONS FOR SUPERLINEAR ELLIPTIC PROBLEMS

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Abstract. In this paper, we focus our attention on the positive solutions to second-order nonlinear ordinary differential equations of the form $u'' + q(t)g(u) = 0$, where q is a sign-changing weight and g is a superlinear function. We exploit the classical shooting approach and the comparison theorem to present non-degeneracy and exact multiplicity results for positive solutions. This completes the multiplicity results obtained by Feltrin and Zanolin. Numerical examples and some related open problems are also discussed.

1. Introduction

The paper investigates the positive solutions to the Dirichlet boundary value problem

$$
\begin{cases} -\Delta u = \omega(x)g(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, ω is a sign-changing weight, and g is a function with a superlinear growth, i.e., $g(u) \sim u^p$ with $p > 1$. This kind of problems is usually called "superlinear indefinite", a terminology that has been popularized starting with [38]. As usual, a solution u of (1.1) is said to be positive if $u > 0$ in Ω.

In the last thirty years a great quantity of existence and multiplicity results for positive solutions to indefinite problems, both in the ODE and PDE cases, have been obtained using different techniques, such as topological and variational methods, see for instance [2, 3, 7, 15, 16, 22, 45, 52], and also [9, 25] for a quite complete panorama of the research done in this framework. An important motivation for the analysis of the superlinear indefinite problem (1.1) comes from the search for stationary solutions to parabolic equations arising in different frameworks, such as reaction-diffusion processes and population dynamics models (see [1, 46] and the references therein).

Our investigation lies on a line of research initiated by the work of López-Gómez [44, 45] concerning the analysis of the number of positive solutions depending on the nodal behaviour of ω. More precisely, we are interested in weight functions ω depending on a real parameter μ which plays the role of strengthening or weakening its positive and negative parts (cf. [5, 45, 46]). More precisely, ω can be expressed in this manner

$$
\omega = h^+ - \mu h^-,
$$

where $\mu > 0$ is a parameter, $h\colon [a,b] \to \mathbb{R}$, with $h^\pm := |h|/2 \pm h/2$ the positive and negative parts of h.

The starting point of our research is the following conjecture proposed by Gómez-Reñasco and López-Gómez in [35] (see also [31]).

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Conjecture (Gómez-Reñasco and López-Gómez). *There exists $\mu^* > 0$ such that for every $\mu > \mu^*$ the Dirichlet problem*

$$\begin{cases} u'' + \bigl(h^+(t) - \mu h^-(t)\bigr)|u|^{p-1}u = 0, \qquad p > 1, \\ u(a) = 0 = u(b), \end{cases} \tag{1.2}$$

possesses at least $2^m - 1$ positive solutions, whenever h has m intervals where it is positive separated by intervals where it is negative.

A first partial positive answer of this conjecture was proposed by Gaudenzi, Habets and Zanolin in [32], where they show the existence of three positive solutions to (1.2) when h has two positivity intervals separated by a negativity one, that is $m = 2$, and μ is large. Later this result was improved by the same authors in [33] for $m = 3$. Both papers are based on a very precise phase-plane analysis and a shooting technique (the possibility of using the shooting method in the general case $m \geqslant 2$ has been recently asserted in [8], see Remark 2.2 therein).

In [29], exploiting a topological degree approach, Feltrin and Zanolin solved the conjecture for every positive integer m and a general superlinear function g. Many other analogous multiplicity results for positive solutions followed in different context: supersublinear nonlinearities, logistic-type nonlinearities, periodic and Neumann boundary conditions, ϕ-Laplacian operators, etc. (cf. [10, 11, 26, 27, 28, 30]).

When analyzing these multiplicity results a natural question arise: Is the number of solutions given by the conjecture (and its variants) optimal? In other words, one should understand whether there are examples of problems of the form (1.2) admitting exactly $2^m - 1$ positive solutions for μ large.

The main result of the present manuscript, as far as we know for the first time in literature, gives a positive answer to the above question and states the following. For simplicity in the exposition, we now present it in a special case (see the general result in Theorem 3.3).

Theorem 1.1. *Let $k \in \mathbb{N}$ with $k \geqslant 2$. Let $h(t) = \sin\bigl(k\pi t/(b-a)\bigr)$. Then, there exists $\mu^{**} > 0$ such that for every $\mu > \mu^{**}$ the Dirichlet problem* (1.2) *admits exactly* $2^m - 1$ *positive solutions, where m is the integer part of* $(k+1)/2$. *These solutions are non-degenerate.*

Roughly speaking, Theorem 1.1 can be considered as a continuation result, in the sense that we are going to explain in the following. We start by remarking that the existence of *at least* $2^m - 1$ positive solutions is given by [29] and so we have to prove that there are *at most* $2^m - 1$ solutions. From [29], we observe that the number $2^m - 1$ comes from the possibility of prescribing, for a positive solution, the behavior in each interval I where h is positive between two possible ones: either the solution is "small" on I or it is "large" on I (the solution which is "small" in every interval of positivity is identically zero, thus one have to subtract 1 from the total).

The first step towards establishing Theorem 1.1 is to analyze the behaviour of each positive solution of (1.2) when $\mu \to +\infty$. In particular, we prove that the limit profile of the solution is zero in the intervals where h is negative and solves the Dirichlet problem in each interval of positivity (see Proposition 3.2). The second step is to prove that the number of limit profiles (as $\mu \to +\infty$) is exactly $2^m - 1$. The last step is a "continuation from infinity" to guarantee that, for $\mu > 0$ sufficiently large, $2^m - 1$ remains the exact number of positive solutions of (1.2).

The determination of the number of limit profiles relies on uniqueness criteria for positive solutions to

$$\begin{cases} u'' + f(t,u) = 0, \\ u(a) = 0 = u(b), \end{cases} \tag{1.3}$$

where $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$. The classical multiplicity result proposed by Moore and Nehari in [47] (they consider $f(t,u)=q(t)u^3$ with q piecewise constant and non-negative) shows that the problem of uniqueness can be of great complexity even in very simple situations (even in the case of smooth functions, see Fig. 9). Looking at the literature, one can notice that the question of uniqueness of positive solutions to (1.3) has received a considerable attention when the nonlinear term f is non-negative, see for instance [18, 20, 23, 41, 48, 56] for the ODE case and also [4, 46, 50] for the PDE case, while the few available results when f is sign-changing are contained in [6, 12, 13, 14, 36, 49]. The approach we adopt in this paper is based on a method introduced by Kolodner [40] and subsequently extended by Coffman [17, 18]. This method is reminiscent of the shooting one since it consists in reducing the boundary value problem for the linearization of (1.3) to an initial value problem and in tracking the point of intersection of the solution with the t-axis (see also [41] for a survey and the references therein for a list of the main applications).

To reach our goal however, the uniqueness of the limit profile with a given number of bumps is not enough. Its non-degeneracy is also essential for the continuation argument to work. In our case, though, the limit profile does *not* solve an ODE (the limit profile is not twice differentiable!) and so the meaning of its non-degeneracy is not clear. To circumvent this difficulty for the one-dimensional case of (1.1), namely

$$\begin{cases} u''+\bigl(h^+(t)-\mu h^-(t)\bigr)g(u)=0,\\ u(a)=0=u(b), \end{cases}\tag{1.4}$$

where h changes sign, we have to work separately on each sub-interval where h is positive and negative. To "compose" the non-degeneracy results on each sub-interval (see the proof of Theorem 3.3), the classical uniqueness results mentioned above need to be extended to precisely track the sign of the solution to the linearized equation (see Propositions 2.10 and 2.11).

For the reader's convenience, we end this introduction by listing the hypotheses used. In the sequel, we assume that $h\colon [a,b]\to\mathbb{R}$ is an L^1-function (it will be specified explicitly when we need h to be more regular) and $g\colon [0,+\infty[\,\to[0,+\infty[$ is a $\mathcal{C}^1$-function such that

$$(g_+)\qquad g(0)=0,\qquad g(s)>0,\quad \text{for all } s\in\,]0,+\infty[.$$

Moreover, we also make use of the following superlinear growth conditions

$$(g_s)\qquad g'(s)>\frac{g(s)}{s},\quad \text{for all } s\in\,]0,+\infty[,$$

$$(g_0)\qquad g'(0)=\lim_{s\to 0^+}\frac{g(s)}{s}=0,$$

and

$$(g_\infty)\qquad \lim_{s\to+\infty}\frac{g(s)}{s}=+\infty.$$

The paper is organized as follows. In Section 2, exploiting the classical Kolodner–Coffman technique, we refine some uniqueness results and present some technical estimates. Those results are then exploited in Section 3 to give the proof of the main exact multiplicity result, based on the study of limit profiles of positive solutions to (1.4). At last, in Section 4, we present some numerical experiments to graphically illustrate the main theorem and to shed some light on the behavior of the branches of solutions as μ moves away from $+\infty$; moreover, some related open questions are discussed.

2. Uniqueness

In this section, we adapt and extend the classical Kolodner–Coffman method [17, 18, 40, 41] to have more information on the linearized equation at a positive solution to the Dirichlet boundary value problem

$$\begin{cases} u'' + q(t)g(u) = 0, \\ u(a) = 0 = u(b), \end{cases} \tag{2.1}$$

where $q \in L^1([a,b],[0,+\infty[)$, $q \not\equiv 0$, and $g \in \mathcal{C}^1([0,+\infty[,[0,+\infty[)$ satisfies (g_+) and (g_s). Thanks to (g_+), we can extend g to a continuous function on the whole real line by setting $g(u) = 0$ whenever $u \in \,]-\infty, 0]$. For $\alpha \in [0,+\infty[$, let $u(\cdot;\alpha)$ be the unique maximal solution to the differential equation in (2.1) satisfying the initial conditions

$$u(a;\alpha) = 0, \qquad u'(a;\alpha) = \alpha.$$

Given that the solution $u(\cdot;\alpha)$ is concave when it is positive and is affine when it is negative, $u(\cdot;\alpha)$ exists on $[a,b]$.

When $\alpha > 0$, let us also denote $B(\alpha)$ the first $t \in \,]a,b]$ such that $u(t;\alpha) = 0$ (if such a time t exists). Let $\operatorname{dom} B \subseteq \,]0,+\infty[$ be the domain of B. Notice that if $\alpha \in \operatorname{dom} B$, then $u'(B(\alpha);\alpha) < 0$ and, since $g \equiv 0$ for negative real numbers, $u(s;\alpha) < 0$ and $u'(s;\alpha) = u'(B(\alpha);\alpha)$ for every $s \in \,]B(\alpha), b]$.

A first criteria ensuring uniqueness of positive solutions of (2.1) is the following.

Lemma 2.1. *Under the above assumptions, if*

$$\forall \alpha \in \operatorname{dom} B, \quad B(\alpha) = b \;\Rightarrow\; \partial_\alpha u(b;\alpha) < 0, \tag{2.2}$$

then problem (2.1) *has at most one positive solution.*

Proof. We divide the proof into two steps.

Step 1. The map B is continuously differentiable.

First of all, it is classical (see e.g. [37, Chapter V] or [53]) that $u(\cdot;\alpha)$ is continuously differentiable with respect to α. Moreover, we have that

$$u\big(B(\alpha);\alpha\big) = 0, \quad u'\big(B(\alpha);\alpha\big) < 0, \quad \text{for all } \alpha \in \operatorname{dom} B.$$

From an application of the implicit function theorem, we deduce that B is continuously differentiable (from one side if α lies on the boundary of its domain) and

$$\partial_\alpha B(\alpha) = -\frac{\partial_\alpha u\big(B(\alpha);\alpha\big)}{u'\big(B(\alpha);\alpha\big)}. \tag{2.3}$$

Step 2. There exists at most one $\alpha \in \operatorname{dom} B$ such that $B(\alpha) = b$.

