DIFFERENTIAL EXPONENTIAL TOPOLOGICAL FIELDS

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Abstract. We axiomatize a class of existentially closed exponential fields equipped with an E-derivation and endowed with a definable V-topology. We apply our results to the field of real numbers endowed with $exp(x)$ the classical exponential function defined by its power series expansion and to the field of p-adic numbers endowed with the function $exp(px)$ defined on the p-adic integers where p is a prime number strictly bigger than 2 (or with $exp(4x)$ when $p = 2$).

1. Introduction

The problem we address here is the following: given an elementary class of existentially closed exponential topological fields of characteristic 0 (where possibly the exponential function E is partially defined) whether the class of existentially closed differential expansions is an elementary class and if this is the case how it can be axiomatized. The model-complete theories of exponential fields we include in our analysis are the theory of the field of real numbers with the exponential function and the field of p-adic numbers with the exponential function restricted to the subring of p-adic integers. The derivations δ we consider are E-derivations, namely $\delta(E(x)) = \delta(x)E(x)$, but δ is not assumed to be continuous. We answer the question above as follows.

We place ourselves in topological fields where the topology is definable and is a V-topology, so either induced by an archimedean absolute value or a non-trivial valuation [27, Section 3]. Given an $\mathcal{L}$-theory T of fields, we denote by T_δ the $\mathcal{L} \cup \{\delta\}$-theory consisting of T together with an axiom expressing that δ is an E-derivation.

Theorem (later Theorem 4.4) Let T be a model-complete complete theory of topological $\mathcal{L}$-fields of characteristic 0 endowed with a definable V-topology. Assume when the topology is induced by an ordering that the models of T satisfy an implicit function theorem $(\mathrm{IFT})_E$ and have the lack of flat functions property $(\mathrm{LFF})_{\mathcal{E}}$ and when the topology is induced by a non-trivial valuation that the models of T satisfy an analytic implicit function theorem $(\mathrm{IFT})_E^{an}$. Then the class of existentially closed models of T_δ is elementary.

Further even if the axiomatization we give has no clear geometric interpretation, we thought worthwhile to enumerate what is needed to describe the existentially closed models.

In the ordered case, we apply our result to $(\mathbb{R}, exp)$ where exp is the classical exponential function and δ an E-derivation, using the result of A. Wilkie on the model-completeness of $(\mathbb{R}, exp)$.

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In the valued case we apply them to $(\mathbb{Q}_p, E_p)$, where $\mathbb{Q}_p$ is the field of p-adic numbers and to $(\mathbb{C}_p, E_p)$, where $\mathbb{C}_p$ is the completion of the algebraic closure of $\mathbb{Q}_p$. In these last cases, the exponential function is only partially defined (on the valuation ring) with $E_2(x) := exp(4x)$ and $E_p(x) := exp(px)$, $p \neq 2$ and there we use model-completeness results due to N. Mariaule [21], [22].

Independently, this question has also been considered by A. Fornasiero and E. Kaplan in the following setting. Given an o-minimal expansion $\mathcal{K}$ of an ordered field which is model-complete and expanded with a *compatible* derivation [13], they show that indeed the class of existentially closed differential expansions is elementary and they provide an axiomatization. A derivation δ is *compatible* with $\mathcal{K}$ if for any 0-definable C^1-function $f\colon U \to K$, where U is an open subset of some cartesian product K^n, we have $\delta f(\bar{u}) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\bar{u})\delta(u_i)$, for any $\bar{u} \in U$. In particular in case $\mathcal{K}$ expands an exponential field, such derivation δ is an E-derivation. Their results apply to o-minimal fields $\mathcal{K}$ extending the field of real numbers $\mathbb{R}$ and admitting an expansion to all restricted analytic functions. In order to show that there is a compatible derivation, they have at their disposal the quantifier elimination result of J. Denef and L. van den Dries on the expansion $\mathbb{R}_{an}$ of $\mathbb{R}$ with all these functions (with restricted division) and its extension by L. van den Dries, A. Macintyre and D. Marker for $\mathbb{R}_{an,exp}$, where exp is the exponential function given by the classical power series [9].

So when δ is a compatible derivation, in case of $(\mathbb{R}, exp)$, by uniqueness of the model-completion, one gets, following either approaches, the same class of existentially closed exponential differential fields. However, it is unclear in an ordered exponential field model of the theory of $(\mathbb{R}, exp)$ whether any E-derivation is compatible. (We cannot apply the argument used by A. Fornasiero and E. Kaplan since we don't have quantifier-elimination in the language of ordered fields together with the exponential function.)

The plan of the paper is as follows.

In section 2, we review the notion of partial exponential fields and of the corresponding closure operator, denoted by ecl-closure (Definition 2.11). It was introduced by A. Macintyre using the work of A. Khovanskii [20], then it plays a crucial role in the proof of A. Wilkie of the model-completeness of $(\mathbb{R}, exp)$. Later in a purely algebraic context, J. Kirby linked the ecl-closure with the cl-closure, defined through E-derivations (Definition 2.8). He showed that the two closure operators coincide using a result of J. Ax on the Schanuel property in differential fields of characteristic 0. We slightly adapt J. Kirby's results on extensions of E-derivations in order to be able to use them in the case of p-adically closed fields, where the exponential function is only defined on the valuation ring.

Then in section 3, we recall the notion of E-varieties, generic points and torsors. We also recall the setting of topological fields [24]. We define the class of exponential fields we will be able to deal with, namely those satisfying the implicit function theorem (see Definition 3.13) and in the ordered case the lack of flat functions (see Definition **??**). Note that both properties hold in o-minimal expansions of real-closed fields (or more generally in definably complete ordered fields). A version of these properties also holds in the classes of valued fields mentioned above as shown by N. Mariaule (see section 3.5).

In section 4, we finally introduce a scheme of axioms $(\mathrm{DL})_E$ that will axiomatize a class of existentially closed differential exponential fields and show our main result. This scheme of axioms can be compared to the axiomatization of M. Singer of the closed ordered differential fields, denoted by CODF. We also give a geometric interpretation of the scheme $(\mathrm{DL})_E$, which is a priori not first-order.

In section 5, we give examples of topological fields to which we may apply our results.

Finally, in the last section, we show how to endow a topological exponential field of cardinality $\aleph_1$ which is first-countable and separable with an E-derivation which satisfies this scheme of axioms. When the topology is induced by an ordering we point out that such ordered field can also be made a model of CODF. This kind of construction (for CODF) may be found in the work of M. Singer, and the theses of C. Michaux and Q. Brouette.

Acknowledgments: Part of these results appeared in the PhD thesis of Nathalie Regnault [28].

2. E-derivations

{prelim}

2.1. **Preliminaries.** We will only consider commutative rings R of characteristic 0 with $1 \neq 0$. Let $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, $R^* := R \setminus \{0\}$. Denote by $I(R)$ the subgroup of the invertible elements of $(R^*, \cdot, 1)$. Given an ordered set $(I, <)$, denote $I_{\geq j} := \{i \in I : i \geq j\}$ (respectively $(I_{>j}) := \{i \in I : i > j\}$).

Let $\mathcal{L}_{rings} := \{+, \cdot, -, 0, 1\}$ be the language of rings; we will work in different expansions $\mathcal{L}$ of $\mathcal{L}_{rings}$ such as $\mathcal{L}_E := \mathcal{L}_{rings} \cup \{E\}$ and $\mathcal{L}_{E,\delta} := \mathcal{L}_{rings} \cup \{E, \delta\}$ where E, δ are unary functions. The $\mathcal{L}$-formulas will be possibly with parameters and when we want to specify them we will use $\mathcal{L}(B)$ with B a set of constants. Similarly $\mathcal{L}$-definable sets will possibly be definable with parameters. Our notation for tuples will be flexible: x (respectively a) will denote a tuple of variables (respectively a tuple of elements) but sometimes in order to stress that we deal with tuples we will use $\bar{x}$, respectively $\bar{a}$, or bold letters e.g. $\mathbf{x}$, $\mathbf{a}$. In this section we will not make the distinction between an $\mathcal{L}$-structure $\mathcal{M}$ and its domain M whereas from subsection 3.4 on, we will distinguish them.