Assume that (2.1) has a positive solution i.e., there exists $\alpha_0 > 0$ such that $\alpha_0 \in \operatorname{dom} B$ and $B(\alpha_0) = b$. We claim that $[\alpha_0,+\infty[\, \subseteq \operatorname{dom} B$ and that for all $\alpha > \alpha_0$, $B(\alpha) \in \,]0,b[$. Let $]\alpha_0,\alpha_1[$ be the largest connected set such that

$$\forall \alpha \in \,]\alpha_0,\alpha_1[, \quad \alpha \in \operatorname{dom} B \text{ and } B(\alpha) < b. \tag{2.4}$$

Clearly, (2.2)–(2.3) implies that $]\alpha_0,\alpha_1[\, \neq \varnothing$. If $\alpha_1 = +\infty$, the claim is proved. If not, let us consider $t_1 \in [a,b]$ a limit point of $B(\alpha)$ as $\alpha \to \alpha_1$. The continuity of $(t,\alpha) \mapsto u(t;\alpha)$ implies that $u(t_1;\alpha_1) = 0$ and that $u(\cdot;\alpha_1) \geqslant 0$ on $[a,t_1]$. One cannot have $t_1 = a$ because otherwise (tracking the maximums on $[a,B(\alpha)]$) that would imply $\alpha_1 = u'(a;\alpha_1) = 0$. Therefore, as all roots of $u(\cdot;\alpha_1)$ must be simple, $u(\cdot;\alpha_1) > 0$ on $]a,t_1[$ and $t_1 = B(\alpha_1)$. If $t_1 = b$, (2.4) implies that $\partial_\alpha B(\alpha_1) \geqslant 0$ which contradicts (2.2). If $t_1 \in \,]a,b[$, the intermediate value theorem implies that α_1 is in the interior of $\operatorname{dom} B$ and the continuity of B implies that $B < b$ on a neighborhood of α_1, contradicting the maximality of α_1.

Note that the previous argument also forbids any $\alpha_1 < \alpha_0$ such that $B(\alpha_1) = b$ as we can replay it with α_0 and α_1 swapped. The claim is thus proved. $\square$

Remark 2.2. Notice that (g_s) was not used in the above proof. Moreover, Lemma 2.1 remains valid for other boundary conditions *mutatis mutandis*. For example for the boundary conditions $u'(a) = 0 = u(b)$ (or $u(a) = 0 = u'(b)$ as we can swap a and b), the function $u(\cdot;\alpha)$ is defined as the unique solution to

$$\begin{cases} u'' + q(t)g(u) = 0, \\ u(a;\alpha) = \alpha, \quad u'(a;\alpha) = 0, \end{cases} \tag{2.5}$$

and $B(\alpha)$ is, as before, the first root of $u(\cdot;\alpha)$ in $]a, b]$. A criterion for uniqueness is again (2.2).

Our aim now is to provide conditions for the applicability of Lemma 2.1, namely sufficient conditions that guarantee $\partial_\alpha u(b;\alpha) < 0$, for every $\alpha \in [0, +\infty[$ yielding a solution. First of all, we recall the following useful version of the well known Sturm's Comparison Theorem [37, 57].

Theorem 2.3 (Sturm's comparison Theorem)**.** *Let $\omega_1, \omega_2 \colon [c, d] \to \mathbb{R}$ be L^1-functions with $\omega_1(t) \leqslant \omega_2(t)$ in $[c, d]$ with a strict inequality on a set of positive measure. Let u_1 and u_2 be nontrivial solutions to*

$$u_i'' + \omega_i(t)u_i = 0 \quad \text{on }]c, d[, \quad i = 1, 2.$$

Assume that one of the following conditions holds

- $u_1(c) = 0 = u_1(d)$*;*
- $u_1'(c) = 0 = u_1(d)$ *and* $u_2'(c) = 0$*.*

Then, u_2 must vanish at some point in $]c, d[$.

Then, we have the following.

Lemma 2.4. *Suppose the assumptions at the beginning of this section hold. Then, for all $\alpha > 0$ such that $\alpha \in \operatorname{dom} B$, the function $\partial_\alpha u(\cdot;\alpha)$ has at least one zero in $]a, B(\alpha)[$.*

Proof. Fix $\alpha > 0$ with $\alpha \in \operatorname{dom} B$, and let $\hat{b} := B(\alpha)$. We apply the Sturm's Comparison Theorem 2.3 to problem (2.1) written in the following form

$$u'' + \frac{q(t)g(u)}{u}u = 0, \qquad u(a) = 0 = u(\hat{b}),$$

and the equation satisfied by $w := \partial_\alpha u(\cdot;\alpha)$, namely

$$w'' + q(t)g'(u(t;\alpha))w = 0.$$

Exploiting (g_s), we deduce that w has at least one zero between the two zeros a and $\hat{b}$ of $u(\cdot;\alpha)$. The lemma is proved. $\square$

Remark 2.5. As before, Lemma 2.4 remains valid *mutatis mutandis* for the boundary conditions $u'(a) = 0 = u(b)$ (and $u(a) = 0 = u'(b)$, swapping a and b).

Given the previous lemma, to establish that $\partial_\alpha u(b;\alpha) < 0$, it is sufficient to show that $\partial_\alpha u(\cdot;\alpha)$ cannot have more than one zero in $]a, b]$. Below we will refine some classical criteria for this. These require to linearize equation (2.1); thus a stronger regularity on q is needed, namely that q is of bounded variation. Recall that the space $\mathrm{BV}(I)$ of functions of bounded variation on an interval I is defined by [43, Definition 7.1]:

$$\mathrm{BV}(I) = \{u \in L^1(I) \mid u' \text{ is a finite signed Radon measure}\}.$$

Here u' denotes the distributional derivative of u. Each $u \in \mathrm{BV}(I)$ can be written as the difference of two bounded nondecreasing functions and thus admits representatives that are discontinuous at an at most countable number of points [43, Theorem 7.3 and Theorem 2.36]. Therefore $\mathrm{BV}(I) \subseteq L^\infty(I)$. Moreover, the left and right limits of such representatives exist at any point x in the closure of I and are independent of the representative [43, Theorem 7.3 and Theorem 2.17]. These limits will respectively be denoted by $u(x-)$ and $u(x+)$. A special representative of u, called the *precise representative* and denoted by $\overline{u}$, averages the left and right limits:

$$\overline{u}(x) := \tfrac{1}{2}\big(u(x-) + u(x+)\big), \qquad \text{for any } x \in I.$$

The following version of the Fundamental Theorem of Calculus holds [43, Theorem 6.25]: for all $a, b \in I$ with $a \leqslant b$,

$$\int_{]a,b]} u'(\mathrm{d}x) = u'\big(]a,b]\big) = u(b+) - u(a+). \tag{2.6}$$

If one integrates over the interval $]a,b[$, $u(b+)$ must be replaced with $u(b-)$. Finally we also need a generalization of the Leibniz rule.

Proposition 2.6 (Leibniz rule)**.** *Let $I \subseteq \mathbb{R}$ be an interval and $u, v \in \mathrm{BV}(I)$. Then $uv \in BV(I)$ and*

$$(uv)' = u'\,\overline{v} + \overline{u}\,v'. \tag{2.7}$$

A proof of Proposition 2.6 is given in [55, pp. 189–191]. We offer an alternative elementary proof for the one-dimensional case for the reader convenience.

Proof. The fact that $uv \in BV(I)$ is asserted on page 45 of [43]. The right hand side of (2.7) defines a measure because u' and v' are Radon measures and $\overline{u}$ and $\overline{v}$ are bounded and continuous except possibly at an at most countable number of points. To establish (2.7), it suffices to prove that the measures on both sides of (2.7) coincide on intervals of the form $]a,b] \subseteq I$ [43, Corollary B.16]. Thanks to (2.6), $(uv)'\big(]a,b]\big) = u(b+)v(b+) - u(a+)v(a+)$. From (2.6), we deduce that

$$\forall x \in\,]a,b], \qquad \overline{v}(x) = v(a+) + \tfrac{1}{2}\Big(\int_{]a,x[} + \int_{]a,x]}\Big) v'(\mathrm{d}y)$$

and similarly for u. Therefore

$$\int_{]a,b]} \overline{v}(x) u'(\mathrm{d}x) = v(a+)\big(u(b+) - u(a+)\big) + \int_{]a,b]} \tfrac{1}{2}\Big(\int_{]a,x[} + \int_{]a,x]}\Big) v'(\mathrm{d}y) u'(\mathrm{d}x).$$

A similar computation followed by a permutation of the integrals yields

$$\int_{]a,b]} \overline{u}(y) v'(\mathrm{d}y) = u(a+)\big(v(b+) - v(a+)\big) + \int_{]a,b]} \tfrac{1}{2}\Big(\int_{]x,b]} + \int_{[x,b]}\Big) v'(\mathrm{d}y) u'(\mathrm{d}x).$$

Summing the last two integrals gives

$$\begin{aligned}
&(u'\,\overline{v} + \overline{u}\,v')\big(]a,b]\big) \\
&\quad = v(a+)\big(u(b+) - u(a+)\big) + u(a+)\big(v(b+) - v(a+)\big) + \int_{]a,b]}\int_{]a,b]} v'(\mathrm{d}y) u'(\mathrm{d}x) \\
&\quad = v(a+)\big(u(b+) - u(a+)\big) + u(a+)\big(v(b+) - v(a+)\big) + \big(v(b+) - v(a+)\big)\big(u(b+) - u(a+)\big) \\
&\quad = (uv)'\big(]a,b]\big).
\end{aligned}$$

□

Notice that assuming $q \in \mathrm{BV}([a, b])$ allows q to be piecewise constant as well as piecewise affine. These simple cases are already not included in the smoothness assumptions of the original version of the next results. Furthermore, let us highlight that the following proof differs from the one of Moroney [48] and shows that this result is a consequence of the property (2.2).

Theorem 2.7 (Moroney)**.** *Suppose the assumptions at the beginning of this section hold. Assume further that $q \in \mathrm{BV}([a, b])$ is non-increasing on $[a, b]$. Then,* (2.2) *holds for the map u defined by* (2.5)*. In particular, the differential equation in* (2.1) *with boundary conditions $u'(a) = 0 = u(b)$ has at most one positive solution.*

Proof. Let $u(\cdot; \alpha)$ be the solution to (2.5). Let $\alpha > 0$ be such that $u(t; \alpha) > 0$ for all $t \in \,]a, b[$ and vanishes for $t = b$. Let $w(t) := \partial_\alpha u(t; \alpha)$. From Lemma 2.1, we know that there is at most one positive solution if $w(b) < 0$ for all such α's. From Lemma 2.4, we know that w has a (simple) root in $]a, b[$. If we show that w has no other root in $]a, b[$, we are done. This will be done following the ideas of Sturm's Separation Theorem [57, Theorem 2.6.2].

Suppose on the contrary that there are two roots of w in $]a, b]$. Thus there exists $c < d$ such that

$$w(c) = 0 = w(d), \quad w < 0 \text{ on }]c, d[, \quad \text{and} \quad w'(c) < 0 < w'(d).$$

Recall that w satisfies $w'' + q(t)g'(u)w = 0$ where $u(\cdot) := u(\cdot; \alpha)$. Now consider $u'(\cdot) := u'(\cdot; \alpha)$. Differentiating the equation of u and using Proposition 2.6, we obtain that u' satisfies

$$u''' + \overline{q}(t)g'(u)u' = -q'(t)\overline{g(u)}$$

in the sense of measures. Note that, the function $g(u)$ being continuous, $\overline{g(u)} = g(u)$. Moreover $\overline{q} = q$ a.e. for the Lebesgue measure (as the number of discontinuity points of q is at most countable) and consequently also for the measure $g'(u)u' \,\mathrm{d}x$. Thus u' satisfies

$$u''' + q(t)g'(u)u' = -q'(t)g(u).$$

Multiplying the equation for w (see (2.10)) by u' and vice versa and subtracting them gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl[u'(t)w'(t) - u''(t)w(t)\bigr] = q'(t)g(u)w(t) \tag{2.8}$$

in the sense of measures. Here we made use of (2.7) as well as the identities $\overline{u'} = u'$, $\overline{w} = w$, $\overline{w'} = w'$, because these functions are (absolutely) continuous, and $\overline{u''}w' = u''w'$ because $w' \in L^1$ and so the measure $w' \,\mathrm{d}x$ does not care about the value of u'' at its points of discontinuity which are at most countable. Then integrating from c to d and using (2.6) yields:

$$u'(d)w'(d) - u'(c)w'(c) = \int_{]c,d]} g(u(t))w(t)\, q'(\mathrm{d}t).$$

Since $q' \leqslant 0$ as a measure on $[a, b]$, the right-hand side is $\geqslant 0$. On the other hand, $u'(t) = \int_a^t -qg(u) \leqslant 0$ so the left-hand side is $\leqslant 0$. Therefore both are null and $0 = u'(c) = \int_a^c -qg(u)$. This in turn implies that $q \equiv 0$ a.e. on $[a, c]$ because $g(u) > 0$ on $]a, b[$ as a consequence of (g_+). Therefore $w'' \equiv 0$ on $]a, c[$ which, given the initial conditions of w, contradicts the fact that c is a root of w. □

The next lemma proposes sufficient conditions on the weight q ensuring the symmetry with respect to the middle point of the interval $[a, b]$ of every positive solution of (2.1) (cf. [34] for an analogous result in the PDE setting under more restrictive regularity assumptions).

Lemma 2.8. *Let $g \in \mathcal{C}^1([0,+\infty[,[0,+\infty[)$ satisfy (g_+). Let $q \in L^1([a,b])$ be such that $q \geqslant 0$, $q(t) = q(a+b-t)$ a.e. on $[a,b]$, and q non-decreasing on $[a,c]$, where $c := (a+b)/2$. Then, every positive solution u of* (2.1) *is symmetric, i.e., $u(t) = u(a+b-t)$, for every $t \in [a,b]$, and is non-decreasing on $[a,c]$.*

Proof. Let u be a positive solution of (2.1). Since u is concave, the maximum of u in $[a,b]$ is attained either at a unique maximum point t^*, or at all the points of a subinterval $J \subseteq [a,b]$ (if $q \equiv 0$ in J), in this latter case let $t^* := \min J$. By contradiction, assume that $t^* \neq c$.

Due to the symmetry of q, $t \mapsto u(a+b-t)$ is another positive solution of (2.1); hence, without loss of generality, we can assume that $t^* \in [a,c[$. Let $\tilde{u}\colon [a,2t^*-a] \to \mathbb{R}$ and $\tilde{q}\colon [a,2t^*-a] \to \mathbb{R}$ be defined as follows

$$\tilde{u}(t) := \begin{cases} u(t), & \text{if } t \in [a,t^*], \\ u(2t^*-t), & \text{if } t \in [t^*, 2t^*-a], \end{cases} \qquad \tilde{q}(t) := \begin{cases} q(t), & \text{if } t \in [a,t^*], \\ q(2t^*-t), & \text{if } t \in [t^*, 2t^*-a]. \end{cases} \tag{2.9}$$

See Figure 1 for a graphical representation. We notice that $\tilde{u}$ and $\tilde{q}$ are symmetric with respect to t^* and $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}'' + \tilde{q}(t)g(\tilde{u}) = 0, & \text{in } [a,2t^*-a] \subseteq [a,b], \\ \tilde{u}(a) = 0 = \tilde{u}(2t^*-a). \end{cases}$$

By the monotonicity and the symmetry of q, we observe that $\tilde{q} \leqslant q$ in $[a,2t^*-a]$. At last, if $\tilde{q} < q$ on a set of positive measure, by the Sturm's Comparison Theorem 2.3, we conclude that u should vanish in $]a,2t^*-a[$, a contradiction. If instead $\tilde{q} \equiv q$ a.e. on $[a,2t^*-a]$, then $\tilde{u} \equiv u$ on $[a,2t^*-a]$ (as they solve the same initial value problem) and so $u(2t^*-a) = \tilde{u}(2t^*-a) = 0$, again a contradiction.

The monotonicity of u is a direct consequence of its concavity. □

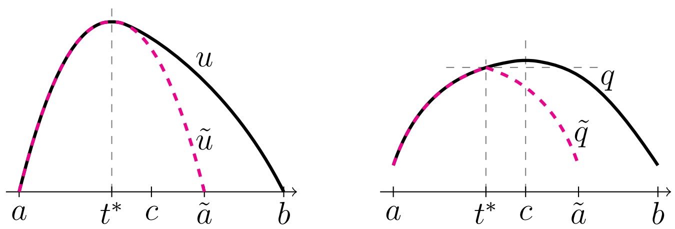

FIGURE 1. Qualitative representation of $\tilde{u}$ and $\tilde{q}$ defined in (2.9), where $\tilde{a} = 2t^* - a$.

Combining Lemma 2.8 with Theorem 2.7, we obtain the first criterion for uniqueness to problem (2.1).

Theorem 2.9. *Suppose the assumptions at the beginning of this section hold. Let $q \in \mathrm{BV}([a,b])$ be such that $q(t) = q(a+b-t)$ a.e. on $[a,b]$ and q is non-decreasing on $[a,c]$, where $c = (a+b)/2$. Then, problem* (2.1) *has at most one positive solution.*

Note that this theorem establishes uniqueness without actually proving that (2.2) holds. However, for the next section, this and actually more information on the solutions of the linearized equation is necessary. This is what we do in the next proposition.

Proposition 2.10. *Under the same assumptions as in Theorem 2.9, if u is a positive solution to* (2.1) *and w is a nontrivial solution to the linearized equation*

$$\begin{cases} w'' + q(t)g'(u)w = 0, \\ w(a) \geqslant 0, \;\; w'(a) \geqslant 0. \end{cases} \tag{2.10}$$

Then, $w(b) < 0$ *and* $w'(b) < 0$.

Proof. Reasoning as in the proof of Lemma 2.4, using Sturm's Comparison Theorem 2.3, we show the existence of $\xi \in \,]a, b[$ such that $w(\xi) = 0$ and $w(t) > 0$, for all $t \in \,]a, \xi[$. We claim that

$$\xi \in \,]c, b[, \tag{2.11}$$

where $c := (a + b)/2$. We suppose by contradiction that $\xi \in \,]a, c]$. Let $\eta \in [a, c[$ be such that $w'(\eta) = 0$. Such a point η exists because of the conditions on w at a in (2.10). Differentiating the equation in (2.1), we obtain, as in the proof of Theorem 2.7, the following equation for u':

$$u''' + q(t)g'(u)u' = -q'(t)g(u)$$

in the sense of measures. Then multiplying the equation of w in (2.10) by u' and vice versa and subtracting them, we obtain (2.8). An integration from η to ξ then yields

$$u'(\xi)w'(\xi) + u''(\eta+)w(\eta) = \int_{]\eta,\xi]} g(u(t))\, w(t)\, q'(\mathrm{d}t).$$

The left-hand side is $\leqslant 0$ since $w(\eta) > 0$, $u''(\eta+) = -q(\eta+)g(u(\eta)) \leqslant 0$, $w'(\xi) < 0$, and $u'(\xi) \geqslant 0$ (thanks to Lemma 2.8). The right-hand side is $\geqslant 0$ thanks to the hypothesis $q' \geqslant 0$ in the sense of measures on $[a, c]$. Therefore both are null and this implies that $q' \equiv 0$ on $]\eta, \xi]$ and $q(\eta+) = 0$. Thus $q \equiv 0$ a.e. on $[\eta, \xi]$, so $w'' \equiv 0$ in $[\eta, \xi]$ and we deduce that $w' \equiv w'(\eta) = 0$. Hence w is constant in $[\eta, \xi]$. This contradiction proves the claim (2.11).

Next, we prove that there are no critical points of w in $[\xi, b]$. By contradiction, let $\gamma \in [\xi, b]$ be such that $w'(\gamma) = 0$ (without loss of generality, assume that γ is the first critical point after ξ). Proceeding as before, we integrate (2.8) from ξ to γ, obtaining

$$0 \geqslant -u''(\gamma+)w(\gamma) - u'(\xi)w'(\xi) = \int_{]\xi,\gamma]} g\big(u(t)\big)\, w(t)\, q'(\mathrm{d}t) \geqslant 0,$$

since $w(\gamma) < 0$, $u''(\gamma+) = -q(\gamma+)g(u(\gamma)) \leqslant 0$, $w'(\xi) < 0$, $u'(\xi) \leqslant 0$ (due to the symmetry, as before). Therefore $q' \equiv 0$ on $]\xi, \gamma]$ and $q(\gamma+) = 0$. Thus $q \equiv 0$ a.e. on $[\xi, \gamma]$, which in turn implies that w' is constant on $[\xi, \gamma]$, contradicting the definition of η. In conclusion, no critical points of w exist in $[\xi, b]$. Hence $w' < 0$ in $[\xi, b]$ and the proof is complete. □

Let us conclude this section by showing that the previous result holds uniformly with respect to perturbations of g'.

Proposition 2.11. *Under the assumptions of Theorem 2.9, let* u *be a positive solution to* (2.1) *and* $(G_n)_n$ *be a sequence in* $\mathcal{C}([a, b])$ *such that* $G_n \to g'(u(\cdot))$ *uniformly on* $[a, b]$. *Further, let* $(w_n)_n$ *be a sequence of solutions to*

$$\begin{cases} w'' + q(t)G_n(t)w = 0, \\ w(a) \geqslant 0, \ w'(a) \geqslant 0. \end{cases}$$

Then, there exists $\varepsilon > 0$ *such that, for all* n *sufficiently large,*

$$w_n(b) \leqslant -\varepsilon\big(w_n(a) + w_n'(a)\big) \quad \text{and} \quad w_n'(b) \leqslant -\varepsilon\big(w_n(a) + w_n'(a)\big).$$

Proof. Let $z_{1,n}$ (resp. $z_{2,n}$) be the solution to

$$\begin{cases} w'' + q(t)G_n(t)w = 0, \\ w(a) = 1, \ w'(a) = 0, \end{cases} \qquad \left(\text{resp. } \begin{cases} w'' + q(t)G_n(t)w = 0, \\ w(a) = 0, \ w'(a) = 1 \end{cases}\right). \tag{2.12}$$

We claim that there exists $\varepsilon > 0$ such that, for all n sufficiently large and $j \in \{1,2\}$,

$$z_{j,n}(b) \leqslant -\varepsilon \text{ and } z'_{j,n}(b) \leqslant -\varepsilon. \tag{2.13}$$

Suppose on the contrary that there exists a $j \in \{1,2\}$ and a subsequence, still denoted $(z_{j,n})_n$, such that

$$\liminf_{n\to\infty} z_{j,n} \geqslant 0 \quad \text{or} \quad \liminf_{n\to\infty} z'_{j,n} \geqslant 0. \tag{2.14}$$

Because $(G_n)_n$ is bounded in $\mathcal{C}([a,b])$ and the initial conditions are also bounded, Grönwall's Lemma on $z_{j,n}^2 + (z'_{j,n})^2$ implies that the sequence $(z_{j,n})_n$ is bounded in $\mathcal{C}^1([a,b])$. Now, using again the equation and invoking Ascoli–Arzelà Theorem, one deduces that $(z_{j,n})_n$ converges (taking if necessary a subsequence) in $\mathcal{C}^1$ and one names its limit $z_{j,\infty}$. Thanks to $G_n \to g'(u)$, $z_{j,\infty}$ is a solution to

$$\begin{cases} z'' + q(t)g'(u(t))z = 0, \\ z(a) = 1, \quad z'(a) = 0, \end{cases} \text{(if } j=1\text{)} \qquad \text{or} \qquad \begin{cases} z'' + q(t)g'(u(t))z = 0, \\ z(a) = 0, \quad z'(a) = 1 \end{cases} \text{(if } j=2\text{)}. \tag{2.15}$$

Moreover, (2.14) implies that $z_{j,\infty}(b) \geqslant 0$ or $z'_{j,\infty}(b) \geqslant 0$. This contradicts Proposition 2.10.

To conclude the proof, it suffices to notice that

$$w_n = w_n(a)z_{1,n} + w'_n(a)z_{2,n} \tag{2.16}$$

and therefore (2.13) implies the claim. □

3. Exact multiplicity

In this section, we focus our attention on the Dirichlet boundary value problem

$$\begin{cases} u'' + \big(h^+(t) - \mu h^-(t)\big)g(u) = 0, \\ u(a) = 0 = u(b), \end{cases} \tag{3.1}$$

where μ is a real parameter, h^+ and h^- denote the positive and, respectively, the negative part of the weight function h, namely $h^\pm := |h|/2 \pm h/2$.

In the present section, our starting point is the following result proposed in [29] for Dirichlet and mixed boundary conditions and subsequently in [30] for the Neumann and periodic boundary value problems. In the framework of problem (3.1) it reads as follows.