Definition 2.1. [6] An E-ring R is a ring equipped with a morphism E from the additive group $(R, +, 0)$ to the multiplicative group $I(R)$ satisfying $E(0) = 1$ and $\forall x \forall y\ (E(x + y) = E(x) \cdot E(y))$. (So an E-ring can be endowed with an $\mathcal{L}_E$-structure.) An E-field is a field which is an E-ring.

We will also consider partial E-fields, and so the corresponding language contains a unary predicate for the domain of the exponential function. We will first define partial E-domains.

Definition 2.2. Let F be an integral domain, namely a commutative ring with no non-zero zero-divisors. A partial E-domain is a two-sorted structure

$$((F, +_F, \cdot_F, 0_F, 1_F), (A, +_A, 0_A), E),$$

where $(A, +_A, 0_A)$ is a group and $E : (A, +_A, 0_A) \to I(F)$ is a group morphism. We identify $(A, +_A, 0_A)$ with an additive subgroup of $(F, +_F, 0_F)$ and to stress it, we will denote it by $A(F)$. When the domain of E is clear from the context, we will also simply use the notation (F, E), even though E is only partially defined.

A partial E-field F is a partial E-domain which is a field. A partial E-subfield F_0 is a partial E-field which is a two-sorted substructure. We denote by $F_0(\bar{a})_E$, where $\bar{a} \subseteq F$, the smallest partial E-subfield of F containing F_0 and $\bar{a}$ and by $F_0\langle\bar{a}\rangle_E$ the smallest partial E-subring generated by F_0 and $\bar{a}$. When $F_0 = \mathbb{Q}$, we denote $\mathbb{Q}\langle\bar{a}\rangle_E$ simply by $\langle\bar{a}\rangle_E$. To make the distinction with the $\mathcal{L}_{rings}$-substructure, we denote by $\mathbb{Q}[\bar{a}]$ the subring generated by $\bar{a}$.

Note that in [18, Definition 2.2], one uses a stronger notion of partial E-fields, namely one requires that $A(F)$ is a $\mathbb{Q}$-vector space, namely one endows $A(F)$ with scalar multiplications $(\cdot q)_{q\in\mathbb{Q}}$. Instead here, given two partial E-fields $F_0 \subseteq F$, we replace that by the condition that $A(F_0)$ is a pure subgroup of $A(F)$.

{pure}

Notation 2.3. Let F_0, F be two partial E-fields with F_0 a substructure of F. Then the subgroup $A(F_0)$ is pure in $A(F)$ iff for any $a \in A(F)$ and $n \in \mathbb{N}^*$, if $na \in A(F_0)$, then $a \in A(F_0)$. We use the notation $A(F_0) \subseteq_1 A(F)$.

In addition, when the field F is endowed with a field topology and when $\lim_{n\to\infty} \sum_{i\geq 0}^{n} \frac{x^i}{i!}$ exists, we can consider the (partial) function $x \mapsto exp(x) := \lim_{n\to\infty} \sum_{i\geq 0}^{n} \frac{x^i}{i!}$. Then the domain of $exp(x)$ is a subgroup and a $\mathbb{Q}$-vector space whenever F is closed under roots.

{example}

Examples 2.1.

(1) Let F be a partial E-field and consider the field of Laurent series $F((t))$ (or more generally a Hahn field (see below)). Then, regardless of whether we put a topology on $F((t))$, we can always define $exp(x) := \sum_{i\geq 0} \frac{x^i}{i!}$ for $x \in tF[[t]]$. Indeed, by Neumann's Lemma, the element $exp(x) \in F[[t]]$ [10, chapter 8, section 5, Lemma]. Then, we extend E on $A(F) \oplus tF[[t]]$ as follows. Write $r \in A(F) \oplus tF[[t]]$ as $r_0 + r_1$ where $r_0 \in A(F)$ and $r_1 \in t.F[[t]]$. Define E on $A(F) \oplus tF[[t]]$ as follows: $E(r_0 + r_1) := E(r_0)exp(r_1)$. So $F((t))$ can be endowed with a structure of a partial E-field with $A(F((t))) := A(F) \oplus tF[[t]]$.

(2) More generally, under the same assumption on F, let $(G, +, -, 0, <)$ be an abelian totally ordered group, then the Hahn field $F((G))$, can be endowed with a structure of a partial E-field defining E on the elements $r \in A(F) \oplus F((G_{>0}))$ similarly, where $G_{>0} := \{g \in G : \ g > 0\}$ (respectively $G_{\geq 0} := \{g \in G : \ g \geq 0\}$). Namely decompose r as $r_0 + r_1$ with $r_0 \in A(F)$ and $r_1 \in F((G_{>0}))$. Then $exp(r_1) \in F((G_{\geq 0}))$ again by Neumann's Lemma and define $E(r) := E(r_0)exp(r_1)$. So $A(F((G))) = A(F) \oplus F((G_{>0}))$.

(3) Let $\bar{\mathbb{R}} := (\mathbb{R}, +, -, \cdot, 0, 1, E)$ where $E(x) = exp(x)$ defined above.

(4) Let $\bar{\mathbb{C}} := (\mathbb{C}, +, -, \cdot, 0, 1, E)$ where $E(x) = exp(x)$.

(5) Let p be a prime number; when $p = 2$ set $E_p(x) := exp(p^2x)$ and when $p > 2$, set $E_p(x) = exp(px)$. Let $\mathbb{C}_p$ be the completion of the algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$ (in $\mathbb{C}$). As examples of partial E-fields, we have the field of p-adic numbers $\bar{\mathbb{Q}}_p := (\mathbb{Q}_p, +, -, \cdot, 0, 1, E_p)$ or $\bar{\mathbb{C}}_p := (\mathbb{C}_p, +, -, \cdot, 0, 1, E_p)$. In these two cases, E_p is defined on the valuation ring $\mathbb{Z}_p$ of $\mathbb{Q}_p$ (respectively on the valuation ring $\mathcal{O}_p$ of $\mathbb{C}_p$).

We will investigate these examples further in section 5.

{E-der}

Definition 2.4. Let R be a (partial) E-ring. An E-derivation δ is a unary function on R satisfying:

(1) $\delta(a + b) = \delta(a) + \delta(b)$,
(2) the Leibnitz rule: $\delta(ab) = \delta(a)b + a\delta(b)$,
(3) $\forall a \in A\ (\delta(E(a)) = \delta(a)E(a))$.

We will denote the differential expansion of R by R_δ.

For example, let F_δ be a differential E-field (δ can be the trivial derivation). We have already seen how to extend E on $F[[t]]$. Then we extend δ on the field of Laurent series $F((t))$ by setting $\delta(t) = 1$ and by requiring it to be strongly additive. Then δ is again an E-derivation on $F((t))$. Indeed, for $x \in tF[[t]]$, we have $\delta(exp(x)) = \sum_{i\geq 0} \delta(\frac{x^i}{i!}) = \delta(x)exp(x)$ and for $x \in F[[t]]$ with $x = r_0 + r_1$ where $r_0 \in A(F)$ and $r_1 \in tF[[t]]$, we have $\delta(E(r_0 + r_1)) = E(r_0)exp(r_1)\delta(r_1) + \delta(r_0)E(r_0)exp(r_1) = \delta(x)E(x)$. This makes $(F((t)), F[[t]], exp, \delta)$ a differential (partial) E-field.