Theorem 3.1 (Feltrin and Zanolin, 2015)**.** *Let $g\colon [0,+\infty[\to [0,+\infty[$ be a continuous function satisfying (g_+), (g_0) and (g_∞). Let $h\colon [a,b] \to \mathbb{R}$ be an L^1-function. Suppose that*

(h_*) *there exist $2m+2$ points*

$$a = \tau_0 \leqslant \sigma_1 < \tau_1 < \cdots < \sigma_i < \tau_i < \cdots < \sigma_m < \tau_m \leqslant \sigma_{m+1} = b,$$

such that $h \succ 0$ on $[\sigma_i, \tau_i]$, for every $i \in \{1,\ldots,m\}$, and $h \prec 0$ on $[\tau_i, \sigma_{i+1}]$, for every $i \in \{0,\ldots,m\}$.

Then, there exists $\mu^* > 0$ such that for every $\mu > \mu^*$ problem (3.1) has at least $2^m - 1$ positive solutions.

The symbol $h \succ 0$ means that $h \geqslant 0$ almost everywhere and $h \not\equiv 0$ on a given interval, and $h \prec 0$ stands for $-h \succ 0$. Moreover, for simplicity in the sequel, we set

$$I_i^+ := [\sigma_i, \tau_i], \quad i \in \{1,\ldots,m\}, \qquad \text{and} \qquad I_i^- := [\tau_i, \sigma_{i+1}], \quad i \in \{0,\ldots,m\},$$

the intervals of positivity and the ones of negativity for the function h. Without loss of generality, from now on, we assume that the points σ_i and τ_i are selected in such a manner that $h \not\equiv 0$ on all left neighborhoods of σ_i and on all right neighborhoods of τ_i (cf. [29, Section 5.2]).

In [29] the authors introduce a topological approach based on an extension of the Leray–Schauder degree for locally compact operators on open (possibly unbounded) sets. Along the proof, they first introduce three constants $0 < r < R$ (with r small and R large) and $\mu^* > 0$ (sufficiently large) such that for every positive solution u of (3.1) with $\mu > \mu^*$, it holds that

$$0 < \max_{t \in I_i^+} |u(t)| < R, \quad \max_{t \in I_i^+} |u(t)| \neq r, \quad \text{for every } i \in \{1, \ldots, m\}. \tag{3.2}$$

Next, based on the above mentioned degree theory, given any nonempty set of indices $\mathcal{I} \subseteq \{1, \ldots, m\}$, they prove that there exists at least one positive solution $u_{\mathcal{I},\mu}$ to (3.1) contained in the set

$$\begin{aligned} \Lambda^{\mathcal{I}} := \Bigg\{ u \in \mathcal{C}([a,b]) \colon r < \max_{t \in I_i^+} |u(t)| < R, \ i \in \mathcal{I} \text{ and} \\ \max_{t \in I_i^+} |u(t)| < r, \ i \in \{1, \ldots, m\} \setminus \mathcal{I} \Bigg\}, \end{aligned} \tag{3.3}$$

namely, $u_{\mathcal{I},\mu}$ is "small" on I_i^+ for $i \notin \mathcal{I}$ and "large" (i.e., $r < \max_{I_i^+} u_{\mathcal{I},\mu} < R$) if $i \in \mathcal{I}$. It is worth noting that $u_{\mathcal{I},\mu}$ is concave in each I_i^+ and convex in each I_i^-, due to the sign condition on h. As a consequence, $\|u\|_\infty < R$. Let us also remark that the precise value of $r > 0$ does not matter as long as it is small enough so that the only non-negative solution in $\Lambda^{\varnothing}$ (i.e., small on all the intervals I_i^+) is the trivial solution (see [29, Lemma 2.2]).

The following result illustrates the convergence of the solutions for $\mu \to +\infty$.

Proposition 3.2. *Let $(\mu_n)_n \subseteq \,]\mu^*, +\infty[$ be such that $\mu_n \to +\infty$, and $(u_{\mu_n})_n$ be a sequence of positive solutions to problem* (3.1)*. Then, there exists a continuous function $u_\infty \colon [a,b] \to [0, +\infty[$ such that, going to a subsequence of $(u_{\mu_n})_n$ if necessary, one has*

$$\lim_{n \to +\infty} u_{\mu_n} = u_\infty \quad \textit{uniformly on } [a,b].$$

Moreover, for all $i \in \{0, \ldots, m\}$, $u_\infty \equiv 0$ on I_i^-, and the restriction $u_\infty|_{I_i^+} \colon I_i^+ \to [0, +\infty[$ is a non-negative solution to

$$\begin{cases} u'' + h^+(t) g(u) = 0, \\ u(\sigma_i) = 0 = u(\tau_i). \end{cases} \tag{3.4}$$

Furthermore, if in addition $(u_{\mu_n})_n \subseteq \Lambda^{\mathcal{I}}$ for some $\mathcal{I} \subseteq \{1, \ldots, m\}$ with $\mathcal{I} \neq \varnothing$, then, for all $i \in \{1, \ldots, m\} \setminus \mathcal{I}$, $u_\infty \equiv 0$ on I_i^+ and, for all $i \in \mathcal{I}$, $u_\infty|_{I_i^+}$ is a positive solution to (3.4).

Proof. Let $(\mu_n)_n \subseteq \,]\mu^*, +\infty[$ be such that $\mu_n \to +\infty$, and $(u_{\mu_n})_n$ be a sequence of positive solutions to problem (3.1). Then, recalling (3.2) and (3.3), we find

$$u_{\mu_n} \in \bigcup_{\mathcal{I} \subseteq \{1,\ldots,m\}, \ \mathcal{I} \neq \varnothing} \Lambda^{\mathcal{I}}, \quad \text{for every } n.$$

The case $\mathcal{I} = \varnothing$ is excluded because u_{μ_n} is nontrivial. Thus, up to a subsequence, $(u_{\mu_n})_n \subseteq \Lambda^{\mathcal{I}}$ for some nonempty $\mathcal{I} \subseteq \{1, \ldots, m\}$. From now on, to highlight this property, we denote $(u_{\mathcal{I},\mu_n})_n \subseteq \Lambda^{\mathcal{I}}$ that subsequence.

The proof borrows some ideas developed in [11, 29, 30]. Let $\mu > \mu^*$ and let $\mathcal{I} \subseteq \{1, \ldots, m\}$ be nonempty. Let $u_{\mathcal{I},\mu} \in \Lambda^{\mathcal{I}}$ be a solution to problem (3.1). Then, $\|u_{\mathcal{I},\mu}\|_\infty \leqslant R$ and

$$|u''_{\mathcal{I},\mu}(t)| \leqslant h^+(t) \max_{s \in [0,R]} g(s), \quad \text{for a.e. } t \in I_i^+, \tag{3.5}$$

for all $i \in \{1, \ldots, m\}$. Next, since $\|u_{\mathcal{I},\mu}\|_\infty \leqslant R$, the mean value theorem implies that there exists $t_i^* \in I_i^+$ such that $|u'_{\mathcal{I},\mu}(t_i^*)| = |u_{\mathcal{I},\mu}(\tau_i) - u_{\mathcal{I},\mu}(\sigma_i)|/|I_i^+| \leqslant R/|I_i^+|$. Then, we have
(3.6)

$$|u'_{\mathcal{I},\mu}(t)| = \left|u'_{\mathcal{I},\mu}(t_i^*) + \int_{t_i^*}^t u''_{\mathcal{I},\mu}(\xi)\,\mathrm{d}\xi\right| \leqslant \frac{R}{|I_i^+|} + \|h\|_{L^1(I_i^+)} \max_{s\in[0,R]} g(s) =: \kappa_i, \quad \text{for every } t \in I_i^+,$$

for all $i \in \{1, \ldots, m\}$. As a consequence of the convexity of $u_{\mathcal{I},\mu}$ on I_i^-, $|u'_{\mathcal{I},\mu}(\cdot)|$ is bounded on $[a, b]$. Then, via Ascoli–Arzelà Theorem, there exists $u_{\mathcal{I},\infty} \in \mathcal{C}([a, b])$ such that, up to a subsequence, $u_{\mathcal{I},\mu_n} \to u_{\mathcal{I},\infty}$ uniformly on $[a, b]$.

The rest of the proof is divided into three steps.

Step 1. We are going to prove that $u_{\mathcal{I},\mu}$ tends uniformly to 0 on all the intervals I_i^-. More precisely, we claim that for every ε with $0 < \varepsilon \leqslant r$, there exists $\mu_\varepsilon^\star \geqslant \mu^*$ such that for every $\mu > \mu_\varepsilon^\star$ and $i \in \{1, \ldots, m\}$, we have $\max_{t\in I_i^-} u_{\mathcal{I},\mu}(t) < \varepsilon$.

Let $\varepsilon \in \,]0, r]$. Let us define

$$\delta_i := \min\left\{\frac{|I_i^-|}{2}, \frac{\varepsilon}{2\kappa_i}\right\} \quad \text{and} \quad \tilde{\delta}_i := \min\left\{\frac{|I_i^-|}{2}, \frac{\varepsilon}{2\kappa_{i+1}}\right\},$$

where κ_i and κ_{i+1} are defined in (3.6), as well as

$$\mu_i^{\text{left}} := \frac{R + \kappa_i\delta_i}{\min_{s\in[\varepsilon/2,R]} g(s) \int_{\tau_i}^{\tau_i+\delta_i} \int_{\tau_i}^t h^-(\xi)\,\mathrm{d}\xi\,\mathrm{d}t},$$

$$\mu_i^{\text{right}} := \frac{R + \kappa_{i+1}\tilde{\delta}_i}{\min_{s\in[\varepsilon/2,R]} g(s) \int_{\sigma_{i+1}-\tilde{\delta}_i}^{\sigma_{i+1}} \int_t^{\sigma_{i+1}} h^-(\xi)\,\mathrm{d}\xi\,\mathrm{d}t}.$$

The denominators are positive because h^- is never identically zero in right neighborhoods of τ_i nor in left neighborhoods of σ_{i+1}. We claim that, for $\mu > \mu_\varepsilon^\star := \max_{i=1,\ldots,m}\{\mu_i^{\text{left}}, \mu_i^{\text{right}}, \mu^*\}$ and $i \in \{1, \ldots, m\}$, we have $u_{\mathcal{I},\mu}(\tau_i) < \varepsilon$ and $u_{\mathcal{I},\mu}(\sigma_{i+i}) < \varepsilon$. If so, by the convexity of $u_{\mathcal{I},\mu}$ on I_i^-, we immediately have $u_{\mathcal{I},\mu}(t) < \varepsilon$ for all $t \in I_i^-$.

Let $\mu > \mu_\varepsilon^*$ and $i \in \{1, \ldots, m\}$. Suppose on the contrary that $u_{\mathcal{I},\mu}(\tau_i) \geqslant \varepsilon$ or $u_{\mathcal{I},\mu}(\sigma_{i+1}) \geqslant \varepsilon$.

Let us first deal with the case $u_{\mathcal{I},\mu}(\tau_i) \geqslant \varepsilon$. By (3.6), $u'_{\mathcal{I},\mu}(\tau_i) \geqslant -\kappa_i$. The convexity of $u_{\mathcal{I},\mu}$ on I_i^- guarantees that $u'_{\mathcal{I},\mu}(t) \geqslant -\kappa_i$ for all $t \in I_i^-$. From the definition of δ_i, it is clear that $u_{\mathcal{I},\mu}(t) \geqslant \varepsilon/2$ for all $t \in [\tau_i, \tau_i + \delta_i]$. An integration on $[\tau_i, t] \subseteq [\tau_i, \tau_i + \delta_i]$ yields

$$u'_{\mathcal{I},\mu}(t) = u'_{\mathcal{I},\mu}(\tau_i) + \mu \int_{\tau_i}^t h^-(\xi) g(u_{\mathcal{I},\mu}(\xi))\,\mathrm{d}\xi \geqslant -\kappa_i + \mu \min_{s\in[\varepsilon/2,R]} g(s) \int_{\tau_i}^t h^-(\xi)\,\mathrm{d}\xi.$$

Next, integrating the above inequality on $[\tau_i, \tau_i + \delta_i]$ and using $\mu > \mu_\varepsilon^* \geqslant \mu_i^{\text{left}}$, we obtain

$$\begin{aligned} u_{\mathcal{I},\mu}(\tau_i + \delta_i) &= u_{\mathcal{I},\mu}(\tau_i) + \int_{\tau_i}^{\tau_i+\delta_i} u'_{\mathcal{I},\mu}(t)\,\mathrm{d}t \\ &\geqslant -\kappa_i\delta_i + \mu \min_{s\in[\varepsilon/2,R]} g(s) \int_{\tau_i}^{\tau_i+\delta_i} \int_{\tau_i}^t h^-(\xi)\,\mathrm{d}\xi\,\mathrm{d}t \\ &> -\kappa_i\delta_i + R + \kappa_i\delta_i = R, \end{aligned}$$

a contradiction with the fact that R is a bound on $u_{\mathcal{I},\mu}$.

The case $u_{\mathcal{I},\mu}(\sigma_{i+1}) \geqslant \varepsilon$ is similar. As above, (3.6) implies that $u'_{\mathcal{I},\mu}(\sigma_{i+1}) \leqslant \kappa_{i+i}$ and so $u'_{\mathcal{I},\mu}(t) \leqslant \kappa_{i+i}$ for all $t \in I_i^-$. By definition of $\tilde{\delta}_i$, we have $u_{\mathcal{I},\mu}(t) \geqslant \varepsilon/2$ for all $t \in [\sigma_{i+1} - \tilde{\delta}_i, \sigma_{i+1}]$.

Integrating twice and using $\mu > \mu^*_\varepsilon \geqslant \mu_i^{\mathrm{right}}$, we obtain the contradiction

$$u_{\mathcal{I},\mu}(\sigma_{i+1} - \tilde{\delta}_i) \geqslant -\kappa_{i+1}\tilde{\delta}_i + \mu \min_{s\in[\varepsilon/2,R]} g(s) \int_{\sigma_{i+1}-\tilde{\delta}_i}^{\sigma_{i+1}} \int_t^{\sigma_{i+1}} h^-(\xi)\,\mathrm{d}\xi\,\mathrm{d}t > R.$$

Step 2. Let $i \in \mathcal{I}$. We are going to prove that $(u_{\mathcal{I},\mu_n})$ tends uniformly, up to a subsequence, to a solution of (3.4) in the interval I_i^+.

We already know that $(u_{\mathcal{I},\mu})_{\mu>\mu^*}$ is bounded on I_i^+ and, due to (3.6), so is $(u'_{\mathcal{I},\mu})_{\mu>\mu^*}$. Then thanks to (3.5), $(u''_{\mathcal{I},\mu})_{\mu>\mu^*}$ is bounded in $L^1(I_i^+)$ and equi-integrable. Then, by combining the Dunford–Pettis Theorem with the Eberlein–Šmulian Theorem, we obtain that (up to a subsequence) $u''_{\mathcal{I},\mu_n} \rightharpoonup v_{\mathcal{I}}$ in $L^1(I_i^+)$, as $n \to +\infty$, for some $v_{\mathcal{I}} \in L^1(I_i^+)$. Now, by Ascoli–Arzelà Theorem, we obtain that $u_{\mathcal{I},\mu_n} \to u_{\mathcal{I},\infty}$ in $\mathcal{C}^1(I_i^+)$, as $n \to +\infty$. Therefore, $u''_{\mathcal{I},\infty} = v_{\mathcal{I}}$ and $u_{\mathcal{I},\infty} \in W^{1,2}(I_i^+)$. By Step 1, we already know that $u_{\mathcal{I},\infty}(\sigma_i) = u_{\mathcal{I},\infty}(\tau_i) = 0$. Finally, passing to the limit on the weak formulation of the equation, we have that $u_{\mathcal{I},\infty}|_{I_i^+}$ is a non-negative solution to (3.4). Moreover, because $i \in \mathcal{I}$, $\max_{I_i^+} u_{\mathcal{I},\infty} \geqslant r$, so $u_{\mathcal{I},\infty}$ is nontrivial and, exploiting the strong maximum principle, one deduces that $u_{\mathcal{I},\infty}|_{I_i^+}$ is a positive solution to (3.4).

Step 3. Let $i \in \{1,\dots,m\} \setminus \mathcal{I}$. Using hypothesis (g_0) and reasoning as in Step 2, $u_{\mathcal{I},\mu_n} \to u_{\mathcal{I},\infty}$ and that $u_{\mathcal{I},\infty}|_{I_i^+}$ is a non-negative solution to (3.4) such that $\max_{I_i^+} u_{\mathcal{I},\infty} \leqslant r$. That implies that $u_{\mathcal{I},\infty} \equiv 0$ in the interval I_i^+ provided that $r > 0$ was chosen small enough. (See [11, Proposition 5.4] for another approach.) □

Proposition 3.2 is the key ingredient of our investigation, since it describes the limit profiles (for $\mu \to +\infty$) of the solutions $u_{\mathcal{I},\mu}$. The goal of the present section is to find conditions such that $2^m - 1$ is the exact number of solutions to (3.1), knowing that $2^m - 1$ is the exact number of limit profiles.

We can thus state and prove our main result.

Theorem 3.3. *Let $g \in \mathcal{C}^1([0,+\infty[,[0,+\infty[)$ be a function satisfying (g_+), (g_s), (g_0) and (g_∞). Let $h\colon [a,b] \to \mathbb{R}$ be an L^1-function satisfying (h_*). Moreover, we suppose that*

- *for every $i \in \{1,\dots,m\}$, $h^+ \in \mathrm{BV}(I_i^+)$, h^+ satisfies $h^+(t) = h^+(\sigma_i + \tau_i - t)$ for a.e. $t \in [\sigma_i,\tau_i]$ and is non-decreasing on $[\sigma_i,\zeta_i]$, where $\zeta_i := (\tau_i + \sigma_i)/2$ is the middle point of I_i^+.*

Then, there exists $\mu^{**} > 0$ such that for every $\mu > \mu^{**}$ problem (3.1) has exactly $2^m - 1$ positive solutions. These solutions are non-degenerate.

Proof. By hypotheses (g_+), (g_0), (g_∞), and (h_*), Theorem 3.1 applies and guarantees the existence of at least $2^m - 1$ positive solutions of (3.1) for every μ large enough. Each of these solutions belongs to set $\Lambda^{\mathcal{I}}$ for every nonempty subset of indices $\mathcal{I} \subseteq \{1,\dots,m\}$, cf. (3.3). We are going to prove that, for μ larger, there exists a unique positive non-degenerate solution to problem (3.1) in each $\Lambda^{\mathcal{I}}$, for every nonempty $\mathcal{I} \subseteq \{1,\dots,m\}$.

Let $\mathcal{I} \subseteq \{1,\dots,m\}$ be non-empty. The arguments for non-degeneracy and uniqueness are similar. If there are degenerate solutions $u_{\mathcal{I},\mu}$ for μ large, that means that there exists a sequence $\mu_n \to +\infty$ such that the linearized equation

$$\begin{cases} w'' + \big(h^+(t) - \mu_n h^-(t)\big) g'\big(u_{\mathcal{I},\mu_n}(t)\big) w = 0, \\ w(a) = 0 = w(b) \end{cases}$$

possesses a nontrivial solution w_n. Analogously, if there are distinct solutions in $\Lambda^{\mathcal{I}}$ for μ large, then there exist a sequence $\mu_n \to +\infty$ and $u_{\mathcal{I},\mu_n} \in \Lambda^{\mathcal{I}}$, $v_{\mathcal{I},\mu_n} \in \Lambda^{\mathcal{I}}$ two different positive solutions to (3.1). Setting $w_n := u_{\mathcal{I},\mu_n} - v_{\mathcal{I},\mu_n}$ provides a nontrivial solution to

$$\begin{cases} w'' + \bigl(h^+(t) - \mu_n h^-(t)\bigr) G_n(t) w = 0, & (3.7)\\ w(a) = 0 = w(b), & (3.8) \end{cases}$$

where

$$G_n(t) := \int_0^1 g'\bigl(s u_{\mathcal{I},\mu_n}(t) + (1-s) v_{\mathcal{I},\mu_n}(t)\bigr)\,\mathrm{d}s. \tag{3.9}$$

Thus, if non-degeneracy (resp. uniqueness) fails for μ large, there exists a sequence $\mu_n \to +\infty$ such that, for all n, system (3.7)–(3.8) with $G_n(t) := g'\bigl(u_{\mathcal{I},\mu_n}(t)\bigr)$ (resp. with G_n given by (3.9)) has a nontrivial solution. We will show hereafter that this is not possible.

Before that, let us collect the properties of G_n (common to both cases). First, Proposition 3.2 says that $u_{\mathcal{I},\mu_n} \to u_{\mathcal{I},\infty}$ uniformly on $[a,b]$. In case we have two solutions, one also has $v_{\mathcal{I},\mu_n} \to v_{\mathcal{I},\infty}$ uniformly on $[a,b]$. Since $u_{\mathcal{I},\infty} \equiv 0 \equiv v_{\mathcal{I},\infty}$ on all I_i^-, $i \in \{0,\dots,m\}$, and I_i^+ with $i \notin \mathcal{I}$ and we are in a situation where we have uniqueness of the positive solution of (3.4) on I_i^+, $i \in \mathcal{I}$ (see Theorem 2.9), one necessarily has $u_{\mathcal{I},\infty} \equiv v_{\mathcal{I},\infty}$ on $[a,b]$. Therefore, G_n satisfies the following properties:

- $G_n \in \mathcal{C}([a,b])$;
- $G_n \to g'(u_{\mathcal{I},\infty}(\cdot))$ uniformly on $[a,b]$;
- for all $t \in [a,b]$, $G_n(t) \geqslant 0$.

In the argument below we will also make use of the following function

$$r_n(t) := \frac{w_n(t)}{w_n'(t)} \quad \text{defined for } t \in [a,b] \text{ such that } w_n'(t) \neq 0.$$

Using equation (3.7), one deduces that r_n satisfies the equation

$$r_n'(t) = \frac{(w_n'(t))^2 - w_n(t) w_n''(t)}{(w_n'(t))^2} = 1 + \bigl(h^+(t) - \mu_n h^-(t)\bigr) G_n(t)\, r_n^2(t), \quad \text{for a.e. } t \in \operatorname{dom} r_n. \tag{3.10}$$

Note that, because w_n is a nontrivial solution to (3.7), w_n and w_n' cannot vanish at the same time and so $|r_n(t)| \to +\infty$ when t approaches the boundary of $\operatorname{dom} r_n$.

Now, let us rule out the existence of nontrivial solutions to (3.7)–(3.8) by examining in turn how boundary signs transfer for each sub-interval.

Step 1. First, let us consider a negativity interval $I_i^- = [\tau_i, \sigma_{i+1}]$ for some fixed $i \in \{0,\dots,m\}$. From equation (3.7) we deduce that $w_n''(t) \leqslant 0$ for a.e. $t \in I_i^-$ such that $w_n(t) < 0$ and $w_n''(t) \geqslant 0$ for a.e. $t \in I_i^-$ such that $w_n(t) > 0$. Therefore, w_n is concave in the interval I_i^- whenever w_n is negative, and convex in the interval I_i^- whenever w_n is positive. Therefore, we have

(3.11) if $w_n(\tau_i) \geqslant 0$ and $w_n'(\tau_i) \geqslant 0$, then $w_n(\sigma_{i+1}) \geqslant w_n(\tau_i)$ and $w_n'(\sigma_{i+1}) \geqslant w_n'(\tau_i)$;

(3.12) if $w_n(\tau_i) \leqslant 0$ and $w_n'(\tau_i) \leqslant 0$, then $w_n(\sigma_{i+1}) \leqslant w_n(\tau_i)$ and $w_n'(\sigma_{i+1}) \leqslant w_n'(\tau_i)$.

In cases (3.11)–(3.12), we claim that, for any $c > 0$, one also has

$$|w_n(\tau_i)| \leqslant c |w_n'(\tau_i)| \;\Rightarrow\; |w_n(\sigma_{i+1})| \leqslant \hat{c} |w_n'(\sigma_{i+1})|, \tag{3.13}$$

where $\hat{c} := c + |I_i^-|$. Indeed the premise of (3.13) implies that $w_n'(\tau_i) \neq 0$ (as w_n is a nontrivial solution). Thus, noticing that (3.7) implies that $|w_n'|$ is non-decreasing on I_i^-, $I_i^- \subseteq \operatorname{dom} r_n$. Then equation (3.10) implies that $r_n' \leqslant 1$ on I_i^- and, integrating, we obtain

$$r_n(\sigma_{i+1}) \leqslant r_n(\tau_i) + |I_i^-|,$$

which proves the claim.