Notation 2.5. Let δ be an E-derivation on R. For $m \geqslant 0$ and $a \in R$, we define

$$\delta^m(a) := \underbrace{\delta \circ \ldots \circ \delta}_{m \text{ times}}(a), \text{ with } \delta^0(a) := a,$$

and $\bar{\delta}^m(a)$ as the finite sequence $(\delta^0(a), \delta(a), \ldots, \delta^m(a)) \in R^{m+1}$.

Similarly, given an element $\mathbf{a} = (a_1, \ldots, a_n) \in R^n$, we write

$$\bar{\delta}^m(\mathbf{a}) := (a_1, \ldots, a_n, \ldots, \delta^m(a_1), \ldots, \delta^m(a_n)) \in R^{(m+1)n}.$$

Denote by $\mathbb{Q}\langle \mathbf{a} \rangle_{E,\delta}$ the E-differential subring of R generated by $\mathbf{a}$ and $\mathbb{Q}$.

In section 2.3, we will consider in general the problem of extending E-derivations but first it is convenient to recall the notion of E-polynomials and differential E-polynomials.

{free}

2.2. **Free exponential rings.** The construction of free E-rings $\mathbb{Z}[\mathbf{X}]^E$ on finitely many variables $\mathbf{X} := (X_1, \ldots, X_n)$ (and more generally free E-rings $R[\mathbf{X}]^E$ over (R, E)) can be found in many places in the literature. It is initially due to B. Dahn. The elements of these rings are called *E-polynomials* in the indeterminates $\mathbf{X}$. Here we will briefly recall their construction, following [6] and [21]. When $n = 1$, we will use the variable X and since we will also use differential E-polynomials, we will also allow $\mathbf{X}$ to denote a tuple of countably many variables.

Let R be an E-ring. Then the ring $R[\mathbf{X}]^E$ is constructed by stages as follows: let $R_{-1} := R$, $R_0 := R[\mathbf{X}]$ and A_0 the ideal generated by $\mathbf{X}$ in $R[\mathbf{X}]$. Then $R_0 = R \oplus A_0$. Let $E_{-1} = E$ on R composed by the embedding of R_{-1} into R_0.

For $k \geq 0$, set $R_k = R_{k-1} \oplus A_k$ and let t^{A_k} be a multiplicative copy of the additive group A_k.

For instance for $k = 1$, we get $R_1 = R_0[t^{A_0}]$ and A_1 is a direct summand of R_0 in R_1.

Then, put $R_{k+1} := R_k[t^{A_k}]$ and let A_{k+1} be the free R_k-submodule generated by t^a with $a \in A_k - \{0\}$. We have $R_{k+1} = R_k \oplus A_{k+1}$.

By induction on $k \geq 0$, one shows the following isomorphism: $R_{k+1} \cong R_0[t^{A_0 \oplus \cdots \oplus A_k}]$, using the fact that $R_0[t^{A_0 \oplus \cdots \oplus A_k}] \cong R_0[t^{A_0 \oplus \cdots \oplus A_{k-1}}][t^{A_k}]$ [21, Lemma 2].

We define the map $E_k : R_k \to R_{k+1}$, $k \geq 0$, as follows: $E_k(r' + a) = E_{k-1}(r')t^a$, where $r' \in R_{k-1}$ and $a \in A_k$.

Finally let $R[\mathbf{X}]^E := \bigcup_{k\geq 0} R_k$ and extend E on $R[\mathbf{X}]^E$ by setting $E(f) := E_k(f)$ for $f \in R_k$. It is easy to check that it is well-defined. Let $f \in R_{k+1}$, then $f = f_k + g$ where $f_k \in R_k$ and $g \in A_{k+1}$. So $E(f) = E(f_k)t^g$. By definition $E(f_k) = E_k(f_k)$ and so if $f_k = f_{k-1} + g_k$ with $f_{k-1} \in R_{k-1}$ and $g_k \in A_k$, we have $E(f_{k-1}) = E_{k-1}(f_{k-1})t^{g_k}$. Unravelling f in this way, we get that $E(f) = E(f_0)t^{g+g_k+\ldots+g_0}$ with $f = f_0 + g_0 + \cdots + g_k + g$, $f_0 \in R$, $g_0 \in A_0, \ldots, g_k \in A_k, g \in A_{k+1}$.

Finally note that the above construction can be extended when R is a partial E-domain, the only change is that we only define $E(f)$ for f as written above when $f_0 \in A(R)$.

Using the construction of $R[\mathbf{X}]^E$ as an increasing union of group rings, one can define on the elements of $R[\mathbf{X}]^E$ an analogue of the degree function for ordinary polynomials which measures the complexity of the elements; it takes its values in the class On of ordinals and was described for instance in [6, 1.9] for exponential polynomials in one variable. Here we deal with exponential polynomials in more than one variable and so we follow [20, section 1.8].

Let us denote by $totdeg_{\mathbf{X}}(p)$ the total degree of p, namely the maximum of $\{\sum_{j=1}^m i_j:$ for each monomial $X_1^{i_1}\cdots X_m^{i_m}$ occurring (nontrivially) in p with $i_1,\ldots,i_m \in \mathcal{N},\ m\in\mathbb{N}_{\geq 1}\}$.

Then one defines a height function h (with values in $\mathbb{N}$) which detects at which stage of the construction the (non-zero) element is introduced.

Let $p(\mathbf{X}) \in R[\mathbf{X}]^E$, then $h(p(\mathbf{X})) = k$, if $p \in R_k \setminus R_{k-1},\ k > 0$ and $h(p(\mathbf{X})) = 0$ if $p \in R[\mathbf{X}]$.

Using the freeness of the construction, one defines a function rk

$$rk : R[\mathbf{X}]^E \to \mathbb{N} :$$

If $p = 0$, set $rk(p) := 0$,
if $p \in R[\mathbf{X}] \setminus \{0\}$, set $rk(p) := totdeg_{\mathbf{X}}(p) + 1$ and
if $p \in R_k$, $k > 0$, let $p = \sum_{i=1}^d r_i.E(a_i)$, where $r_i \in R_{k-1}$, $a_i \in A_{k-1}\setminus\{0\}$. Set $rk(p) := d$.
Finally, one defines the complexity function ord

$$ord : R[\mathbf{X}]^E \to On$$

as follows. Write $p \in R_k$ as $p = p_0 + p_1 + \cdots + p_k$ with $p_0 \in R_0$, $p_i \in A_i$, $1 \leq i \leq k$. Define $ord(p) := \sum_{i=0}^k \omega^i.rk(p_i)$.

Note that if $p_0 = 0$, then there is $q \in R[\mathbf{X}]^E$ such that $ord(E(q).p) < ord(p)$ (the proof is exactly the same as the one in [6, Lemma 1.10]).

On $R[\mathbf{X}]^E$, we define n E-derivations ∂_{X_i} as follows: $\partial_{X_i} \restriction R = 0$ and $\partial_{X_i} X_j = \delta_{ij}$, where δ_{ij} is the Kronecker symbol, $1 \leq i,j \leq n$.

{dfninitial}

Notation 2.6. Assume that δ is an E-derivation on R. Let $\mathbf{X} := (X_1,\ldots,X_n)$, denote by $R\{\mathbf{X}\}^E$ the ring of differential E-polynomials over R in n differential indeterminates $X_1,\cdots,X_n$, namely it is the E-polynomial ring in indeterminates $\delta^j(X_i)$, $1 \leq i \leq n$, $j \in \omega$, with by convention $\delta^0(X_i) := X_i$. Let $p(\mathbf{X}) \in R\{\mathbf{X}\}^E$. Let $m \in \mathbb{N}$ be the (differential) order of p (denoted by δ-$ord(p)$) as classically defined in differential algebra [19, page 75] (if $m = 0$, then p is an ordinary E-polynomial). In particular we have that p can be written as $p^*(\bar{\delta}^m(\mathbf{X}))$ with $\bar{\delta}^m(\mathbf{X}) = (X_1,\ldots,X_n,\delta(X_1),\ldots,\delta(X_n),\ldots,\delta^m(X_1),\ldots,\delta^m(X_n))$ and p^* an ordinary E-polynomial.