Step 2. Let us now deal with intervals $I_i^+ = [\sigma_i, \tau_i]$ with $i \notin \mathcal{I}$. We first claim that, for n large enough and for any $c > 0$,

$$|w_n(\sigma_i)| \leqslant c|w_n'(\sigma_i)| \;\Rightarrow\; |w_n(\tau_i)| \leqslant \hat{c}|w_n'(\tau_i)| \tag{3.14}$$

where $\hat{c} := c + 2|I_i^+|$. To prove that, note that the premise of the implication reads $|r_n(\sigma_i)| \leqslant c$ ($w_n'(\sigma_i) \neq 0$ because the premise would otherwise imply that w_n is the trivial solution). We will establish that, for n sufficiently large,

$$\forall t \in I_i^+, \qquad t \in \operatorname{dom} r_n \quad \text{and} \quad |r_n(t)| \leqslant \hat{c} = c + 2|I_i^+|, \tag{3.15}$$

from which (3.14) follows. Let ϑ_i be such that

$$0 < \vartheta_i < \frac{|I_i^+|}{\|h\|_{L^1(I_i^+)}\, \hat{c}^2}.$$

Given that $G_n \to g'(u_{\mathcal{I},\infty}) = 0$ uniformly on I_i^- (remembering (g_0)), we can consider n be sufficiently large so that $G_n(t) \leqslant \vartheta_i$ for all $t \in I_i^+$. The properties in (3.15) are plainly satisfied whenever t is close to σ_i. Let $[\sigma_i, t^*[\subseteq [\sigma_i, \tau_i]$ be the maximal interval where the properties in (3.15) are valid. If $t^* = \tau_i$ we are done. Otherwise, $|r_n(t^*)| = \hat{c}$ and $t^* \in \operatorname{dom} r_n$ (recall that $|r_n|$ blows up when t approaches the boundary of $\operatorname{dom} r_n$). Then (3.10) yields the following contradiction:

$$\begin{aligned} 2|I_i^+| = \hat{c} - c \leqslant |r_n(t^*)| - |r_n(\sigma_i)| \leqslant |r_n(t^*) - r_n(\sigma_i)| = \left| \int_{\sigma_i}^{t^*} r_n'(t)\,\mathrm{d}t \right| \\ = \left| \int_{\sigma_i}^{t^*} 1 + h^+(t) G_n(t) r_n^2(t)\,\mathrm{d}t \right| \leqslant |I_i^+| + \|h\|_{L^1(I_i^+)}\, \vartheta_i\, \hat{c}^2 < 2|I_i^+|. \end{aligned}$$

Now we claim that, for n large (how large depends on the c that will be chosen later to apply (3.14)),

$$\text{if } w_n(\sigma_i) \geqslant 0 \text{ and } w_n'(\sigma_i) \geqslant 0, \text{ then } w_n(\tau_i) \geqslant \frac{1}{2} w_n(\sigma_i) \text{ and } w_n'(\tau_i) \geqslant \frac{1}{2} w_n'(\sigma_i); \tag{3.16}$$

$$\text{if } w_n(\sigma_i) \leqslant 0 \text{ and } w_n'(\sigma_i) \leqslant 0, \text{ then } w_n(\tau_i) \leqslant \frac{1}{2} w_n(\sigma_i) \text{ and } w_n'(\tau_i) \leqslant \frac{1}{2} w_n'(\sigma_i). \tag{3.17}$$

Let $z_{1,n}$ (resp. $z_{2,n}$) be as in (2.12) with $[a, b]$ replaced by $I_i^+ = [\sigma_i, \tau_i]$. As in that proof, we have (2.16) and $(z_{1,n})$ (resp. $(z_{2,n})$) converges, taking if necessary a subsequence, in $\mathcal{C}^1(I_i^+)$ to a function $z_{1,\infty}$ (resp. $z_{2,\infty}$) which is a solution to (2.15) with $u \equiv u_{\mathcal{I},\infty} \equiv 0$ on I_i^+. Therefore $z_{1,\infty}(t) = 1$ and $z_{2,\infty}(t) = t - \sigma_i$. Thus, for n sufficiently large, $z_{1,n}(\tau_i) \geqslant 1/2$, $z_{1,n}'(\tau_i) \geqslant -1/(4c)$, $z_{2,n}(\tau_i) \geqslant |I_i^+|/2$, and $z_{2,n}'(\tau_i) \geqslant 3/4$.

If $w_n(\sigma_i) \geqslant 0$ and $w_n'(\sigma_i) \geqslant 0$, then the previous inequalities and (3.14) yield

$$\begin{aligned} w_n(\tau_i) &= w_n(\sigma_i) z_{1,n}(\tau_i) + w_n'(\sigma_i) z_{2,n}(\tau_i) \geqslant \frac{1}{2} w_n(\sigma_i) + \frac{|I_i^+|}{2} w_n'(\sigma_i) \geqslant \frac{1}{2} w_n(\sigma_i), \\ w_n'(\tau_i) &= w_n(\sigma_i) z_{1,n}'(\tau_i) + w_n'(\sigma_i) z_{2,n}'(\tau_i) \geqslant -\frac{1}{4c} w_n(\sigma_i) + \frac{3}{4} w_n'(\sigma_i) \geqslant \frac{1}{2} w_n'(\sigma_i). \end{aligned}$$

Assertion (3.17) is established in a similar way.

Step 3. At last, let us focus on the intervals $I_i^+ = [\sigma_i, \tau_i]$ for some $i \in \mathcal{I}$. We claim that, for any $c > 0$, there exists $\varepsilon > 0$ (independent of n) such that, for n large enough,

$$\begin{aligned}
&\text{if } w_n(\sigma_i) \geqslant 0,\ w_n'(\sigma_i) \geqslant 0 \text{ and } |w_n(\sigma_i)| \leqslant c|w_n'(\sigma_i)| \\
&\text{then } w_n(\tau_i) \leqslant -\varepsilon\big(w_n(\sigma_i) + w_n'(\sigma_i)\big) < 0,\ w_n'(\tau_i) \leqslant -\varepsilon\big(w_n(\sigma_i) + w_n'(\sigma_i)\big) < 0, \\
&\text{and } |w_n(\tau_i)| \leqslant \hat{c}|w_n'(\tau_i)| \text{ for some } \hat{c} \text{ independent on } n;
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\text{if } w_n(\sigma_i) \leqslant 0,\ w_n'(\sigma_i) \leqslant 0 \text{ and } |w_n(\sigma_i)| \leqslant c|w_n'(\sigma_i)| \\
&\text{then } w_n(\tau_i) \geqslant -\varepsilon\big(w_n(\sigma_i) + w_n'(\sigma_i)\big) > 0,\ w_n'(\tau_i) \geqslant -\varepsilon\big(w_n(\sigma_i) + w_n'(\sigma_i)\big) > 0, \\
&\text{and } |w_n(\tau_i)| \leqslant \hat{c}|w_n'(\tau_i)| \text{ for some } \hat{c} \text{ independent on } n.
\end{aligned} \tag{3.19}$$

Let us prove (3.18), (3.19) being similar. Let $c > 0$ be fixed. Proposition 2.11 implies that $w_n(\tau_i) \leqslant -\varepsilon\big(w_n(\sigma_i) + w_n'(\sigma_i)\big)$ and $w_n'(\tau_i) \leqslant -\varepsilon\big(w_n(\sigma_i) + w_n'(\sigma_i)\big)$ for n large enough and for some $\varepsilon > 0$ independent of n. Given that w_n can be written as a linear combination of $z_{1,n}$ and $z_{2,n}$ (see (2.16) where $[a,b]$ is here $[\sigma_i, \tau_i]$), one has

$$\frac{w_n(\tau_i)}{w_n'(\tau_i)} = r_n(\tau_i) = \frac{r_n(\sigma_i) z_{1,n}(\tau_i) + z_{2,n}(\tau_i)}{r_n(\sigma_i) z_{1,n}'(\tau_i) + z_{2,n}'(\tau_i)}$$

with $r_n(\sigma_i) \in [0, c]$ (recall that, as above, $w_n'(\sigma_i) \neq 0$). Note that

$$\frac{\gamma z_{1,n}(\tau_i) + z_{2,n}(\tau_i)}{\gamma z_{1,n}'(\tau_i) + z_{2,n}'(\tau_i)} \xrightarrow[n \to +\infty]{} \frac{\gamma z_{1,\infty}(\tau_i) + z_{2,\infty}(\tau_i)}{\gamma z_{1,\infty}'(\tau_i) + z_{2,\infty}'(\tau_i)} \quad \text{uniformly w.r.t. } \gamma \in [0, c],$$

where $z_{1,\infty}$ and $z_{2,\infty}$ are defined in the proof of Proposition 2.11. As $z_{1,\infty}'(\tau_i) < 0$ and $z_{2,\infty}'(\tau_i) < 0$ by Proposition 2.10, the limit is bounded independently of $\gamma \in [0, c]$ and thus, for n large enough, so is $r_n(\tau_i)$. In other words, there exists a constant $\hat{c}$ independent of n such that, for n large enough, $|w_n(\tau_i)| \leqslant \hat{c}|w_n'(\tau_i)|$. Incidentally, note that this shows that the denominator does not vanish for n large and so that $r_n(\tau_i)$ is well defined.

Step 4. To conclude the proof of the non-degeneracy and uniqueness, let us derive a contradiction. For this, we will consecutively examine the sub-intervals I_i^- and I_i^+ starting from a and show that, for n large, $w_n(b) \neq 0$.

Without loss of generality, we can suppose that $w_n'(a) = 1$. If there is a first nontrivial negativity interval $I_0^- = [a, \sigma_1]$, (3.11) implies that $w_n(\sigma_1) \geqslant 0$, $w_n'(\sigma_1) \geqslant 1$, and $|w_n(\sigma_1)| \leqslant c_1|w_n'(\sigma_1)|$ with $c_1 := 1 + |I_0^-|$. If there is no such first interval, then $\sigma_1 = \tau_0 = a$ and we have $w_n(\sigma_1) = 0$, $w_n'(\sigma_1) = 1$, and $|w_n(\sigma_1)| \leqslant c_1|w_n'(\sigma_1)|$ with $c_1 := 1$. Thus, in both cases we have, for n large, that

$$w_n(\sigma_1) \geqslant 0, \quad w_n'(\sigma_1) \geqslant 1, \quad \text{and} \quad |w_n(\sigma_1)| \leqslant c_1|w_n'(\sigma_1)|.$$

Next we have the interval $I_1^+ = [\sigma_1, \tau_1]$. If $1 \notin \mathcal{I}$, then (3.14) and (3.16) imply that, for n possibly larger,

$$w_n(\tau_1) \geqslant 0, \quad w_n'(\tau_1) \geqslant \frac{1}{2}, \quad \text{and} \quad |w_n(\tau_1)| \leqslant c_2|w_n'(\tau_1)|, \tag{3.20}$$

with $c_2 := c_1 + 2|I_1^+|$. If otherwise $1 \in \mathcal{I}$, then (3.18) yield

$$w_n(\tau_1) \leqslant -\varepsilon_1, \quad w_n'(\tau_1) \leqslant -\varepsilon_1, \quad \text{and} \quad |w_n(\tau_1)| \leqslant c_2|w_n'(\tau_1)|, \tag{3.21}$$

for some $\varepsilon_1 > 0$ and $c_2 > 0$ (independent of n).

The next interval is $I_1^- = [\tau_1, \sigma_2]$. If (3.20) occurred, then (3.11) and (3.13) yield, possibly taking n larger,

$$w_n(\sigma_2) \geqslant 0, \quad w_n'(\sigma_2) \geqslant \frac{1}{2}, \quad \text{and} \quad |w_n(\sigma_2)| \leqslant c_3 |w_n'(\sigma_2)|,$$

where $c_3 := c_2 + |I_1^-|$. If (3.21) occurred, then (3.12) and (3.13) yield

$$w_n(\sigma_2) \leqslant -\varepsilon_1, \quad w_n'(\sigma_2) \leqslant -\varepsilon_1, \quad \text{and} \quad |w_n(\sigma_2)| \leqslant c_3 |w_n'(\sigma_2)|,$$

where again $c_3 := c_2 + |I_1^-|$.

Continuing this procedure through the finitely many sub-intervals I_i^+ and I_i^-, we have to go through (at least) one interval I_i^+ with $i \in \mathcal{I}$ and thus $|w_n(\sigma_i)| \geqslant \varepsilon > 0$, for n large enough and some $\varepsilon > 0$ (independent of n). As a consequence, one ends up with the fact that $w_n(b) \neq 0$ for n large enough. This contradiction concludes the proof. □

4. Numerical experiments and open problems

This section is devoted to the numerical exploration of questions related to Theorem 3.3 as well as discussions about possible future investigations. Throughout this section, we will specialize the function g to the representative case $g(u) = u^3$ and take $[a, b] = [0, 1]$.

In the previous section, the exact number of positive solutions of (3.1) for μ large was determined. A natural question is for what range of μ does this number of solutions persist. On Fig. 2 and 3, one can see the branches of positive solutions to (3.1) for $h(t) = \sin(3\pi t)$ (which corresponds to $m = 2$ in Theorem 1.1) as well as the graphs of these solutions for various values of μ. In this situation, we observe that there are 3 positive solutions for all $\mu \geqslant 0$ and the 3 branches collapse to a single point when $\mu \approx -0.21$. Below this value, a unique positive solution (whence symmetric) exists. This bifurcation point can be seen as a symmetry breaking of the ground state: numerical evidence suggests that, on the right of the bifurcation point, the value $\mathcal{A}(u)$ of the action functional

$$\mathcal{A}(u) := \frac{1}{2} \int_a^b (u')^2 \, \mathrm{d}t - \int_a^b \big(h^+(t) - \mu h^-(t)\big) G(u) \, \mathrm{d}t,$$

where G is a primitive of g, is lower on the "external" branches (the solutions on these branches are symmetric to each other) and higher on the "central" branch made of symmetric solutions.

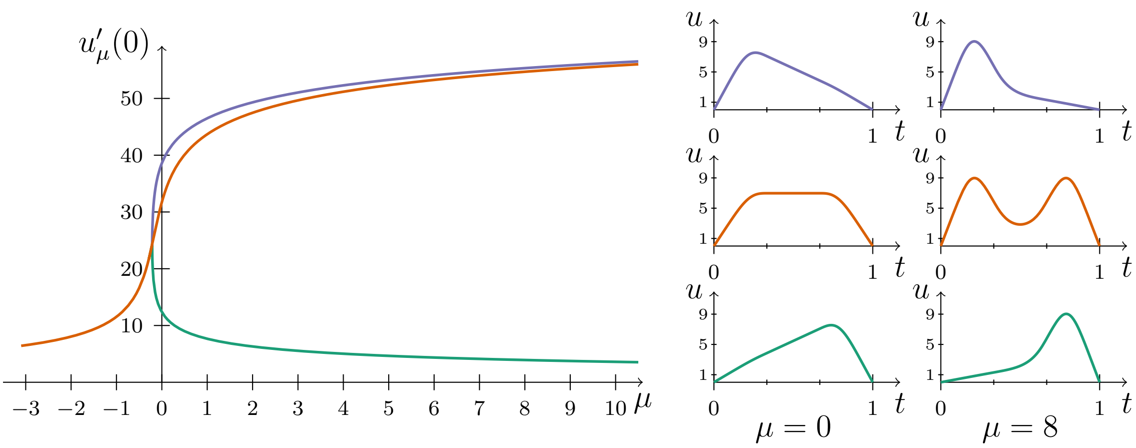

FIGURE 2. The branches of positive solutions to (3.1) with $h(t) = \sin(3\pi t)$ in $[0, 1]$ and $g(u) = u^3$ (on the left) and graphs of solutions for some μ (on the right).

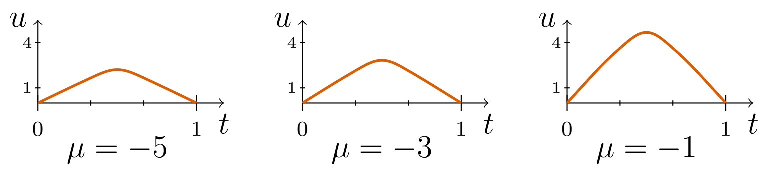

FIGURE 3. Graphs of the unique positive solution to (3.1) with $h(t) = \sin(3\pi t)$ in $[0, 1]$ and $g(u) = u^3$ for various $\mu < 0$.

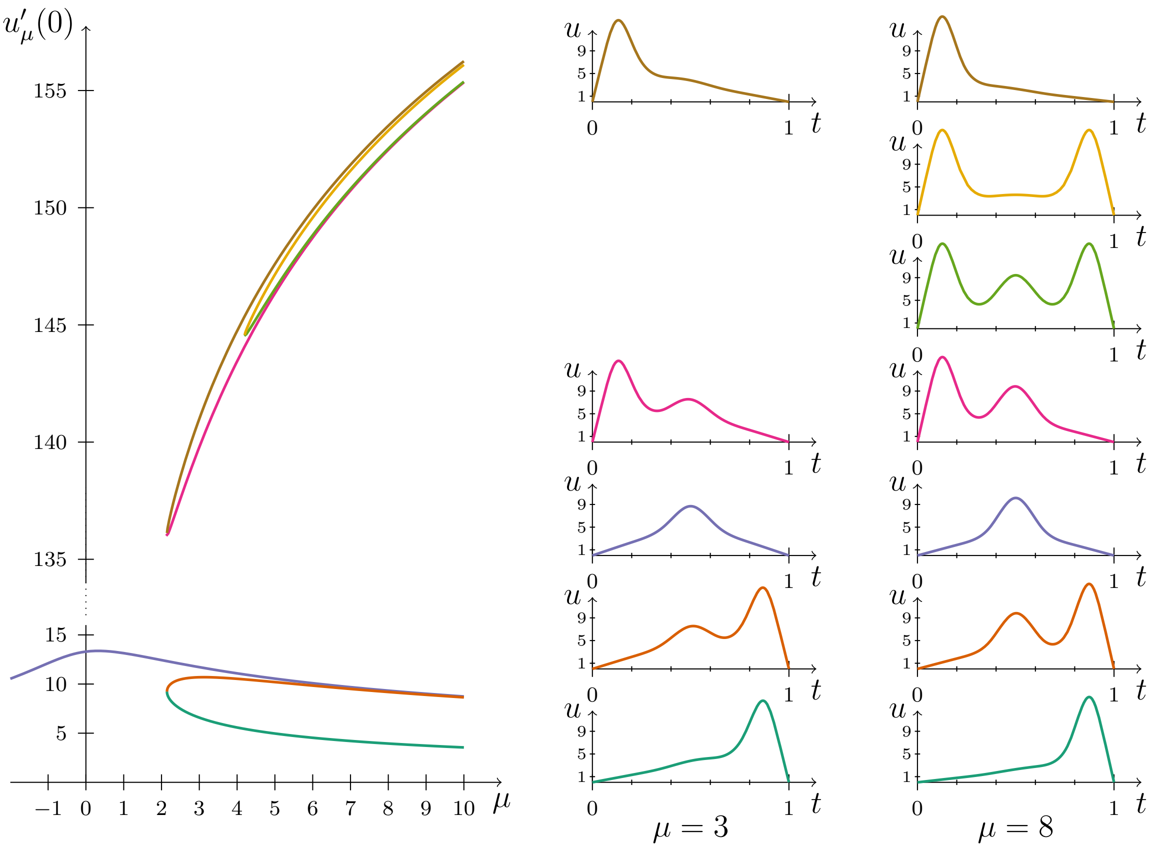

FIGURE 4. The branches of positive solutions to (3.1) with $h(t) = \sin(5\pi t)$ in $[0, 1]$ and $g(u) = u^3$ (on the left) and graphs of solutions for some μ (on the right).

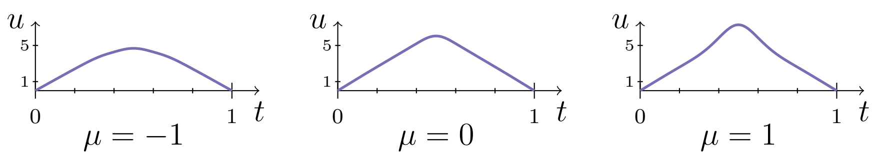

FIGURE 5. Graphs of the unique positive solution to (3.1) with $h(t) = \sin(5\pi t)$ in $[0, 1]$ and $g(u) = u^3$ for various $\mu \leqslant 1$.

Notice that this single bifurcation point is a rare occurrence and is not a consequence solely of the symmetry of the weight h. Indeed, on Fig. 4, six of the seven branches of positive solutions for the symmetric weight $h(t) = \sin(5\pi t)$ collapse two by two. Moreover, all degenerate turning points occur at positive values of μ (those occurring at the same μ are due to the fact that the solutions on these branches are symmetric to each other). The fact that all the branches but one collapse two by two also occurs in asymmetric cases such as $h(t) = \sin(4\pi t)$ (see Fig. 6) and for small perturbations of $h(t) = \sin(3\pi t)$ (see Fig. 8).

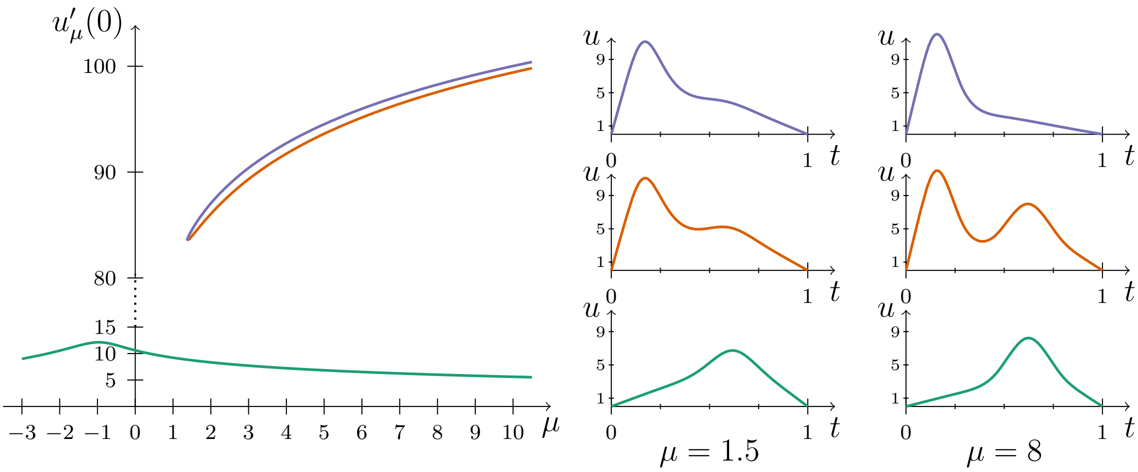

FIGURE 6. The branches of positive solutions to (3.1) with $q(t) = \sin(4\pi t)$ in $[0, 1]$ and $g(u) = u^3$ (on the left) and graphs of solutions for some μ (on the right).

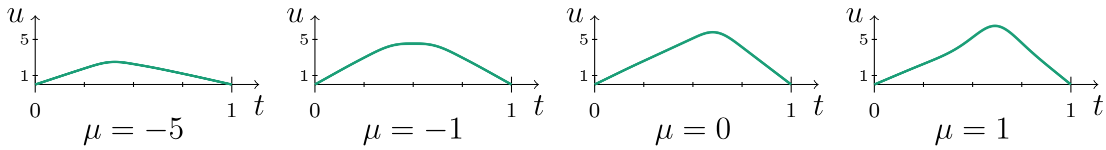

FIGURE 7. Graphs of the unique positive solution to (3.1) with $q(t) = \sin(4\pi t)$ in $[0, 1]$ and $g(u) = u^3$ along the lower branch of solutions.

In all the above examples (Figs. 2, 4, 6, and 8), one branch of positive solutions exists for all values of μ. This is expected as the existence of at least one positive solution for all $\mu \in \mathbb{R}$ has been established [29, Theorem 5.1] (extending several earlier results [21, 24, 51] solely dealing with the case $\mu < 0$). Moreover, numerical experiments show that the ground states of $\mathcal{A}$ form such a branch and that this branch is made of unimodal solutions (i.e., increasing then decreasing functions as they are non-negative). When h is symmetric (i.e., is even with respect to the center of the interval $]a, b[$), the symmetric solutions with the lower action $\mathcal{A}$ also form a branch living for all μ but that branch may not coincide with the previous one as a symmetry breaking may occur (see Fig. 2), in which case functions along that branch may not be unimodal for all $\mu \geqslant 0$.