{der-multi}

Lemma 2.7. *Let δ be an E-derivation on R. Let $p \in R[\mathbf{X}]^E$. Then there exists $p^\delta \in R[\mathbf{X}]^E$ such that in the ring $R\{\mathbf{X}\}^E$, $\delta(p(\mathbf{X})) = \sum_{j=1}^n \delta(X_j)\partial_{X_j} p + p^\delta$. Moreover there is a tuple $\bar{e}$ of elements of R such that $p \in \langle \bar{e}\rangle_E[\mathbf{X}]^E$ and $p^\delta \in \mathbb{Q}(\bar{e}, \langle\delta(\bar{e})\rangle_E)[\mathbf{X}]^E$. Furthermore whenever δ is trivial on R, $p^\delta = 0$.*

Proof: Decompose p as: $p = p_0 + \sum_{i=1}^k p_i$, with $p_0 \in R[\mathbf{X}]$ and $p_i \in A_i$, $i > 0$. We proceed by induction on $ord(p)$, namely we assume that for all $q \in R[\mathbf{X}]^E$ with $ord(q) < ord(p)$, we have $\delta(q(\mathbf{X})) = \sum_{j=1}^n \delta(X_j)\partial_{X_j} q + q^\delta$ with q^δ satisfying the conditions of the statement of the lemma.

If $ord(p) \in \omega$, namely $p \in R[\mathbf{X}]$, the statement of the lemma is well-known. Write $p(\mathbf{X}) = \sum a_{i_1,\cdots,i_n} X_1^{i_1}\cdots X_n^{i_n}$, define $p^\delta := \sum \delta(a_{i_1,\cdots,i_n}) X_1^{i_1}\cdots X_n^{i_n}$. Then $\delta(p(\mathbf{X})) =$

$\sum_{j=1}^{n} \delta(X_j)\partial_{X_j} p + p^\delta$. Note that $p^\delta \in \delta(R)[\mathbf{X}]$ and $ord(p^\delta) \leq ord(p)$. If p is monic and $n = 1$, then $ord(p^\delta) < ord(p)$.

Now assume that $ord(p) \geq \omega$ and that the induction hypothesis holds.

Let $k > 0$ and $p \in R_k \setminus R_{k-1}$. By additivity of the derivation, the way ord has been defined and the induction hypothesis, it suffices to prove it for $p \in A_k$. So, write $p = \sum_{i=1}^{d} r_i E(a_i)$ with $r_i \in R_{k-1}$ and $a_i \in A_{k-1} \setminus \{0\}$; so $ord(p) = \omega^k d$. We have that $\delta(p) = \sum_{i=1}^{d}(\delta(r_i) + r_i\delta(a_i))E(a_i)$.

By induction hypothesis, $\delta(r_i) = \sum_{j=1}^{n} \delta(X_j)\partial_{X_j} r_i + {r_i}^\delta$ and $\delta(a_i) = \sum_{j=1}^{n} \delta(X_j)\partial_{X_j} a_i + {a_i}^\delta$. So we get that $\delta(p) = \sum_{j=1}^{n} \delta(X_j)\partial_{X_j}(\sum_{i=1}^{d} E(a_i)r_i) + \sum_{i=1}^{d} E(a_i)({r_i}^\delta + r_i{a_i}^\delta)$. Put $p^\delta := \sum_{i=1}^{d} E(a_i)({r_i}^\delta + r_i{a_i}^\delta)$ (†).

Let $\mathbf{e}_i$, $\mathbf{c}_i$ be tuples of elements of R such that $r_i \in \langle \mathbf{e}_i \rangle_E[\mathbf{X}]^E$, $a_i \in \langle \mathbf{c}_i \rangle_E[\mathbf{X}]^E$. Then by induction hypothesis, $r_i^\delta \in \mathbb{Q}(\langle \mathbf{e}_i \rangle_E, \delta(\mathbf{e}_i))[\mathbf{X}]^E$, $a_i^\delta \in \mathbb{Q}(\langle \mathbf{c}_i \rangle_E, \delta(\mathbf{c}_i)[\mathbf{X}]^E$. Let $\bar{e} := (\mathbf{e}_1, \dots, \mathbf{e}_d)$ and $\bar{c} := (\mathbf{c}_1, \dots, \mathbf{c}_d)$. We have that $p \in \langle \bar{e}, \bar{c} \rangle_E[\mathbf{X}]^E$ and by (†), $p^\delta \in \mathbb{Q}(\langle \bar{e}, \bar{c} \rangle_E, \delta(\bar{e}), \delta(\bar{c}))[\mathbf{X}]^E$ and if δ is trivial on R, then $p^\delta = 0$.

□

{Khov}

2.3. **Khovanskii systems.** Let F_δ be an expansion of a partial E-field by an E-derivation δ (see Definition 2.4). Note that in [6], the condition of being an E-derivation was relaxed to: $\delta(E(x)) = r\delta(x)E(x)$, for some $r \in R^*$. However if δ is an E-derivation, then $r\delta$ is also an E-derivation, with $r \in R$. More generally, the set of E-derivations on R forms a R-module. Using E-derivations, J. Kirby defined a closure operator cl in E-rings and he showed that cl induces a pregeometry on subsets of R [18, Lemma 4.4, Proposition 4.5].

Definition 2.8. [18, Definition 4.3] Let R be a partial E-ring and let A be a subset of R. Then, {cl}

$$\mathrm{cl}^R(A) := \{u \in R \colon \delta(u) = 0 \text{ for any } E - \text{derivation } \delta \text{ vanishing on } A\}.$$

If $A \subseteq R$, then $\mathrm{cl}^R(A)$ is an E-subring and if R is field, it is an E-subfield.

Note that in the algebraic case, when an element a is algebraic over a subfield endowed with a trivial derivation δ, then $\delta(a) = 0$ as well. Later, we will see an analog of this property in the case of E-derivation working with a notion of E-algebraicity (see Lemma 2.15).

Notation 2.9. Let R be an E-ring. In section 2.2, we recalled the construction of the ring of E-polynomials in $\mathbf{X} := (X_1, \dots, X_n)$ over R. These E-polynomials induce functions from R^n to R and we will denote the corresponding ring of functions by $R[\mathbf{x}]$, where $\mathbf{x} := (x_1, \dots, x_n)$ [6].

Note that when R is a partial E-domain, we get the same ring of E-polynomials but with an E-polynomial we can only associate a partially defined function on R (since E is only defined on $A(R)$).

In [6, section 4], one can find a necessary condition on R under which the map sending an E-polynomial $p(\mathbf{X})$ to the corresponding function $p(\mathbf{x})$ is injective. The condition is as follows: there exist n E-derivations ∂_i on $R[\mathbf{x}]^E$, which are trivial on R and satisfy $\partial_i(x_j) = \delta_{ij}$ [6, Proposition 4.1]. Let $f \in R[\mathbf{x}]^E$, we denote by $\partial_i f$, the function corresponding to the differential E-polynomial $\partial_{X_i} f$.

{Jac}

Notation 2.10. Given $f_1,, f_n \in R[\mathbf{X}]^E$, $\bar{f} := (f_1, \ldots, f_n)$, we will denote by $J_{\bar{f}}(\mathbf{X})$, the Jacobian matrix: $\begin{pmatrix} \partial_{X_1} f_1 & \cdots & \partial_{X_n} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{X_1} f_n & \cdots & \partial_{X_n} f_n \end{pmatrix}$.

As usual, we denote by $det(J_{\bar{f}}(\mathbf{X}))$ the determinant of the matrix $J_{\bar{f}}(\mathbf{X})$; note that it is an E-polynomial. When we evaluate either $J_{\bar{f}}(\mathbf{X})$ or its determinant at an n-tuple $\mathbf{b} \in R^n$, we denote the corresponding values by $J_{\bar{f}}(\mathbf{b})$, respectively $det(J_{\bar{f}}(\mathbf{b}))$.

{Kho}

Definition 2.11. [18, Definition 3.1] Let $B \subseteq R$ be partial E-domains. We will adopt the following convention. A *Khovanskii system over* B is a quantifier-free $\mathcal{L}_E(B)$-formula in free variables $\mathbf{x} := (x_1, \ldots, x_n)$ of the form

$$H_{\bar{f}}(\mathbf{x}) := \bigwedge_{i=1}^{n} f_i(\mathbf{x}) = 0 \ \wedge \ det(J_{\bar{f}}(\mathbf{x})) \neq 0,$$

for some $f_1,, f_n \in B[\mathbf{X}]^E$. (We will sometimes omit the subscript $\bar{f}$ in the above formula and possibly make explicit the coefficients $\bar{c} \in B$ of the E-polynomials $\bar{f}$ in which case, we will use $H_{\bar{c}}(\mathbf{x})$.)

Let $a \in R$. Then $a \in \mathrm{ecl}^R(B)$ if

$$\text{for some Khovanskii system } H_{\bar{f}} \text{ and some } a_2, \ldots, a_n \in R, H_{\bar{f}}(a, a_2, \ldots, a_n) \ \text{ holds}$$

with $f_1, \ldots, f_n \in B[\mathbf{X}]^E$ (assuming that $a_i \in A(R)$, $1 \leq i \leq n$, if needed for the f_i's to be defined).

The operator ecl is a well-behaved E-algebraic closure operator, satisfying the exchange property [20], [17], [18, Lemma 3.3, Theorem 1.1]. A. Wilkie used it in his proof of the model-completeness of the theory of $(\bar{\mathbb{R}}, exp)$, where $\bar{\mathbb{R}}$ denotes the ordered field of real numbers. Then J. Kirby extracted ecl from this o-minimal setting and showed that it coincides with the closure operator cl defined above (see Definition 2.8) [18, Propositions 4.7, 7.1]. Since the operator cl^F on subsets of an E-field F induces a pregeometry, we get a notion of dimension $\dim^F$ as follows:

{dim}

Definition 2.12. Let F be a partial E-field, let $\mathbf{x} := (x_1, \ldots, x_n)$ and let $C \subseteq A(F)$ with $C = \mathrm{cl}^F(C)$, then for $m \leq n$,

$\dim^F(\mathbf{x}/C) = m$ if there exist $x_{i_1}, \ldots, x_{i_m}$ with $1 \leq i_1 < \ldots < i_m \leq n$ such that

$$x_{i_j} \notin \mathrm{cl}(x_{i_\ell}, C; 1 \leq \ell \neq j \leq m) \text{ and } x_i \in \mathrm{cl}^F(x_{i_1}, \ldots, x_{i_m}, C), 1 \leq i \leq n.$$

In order to show that cl $\subseteq$ ecl, J. Kirby uses a result of J. Ax on the Schanuel property in differential fields of characteristic 0 [18, Theorem 5.1], in order to show the following inequality:

$$\mathrm{td}(\mathbf{x}, E(\mathbf{x})/C) - \ell\mathrm{dim}_{\mathbb{Q}}(\mathbf{x}/C) \geq \dim(\mathbf{x}/C), \quad (\dagger)$$

where $\mathrm{td}(\mathbf{x}, E(\mathbf{x})/C)$ denotes the transcendence degree of the field extension $\mathbb{Q}(\mathbf{x}, E(\mathbf{x}), C)$ of $\mathbb{Q}(C)$ and $\ell\mathrm{dim}_{\mathbb{Q}}(\mathbf{x}/C)$ the dimension of the quotient $\langle \mathbf{x}, C\rangle_{\mathbb{Q}}/\langle C\rangle_{\mathbb{Q}}$ of the $\mathbb{Q}$-vector spaces: $\langle \mathbf{x}, C\rangle_{\mathbb{Q}}$ generated by $\mathbf{x}$ and C by $\langle C\rangle_{\mathbb{Q}}$ generated by C. (When $C = \emptyset$, $\ell\mathrm{dim}_{\mathbb{Q}}(\mathbf{x}/C)$ is simply the linear dimension of the $\mathbb{Q}$-vector-space generated by $\mathbf{x}$.

From now on we will also denote by $\dim^F(\cdot/C)$ the dimension induced by the closure operator $\mathrm{ecl}^F(\cdot/C)$ and by ℓdim the linear dimension of a vector-space. As usual we define the dimension of a subset as the maximum of the dimension of finite tuples contained in that subset (see Definition 3.3).

Definition 2.13. [18, Definition 5.3] Let F be a partial E-field and F_0 be a partial E-subfield of F. For any $C \subseteq A(F)$, let

$$d(\mathbf{x}/C) := \mathrm{td}(\mathbf{x}, E(\mathbf{x})/C, E(C)) - \ell\dim_{\mathbb{Q}}(\mathbf{x}/C).$$

Then $F_0 \lhd F$ if for every tuple $\mathbf{x}$ in $A(F)$, $d(\mathbf{x}/F_0) \geq 0$.

Let $\mathcal{M}_0 \subseteq \mathcal{M}_1$ be two $\mathcal{L}$-structures. Recall that the notation $\mathcal{M}_0 \subseteq_{ec} \mathcal{M}_1$ means that any existential formula with parameters in M_0 satisfied in $\mathcal{M}_1$ is also satisfied in $\mathcal{M}_0$. Let us note some straightforward properties of the ecl^F relation (and how it depends on F).

Remark 2.14. Let $F_0 \subseteq F_1$ be two partial E-fields. Suppose that $F_0 \subseteq_{ec} F_1$, then

(1) $A(F_0) \subseteq_1 A(F)$ (see Notation 2.3),
(2) $\mathrm{ecl}^{F_1}(F_0) = F_0$, provided the number of solutions to a Khovanskii system in F_1 is finite, and
(3) let $\varphi(x_1, \ldots, x_k, \bar{y})$ be an existential formula, let $\mathbf{a} \in F_0$, then if $\dim^{F_0}(\varphi(F_0, \mathbf{a})/\langle \mathbf{a}\rangle_E) \geq k$, then $\dim^{F_1}(\varphi(F_1, \mathbf{a})/\langle \mathbf{a}\rangle)_E) \geq k$.

Proof: All these properties are rather straightforward. For convenience of the reader, we will indicate a proof for (2) and (3).

(2) Suppose that $u \in \mathrm{ecl}^{F_1}(F_0)$. So we can find $u_1, \ldots, u_n \in F_1$ and $n+1$ E-polynomials $f_1, \ldots, f_{n+1}$ with coefficients in F_0 such that $H_{\bar{f}}(u, u_1, \ldots, u_n)$ holds. Suppose the number of $n+1$-tuples solution of the Khovanskii system $H_{\bar{f}}$ is equal to ℓ. Then we can express by an existential formula with parameters in F_0 that it has at least ℓ solutions (in F_1). Since $F_0 \subseteq_1 F_1$, it holds in F_0, so the tuple $(u, u_1, \ldots, u_n)$ should appear among the solutions, otherwise we would get one more solution in F_1, a contradiction.