The fact that other branches extend up to $\mu = 0$ is a delicate question. Clearly this is linked to the question of the uniqueness of the positive solutions for $\mu = 0$. In this case, the weight in (3.1) is positive in each I_i^+ and identically zero in every interval I_i^-. The multiplicity of positive solutions of (3.1) depends on the length of the intervals I_i^-: if the intervals I_i^- are sufficiently small, uniqueness holds (see, for instance, [19, 39, 47]).

When μ is negative enough, Figs. 2, 4, 6, and 8 indicate that the problem admits a single solution. For $\mu < 0$, $h^+ - \mu h^- \geqslant 0$ but this uniqueness is not a consequence of the known criteria (neither the one presented in Section 2, nor those found in the literature, see for example [17, 41, 42, 50, 54]). This solution is trivially unimodal as the solutions are concave

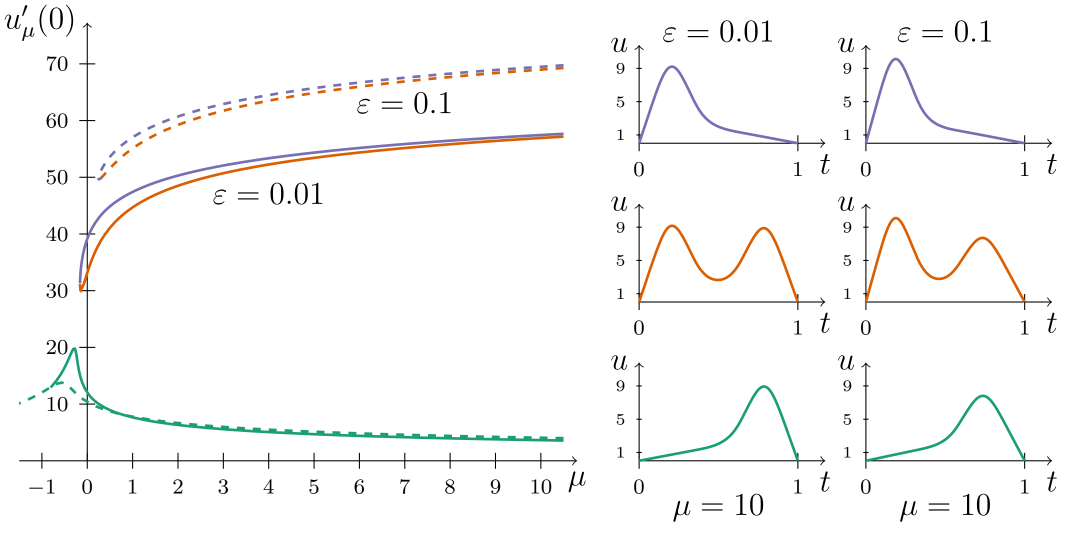

FIGURE 8. The branches of positive solutions to (3.1) with $h(t) = \sin(3\pi t/(1-\varepsilon))$ in $[0, 1-\varepsilon]$, $h(t) = 0$ in $[1-\varepsilon, 1]$ and $g(u) = u^3$ (on the left) and the graphs of solutions for $\mu = 10$ (on the right).

for $\mu \leqslant 0$ (and is symmetric when h is). As far as we know, the question of the uniqueness of positive solutions for large negative μ is open.

It is well known that there are non-negative weights q such that the Dirichlet problem (2.1) possesses several positive solutions. In 1959, Moore and Nehari [47] gave such an example as a smooth perturbation of a piece-wise constant function. Here we have chosen the simple $\mathcal{C}^\infty$ symmetric function

$$q\colon [0,1] \to \mathbb{R},\ q(t) = (2t-1)^2.$$

Still considering $g(u) = u^3$, Fig. 9 shows on the left the graphs of the three positive solutions u_i, $i \in \{0,1,2\}$, to problem (2.1). If we denote $u(\cdot;\alpha)$ the solution to the differential equation of (2.1) for the initial conditions (see Section 2)

$$u(0;\alpha) = 0, \qquad u'(0;\alpha) = \alpha,$$

then $u_i(\cdot) = u(\cdot;\alpha_i)$ for some $\alpha_i > 0$. The right graphs of Fig. 9 show the derivatives $t \mapsto \partial_\alpha u(t;\alpha_i)$ of the solutions with respect to the initial velocity α. Based on the numerical evidence of these graphs, we posit that $\partial_\alpha u(1;\alpha_i) \neq 0$ and so that the three solutions u_i are non-degenerate. Therefore a branch should emanate from each of them if h has the same shape as the above q on an interval I^+_i. More precisely, let us consider the function

$$h\colon [0,1] \to \mathbb{R},\ h(t) = \begin{cases} (t-1/4)^2 & \text{if } t \in [0,1/2[, \\ -\sin(4\pi t) & \text{if } t \in [1/2,1] \end{cases} \tag{4.1}$$

for problem (3.1). Since for this h there are three positive limit profiles to choose from on $I^+_1 = [0,1/2]$ and one positive limit profile on $I^+_2 = [3/4,1]$, we expect $4 \cdot 2 - 1 = 7$ branches of positive solutions. This is confirmed by our numerical experiments, see Figs. 10 and 11. Four of theses branches live until $\mu \approx 26250$ while two persist until $\mu \approx 0.56$ and the last one lives for all μ (and, according to the graphs, is made of unimodal functions). We believe that the techniques developed in this paper can be extended to prove the existence of these seven branches and more generally to establish the exact multiplicity result under a sole condition of the non-degeneracy of the positive limit profiles.

Let us conclude with a more technical remark. Notice that the vertical axis on Figs. 10 and 11 reports $\|u'_\mu\|_{L^2}$ and not $u'_\mu(0)$. The reason is that the initial velocities of some solutions are really close, especially for the large values of μ we need to tackle to see all branches. This is particularly true for solutions having the same limit profile on I_1^+ but differing on I_2^+. Thus, instead of tracking $u'_\mu(0)$, we use a finite element discretization of (3.1) and start a branch continuation algorithm at $\mu = 10^6$ from the limit profiles (the dashed graphs on Figs. 10 and 11) refined thanks to a damped Newton's method.

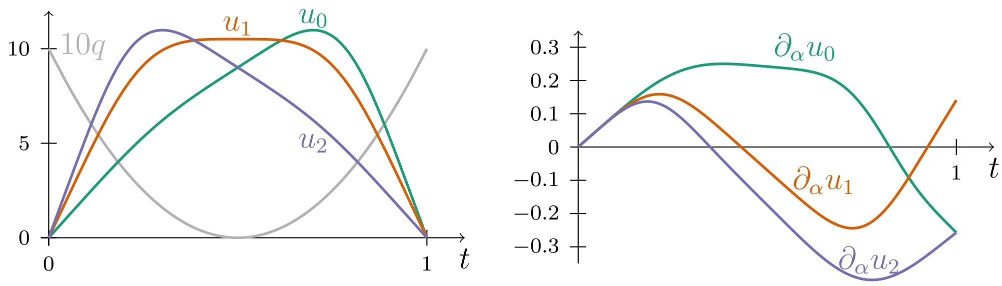

FIGURE 9. The graphs of the three positive solutions u_i to the Dirichlet problem (2.1) when $q(t) = (2t-1)^2$, $t \in [0,1]$, and $g(u) = u^3$, $u \in [0,+\infty[$ (on the left). The linearization with respect to the initial velocity (on the right).

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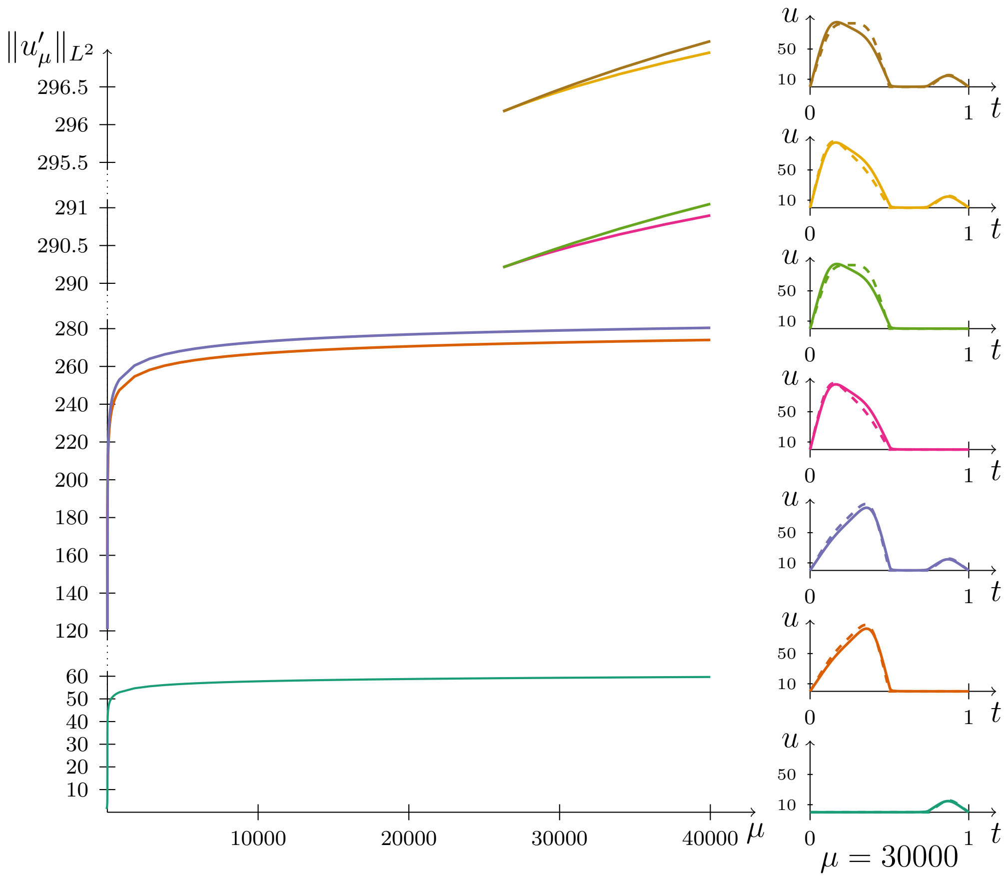

FIGURE 10. The branches (μ, u) of positive solutions when the limit problem possesses three positive solutions on the first interval I_1^+ when h is given by (4.1) and $g(u) = u^3$ (on the left). The graphs of solutions for $\mu = 30000$ as well as those of the limit profiles, dashed (on the right).

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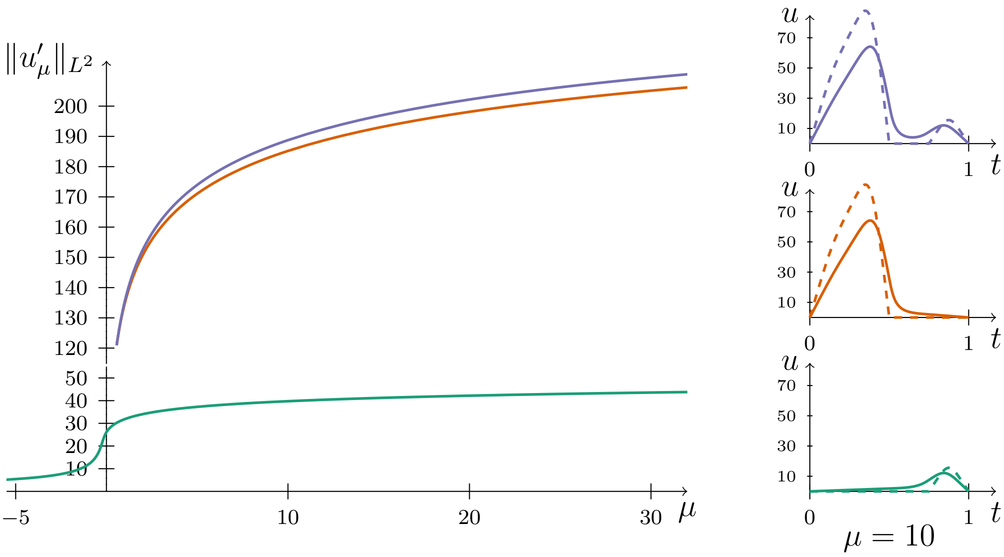

FIGURE 11. The branches (μ, u) of positive solutions when the limit problem possesses three positive solutions on the first interval I_1^+ when h is given by (4.1) and $g(u) = u^3$ (on the left) and the graphs of solutions for $\mu = 10$ as well as those of the limit profiles, dashed (on the right).

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