(3) Let $b_1, \ldots, b_k \in F_0$, $k > 1$, be ecl^{F_0}-independent over $\langle \mathbf{a}\rangle_E$ and be such that $\varphi(b_1, \ldots, b_k, \mathbf{a})$ holds. Let us show that $b_1, \ldots, b_k$ remain ecl^{F_1}-independent over $\langle \mathbf{a}\rangle_E$. We proceed by contradiction assuming for instance that $b_k \in \mathrm{ecl}^{F_1}(b_1, \ldots, b_{k-1}, \langle \mathbf{a}\rangle_E)$. So there are E-polynomials $f_1, \ldots, f_\ell$ with coefficients in $\langle b_1, \ldots, b_{k-1}, \mathbf{a}\rangle_E$ and $u_2, \ldots, u_\ell \in F_1$ such that $H_{\bar{f}}(b_k, u_2, \ldots, u_\ell)$ holds. Since $F_0 \subseteq_{ec} F_1$, we can find $u'_2, \ldots, u'_\ell \in F_0$ such that $H_{\bar{f}}(b_k, u'_2, \ldots, u'_\ell)$ holds, witnessing that $b_k \in \mathrm{ecl}^{F_0}(b_1, \ldots, b_{k-1}, \langle \mathbf{a}\rangle_E)$, a contradiction. □

Recall that A. Khovanskii showed that in the field of real numbers expanded with Pfaffian chain of function, the number of solutions of a Khovanskii system is not only finite but it is bounded independently of the coefficients of the system [33, Proposition 3.1].

{der}

Lemma 2.15. *Let $F_0 \subseteq F_1$, where F_1 is a partial E-field and F_0 is a partial E-domain endowed with an E-derivation δ. Then, given $u \in \mathrm{ecl}^{F_1}(F_0)$, we can extend δ to an E-derivation on u in a unique way.*

Proof: Let $u \in \mathrm{ecl}^{F_1}(F_0)$, so for some n, there exist $u_1 = u, u_2, \ldots, u_n \in F_1$ such that $H(u_1, \ldots, u_n)$ holds in F_1, for some Khovanskii system over F_0. Set $\mathbf{u} := (u_1, \ldots, u_n)$ and $\mathbf{X} := (X_1, \ldots, X_n)$. Let $f_1, \ldots, f_n \in F_0[\mathbf{X}]^E$ be such that

$$\bigwedge_{i=1}^{n} f_i(\mathbf{u}) = 0 \ \wedge \ det(J_{\bar{f}}(\mathbf{u})) \neq 0. \tag{1}$$

Applying δ to $f_1(\mathbf{u}), \ldots, f_n(\mathbf{u})$, and using the E-polynomials $f_1^\delta, \ldots, f_n^\delta$ obtained in Lemma 2.7, we get

$$\begin{pmatrix} f_1^\delta(\mathbf{u}) \\ \vdots \\ f_n^\delta(\mathbf{u}) \end{pmatrix} + J_{\bar{f}}(\mathbf{u}) \begin{pmatrix} \delta(u_1) \\ \vdots \\ \delta(u_n) \end{pmatrix} = \begin{pmatrix} \delta(f_1(\mathbf{u})) \\ \vdots \\ \delta(f_n(\mathbf{u})) \end{pmatrix} = 0 \tag{2}$$

So,

$$\begin{pmatrix} \delta(u_1) \\ \vdots \\ \delta(u_n) \end{pmatrix} = -J_{\bar{f}}(\mathbf{u})^{-1} \cdot \begin{pmatrix} f_1^\delta(\mathbf{u}) \\ \vdots \\ f_n^\delta(\mathbf{u}) \end{pmatrix} \tag{3}$$

Note that $J_{\bar{f}}(\mathbf{X})^{-1} = J_{\bar{f}}^*(\mathbf{X})(det(J_{\bar{f}}(\mathbf{X}))^{-1}$, so $J_{\bar{f}}(\mathbf{X})^{-1}$ is a matrix whose entries are rational E-functions with denominator $det(J_{\bar{f}}(\mathbf{X}))$.

Since ecl has finite character, we may assume that $f_i \in \langle \mathbb{Q}(\mathbf{e}_i)\rangle_E[\mathbf{X}]^E$ for some tuple $\mathbf{e}_i$ and $f_i^\delta \in \mathbb{Q}(\langle \mathbf{e}_i\rangle_E, \delta(\mathbf{e}_i))[\mathbf{X}]^E$ (see Lemma 2.7). Let $\bar{e} := (\mathbf{e}_1, \ldots, \mathbf{e}_n)$; so we can express each $\delta(u_i)$, $1 \le i \le n$, as an E-rational function $t_{i,\bar{f}}(\mathbf{u})$ with coefficients in $\langle \mathbb{Q}(\bar{e}, \delta(\bar{e}))\rangle_E$. Then we extend δ to the E-subfield generated by $F_0, u_1, \ldots, u_n$. Since ecl = cl there is only one such E-derivation extending δ on F_0.

We can also express the successive derivatives $\delta^\ell(u_i)$, $1 \le i \le n$, $\ell \in \mathbb{N}$, $\ell \ge 2$, as E-rational function $t_{i,\bar{f}}^\ell(\mathbf{u})$ with coefficients in $\mathbb{Q}\langle \bar{\delta}^\ell(\bar{e})\rangle_E$. Note that the E-polynomial appearing in the denominator is a power of $det(J_{\bar{f}}(\mathbf{X}))$. We set $t_{i,\bar{f}}^1(\mathbf{u}) = t_{i,\bar{f}}(\mathbf{u})$. □

For later use, we need to make explicit the form of the rational functions $t_{i,\bar{f}}^\ell(\mathbf{u})$ as a function of $\mathbf{u}$ but also of the coefficients of $\bar{f}$ (see section 4.4).

{rat} **Notation 2.16.** By equation (3), we have $\delta(y_i^0) := t_{i,\bar{f}}(\mathbf{y}^0)$ where $\mathbf{y}^0 := (y_1^0, \ldots, y_n^0)$ and $t_{i,\bar{f}}(\mathbf{y}^0)$ is obtained by multiplying the matrix $-J_{\bar{f}}(\mathbf{y}^0)^{-1}$ by the column vector $\begin{pmatrix} f_1^\delta(\mathbf{y}^0) \\ \vdots \\ f_n^\delta(\mathbf{y}^0) \end{pmatrix}$.

Now by Lemma 2.7, there are tuples $\mathbf{x}_i^0 \in F_0$ such that f_i belongs to $\langle \mathbf{x}_i^0\rangle_E[\mathbf{X}]^E$ and $f_i^\delta \in \mathbb{Q}(\langle \mathbf{x}_i^0\rangle_E, \delta(\mathbf{x}_i^0))[\mathbf{X}]^E$. To f_i^δ, we associate an E-rational function $f_i^{\delta,*}$ by replacing $\delta(\mathbf{x}_i^0)$ by the tuple $\mathbf{x}_i^1$.

Let $\bar{x}^j := (\mathbf{x}_1^0, \ldots, \mathbf{x}_n^0)$ with $0 \le j$. Then we re-write $t_{i,\bar{f}}(\mathbf{y}^0)$ as an E-rational function with coefficients in $\mathbb{Q}$, namely as $t_{i,\bar{f}}^*(\mathbf{y}^0; \bar{x}^0, \bar{x}^1))$, $1 \le i \le n$. Set $t_{i,\bar{f}}^{1,*} := t_{i,\bar{f}}^*$ and $\mathbf{t}_{\bar{f}}^{1,*} := (t_{1,\bar{f}}^{1,*}, \ldots, t_{n,\bar{f}}^{1,*})$. Then we define $t_{i,\bar{f}}^{2,*}$ by applying δ and substituting $t_{j,\bar{f}}(\mathbf{y}^0)$ to $\delta(y_j^0)$, $1 \le j \le n$, and $\bar{x}^j$ to $\delta(\bar{x}^{j-1})$, $2 \ge j \ge 1$. So we get an E-rational function $t_{i,\bar{f}}^{2,*}(\mathbf{y}^0; \bar{x}^0, \bar{x}^1, \bar{x}^2)$, $1 \le i \le n$. We iterate this procedure, namely we apply δ to $t_{i,\bar{f}}^{\ell,*}$, we substitute $t_{k,\bar{f}}^{1,*}(\mathbf{y}^0)$ to $\delta(y_k^0)$, $1 \le k \le n$, and $\bar{x}^{j+1}$ to $\delta(\bar{x}^j)$, $j \ge 0$, to obtain $t_{i,\bar{f}}^{\ell+1,*}(\mathbf{y}^0; \bar{x}^0, \ldots, \bar{x}^{\ell+1})$, $1 \le i \le n$. We denote $\mathbf{t}_{\bar{f}}^{\ell+1,*} := (t_{1,\bar{f}}^{\ell+1,*}, \ldots, t_{n,\bar{f}}^{\ell+1,*})$.

Proposition 2.17. *Let $F_0 \subseteq F$ be two partial exponential fields and assume that $A(F_0)$ is pure in $A(F)$, that F_0 is generated as a field by $A(F) \cup E(A(F))$ and that $F_0 \lhd F$, then every E-derivation on F_0 extends to F.*

Proof: This is essentially [18, Theorem 6.3] but there the running assumption on partial E-fields is that $A(F)$ is a $\mathbb{Q}$-vector space. Therefore, in adapting the proof [18, Proposition 5.6], we assume, by induction, that $A(F_\beta)$ is a pure subgroup of $A(F)$. When defining $F_{\beta+1}$, we take the divisible hull in $A(F)$ of the subgroup generated by $A(F_\beta)$ and $\bar{x}$, where r_β belongs to $\bar{x}$ and $d(\bar{x}/F_\beta)$ is minimal. Let us also denote by $\langle A(F_\beta)\rangle_{\mathbb{Q}}$ the $\mathbb{Q}$-vector space generated by $A(F_\beta)$ in F.

Then we slightly modify the proof of [18, Theorem 6.3], by assuming that $A(F_1) \subseteq_1 A(F_2)$ and we choose $a_1, \dots, a_n \in A(F_2)\backslash A(F_1)$ maximal $\mathbb{Z}$-independent over $A(F_1)$ and generating $A(F_2)$ over $A(F_1)$ in the following way: for any $b \in A(F_2)$ there are $z_1, \dots z_n \in \mathbb{Z}$, $u \in A(F_1)$ and $n \in \mathbb{N}^*$ such that $nb = \sum_{i=1}^n z_i a_i + u$. Note that if $\sum_i z_i a_i \in \langle A(F_1)\rangle_{\mathbb{Q}}$, then for some $n \in \mathbb{N}^*$, $n\sum_i z_i a_i \in A(F_1)$. Since $\sum_i z_i a_i \in A(F_2)$ and $A(F_1) \subseteq_1 A(F_2)$, then $\sum_i z_i a_i \in A(F_1)$. So the element b in [18, Fact 6.4], does not belong to $\langle A(F_1)\rangle_{\mathbb{Q}}$ either. The rest of the proof of [18, Theorem 6.3] is similar since it only involves the spaces of derivations over F_2. □

Note that if F_0 is a subfield of F and if $A(F)$ =domain(E), $A(F_0)$ =domain(E)$\cap F_0$, then $A(F_0)$ is pure in domain(E). Indeed, let $n \in \mathbb{N}^*$ and assume that $u \in A(F)$ and $n.u \in A(F_0)$. So $u \in F_0$ and so $u \in A(F_0) = A(F) \cap F_0$.

Proposition 2.18. *Let $F_0 \subseteq F$ be two partial exponential fields. Assume that we have an E-derivation on F_0, then it extends to F.* {der-ext}

Proof: Consider the subfield $C := \mathrm{ecl}^F(F_0)$ of F; we have shown already that any E-derivation on F_0 extends to C (see Lemma 2.15). By [18, Propositions 4.7, 7.1], $C = \mathrm{cl}^F(F_0)$.

Let F_1 be the subfield generated by $(A(F) \cap C) \cup (E(A(F) \cap C))$. We will show that $F_1 \lhd F$ which will enable us to apply the result of J. Kirby recalled above.

Note that F_1 is a subfield of C (C is a partial exponential subfield [18, Lemma 3.3]). Set $A(F_1) := A(F) \cap F_1$, then $A(F_1) \subseteq_1 A(F)$. Note that if $u \in A(F_1)$, then $E(u) \in E(A(F)) \cap C \subseteq F_1$.

In order to show that $F_1 \lhd F$, we take a finite tuple $\mathbf{a} \in A(F)$ and we calculate $d(\mathbf{a}/A(F_1)) := \mathrm{td}(\mathbf{a}, E(\mathbf{a})/A(F_1) \cup E(A(F_1))) - \ell\dim(\mathbf{a}/A(F_1))$, where:

$\mathrm{td}(\mathbf{a}, E(\mathbf{a})/A(F_1) \cup E(A(F_1)))$ denotes the transcendence degree of the subfield of F, generated by $\mathbf{a}, E(\mathbf{a})$ over the subfield generated by $A(F_1) \cup E(A(F_1))$ and

$\ell\dim(\mathbf{a}/A(F_1))$ is the dimension of the quotient of two $\mathbb{Q}$-vector spaces, the first one generated by $\mathbf{a}$ and $A(F_1)$ and the second one by $A(F_1)$.

By Ax's theorem [18, Theorem 5.1, Corollary 5.2],

$$\mathrm{td}(\mathbf{a}, E(\mathbf{a})/C) \geq \ell\mathrm{dim}(\mathbf{a}/C)) + \dim(\mathbf{a}/C),$$

Moreover we have $\mathrm{td}(\mathbf{a}, E(\mathbf{a})/A(F_1) \cup E(A(F_1))) \geq \mathrm{td}(\mathbf{a}, E(\mathbf{a})/C)$.

Now let us show that $\ell\mathrm{dim}(\mathbf{a}/C) = \ell\mathrm{dim}(\mathbf{a}/A(F_1))$. Suppose we have a $\mathbb{Q}$-linear combination u of elements of $\mathbf{a}$ belonging to C. So for some nonzero natural number $n \in \mathbb{N}^*$, we have that nu also belongs to $A(F)$ (since $u \in A(F)$). So, we get that $nu \in A(F) \cap C$ and so $nu \in F_1 \cap A(F) = A(F_1)$, namely $u \in \langle A(F_1)\rangle_{\mathbb{Q}}$. Therefore, $\ell\mathrm{dim}(\mathbf{a}/C) = \ell\mathrm{dim}(\mathbf{a}/A(F_1))$ and $d(\mathbf{a}/F_1) \geq 0$. □

Corollary 2.19. *Let $F_0 \subseteq F$ be two partial exponential fields and let δ be an E-derivation on F_0. Assume that we have ℓ elements $c_1, \dots, c_\ell \in F$ ecl-independent over F_0 and $d_1, \dots, d_\ell \in F$. Then there is an E-derivation $\tilde{\delta}$ on F, extending δ and such that $\tilde{\delta}(c_i) = d_i$, $1 \leq i \leq \ell$.* {der-ext-gener}

Proof: Since $c_1, \ldots, c_\ell \in F$ are ecl-independent over F_0, there are ℓ E-derivations δ_i on F which are zero on F_0 and such that $\delta_i(c_j) = \delta_{ij}$, $1 \le i,\ j \le \ell$. By the preceding proposition, we have a derivation D on F extending δ. Consider $D + \sum_{i=1}^{\ell} f_i\delta_i$ with $f_i \in F$. Since the set of E-derivations on F forms an F- module, this is an E-derivation which extends δ by construction. We define $\tilde{\delta}$ as $D + \sum_{i=1}^{\ell}(d_i - D(c_i))\delta_i$ (setting in the above expression the coefficients f_i to be equal to $d_i - D(c_i)$, $1 \le i \le \ell$). □

3. E-varieties and topological exponential fields

3.1. **E-varieties.** Let K be a (partial) exponential E-field. Let $\mathbf{X} := (X_1, \ldots, X_n)$, $f \in K[\mathbf{X}]^E$ and $\mathbf{a} \in K^n$, denote by $\nabla f := (\partial_{X_1} f(\mathbf{X}), \ldots, \partial_{X_n} f(\mathbf{X}))$. and $\nabla f(\mathbf{a}) := (\partial_{X_1} f(\mathbf{a}), \ldots, \partial_{X_n} f(\mathbf{a}))$.

{var}

Definition 3.1. Let $g_1, \ldots, g_m \in K[\mathbf{X}]^E$ and let

$$V_n(g_1, \ldots, g_m) := \{\mathbf{a} \in K^n \colon \bigwedge_{i=1}^{m} g_i(\mathbf{a}) = 0\}.$$

An E-variety will be a definable subset of some K^n of the form $V_n(\bar{g})$ for some $\bar{g} \in K[\mathbf{X}]^E$. Sometimes we will need to consider the elements of an E-variety in an extension of K; in this case we will say that it is defined over K. Let V be an E-variety, then $\mathbf{a}$ is a regular point of V if for some $\bar{g}$, $V = V_n(\bar{g})$ and $\nabla g_1(\mathbf{a}), \ldots, \nabla g_m(\mathbf{a})$ are linearly independent over K (note that this implies that $m \le n$).

In the following, we will make a partition of variables of the g_i's and consider the regular zeroes with respect to a subset of the set of variables.

{gradient}

Notation 3.2. Let $0 < n_0 \le n$ and let $f \in K[\mathbf{X}]^E$. Denote by

$$\nabla_{n_0} f := (\partial_{X_{n-n_0+1}} f, \ldots, \partial_{X_n} f). \tag{4}$$

Consider the following subset of $V_n(\bar{g})$, with $m \le n_0$:

(5)

$$V^{reg}_{n,n_0}(\bar{g}) := \{\mathbf{b} \in K^n \colon \bigwedge_{i=1}^{m} g_i(\mathbf{b}) = 0 \ \&\ \nabla_{n_0} g_1(\mathbf{b}), \ldots, \nabla_{n_0} g_m(\mathbf{b}) \text{ are } K\text{−linearly independent}\}.$$

In case $n_0 = n$, we simply denote $V^{reg}_{n,n}(\bar{g})$ by $V^{reg}_n(\bar{g})$.

Furthermore, we need the following variant. Let $\bar{i} := (i_1, \ldots, i_{n_0})$ be a strictly increasing tuple of natural numbers between 1 and n (of length $1 \le n_0 \le n$). Then for $f \in K[\mathbf{X}]^E$, we denote by

$$\nabla_{\bar{i}} f := (\partial_{X_{i_1}} f, \ldots, \partial_{X_{i_{n_0}}} f). \tag{6}$$

We consider the subset of $V_n(\bar{g})$:

(7)

$$V^{reg}_{n,\bar{i}}(\bar{g}) := \{\mathbf{b} \in K^n \colon \bigwedge_{i=1}^{m} g_i(\mathbf{b}) = 0 \ \&\ \nabla_{\bar{i}} g_1(\mathbf{b}), \ldots, \nabla_{\bar{i}} g_m(\mathbf{b}) \text{ are } K-\text{linearly independent}\}.$$

Note that in order to be non-empty we need that $m \le n_0 = |\bar{i}|$.

3.2. **Generic points.** Let $K \subseteq L$ be partial E-fields. In section 2.3, we have seen that ecl^L is a closure operator which coincides with cl^L to which we associated the dimension function $\dim^L(\cdot/K)$ (see Definition 2.12). As usual one defines the dimension of a definable subset $B \subseteq L^n$ and the notion of generic points in B (see for instance [17]).

{dim2}

Definition 3.3. Let B be a definable subset of L^n defined over K. The dimension of B over K is defined as $\dim^L(B/K) := \sup\{\dim^L(\mathbf{b}/K) \ : \ \mathbf{b} \in B\}$. Let $\mathbf{b} \in B$, then $\mathbf{b}$ is a generic point of B over K if $\dim^L(\mathbf{b}/K) = \dim^L(B/K)$.

We will need the following notion of subtuples.

{subtuple}

Notation 3.4. Let $\mathbf{a} := (a_1, \ldots, a_n)$ be an n-tuple in K and let $\mathbf{X} := (X_1, \ldots, X_n)$. Let $0 < m < n$ and let $\{i_1, \ldots, i_m\} \dot\cup \{j_1, \ldots, j_{n-m}\}$ be a partition of $\{1, \ldots, n\}$, with $1 \leq i_1 < \ldots < i_m \leq n$ and $1 \leq j_1 < \ldots < j_{n-m} \leq n$.

A *m-subtuple* of $\mathbf{a}$ is a m-tuple denoted by $\mathbf{a}_{[m]}$ of the form $(a_{i_1}, \ldots, a_{i_m})$ and we denote by $\mathbf{a}_{[n-m]} := (a_{j_1}, \ldots, a_{j_{n-m}})$.

Given an E-polynomial $f(\mathbf{X}) \in K[\mathbf{X}]^E$, we denote either by $f(\mathbf{a}_{[n-m]}, X_{i_1}, \ldots, X_{i_m})$ or by $f_{\mathbf{a}_{[n-m]}}(X_{i_1}, \ldots, X_{i_m})$ the E-polynomial obtained from f when substituting for X_{j_i} the element a_{j_i}, $1 \leq i \leq n-m$. We adopt the same convention for $\mathcal{L}_E$-terms.

{polmin}

Remark 3.5. Let $\bar{f} = (f_1, \ldots, f_m) \subseteq K[\mathbf{X}]^E$, $\mathbf{a} := (a_1, \ldots, a_n) \in V_n^{reg}(\bar{f}) \subseteq L^n$, $1 \leq m \leq n$. Then:

(1) There is a m-subtuple $\mathbf{a}_{[m]}$ of $\mathbf{a}$ and a Khovanskii system over $K\langle \mathbf{a}_{[n-m]} \rangle^E$ such that $H_{\bar{f}_{\mathbf{a}_{[n-m]}}}(\mathbf{a}_{[m]})$ holds.
(2) In particular $\dim^L(\mathbf{a}/K) \leq n-m$ and if $V_n(\bar{f}) = V_n^{reg}(\bar{f})$, then $\dim^L(V_n(\bar{f})/K) \leq n-m$.

3.3. **E-ideals and differentiation.** Let R be a partial E-ring. Let $\mathbf{X} := (X_1, \ldots, X_n)$ and $\mathbf{X}_{\hat{i}}$ be the tuple $\mathbf{X}$ where X_i is removed, $1 \leq i \leq n$. Similarly for $\mathbf{a} \in R^n$, we denote $\mathbf{a}_{\hat{i}} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$.

Definition 3.6. Let $I \subseteq R$ be an ideal of R. Then I is an *E-ideal* if

$$(r \in I \rightarrow E(r) - 1 \in I).$$

A prime E-ideal is a prime ideal which is an E-ideal.

In $R[\mathbf{X}]^E$, an example of a prime E-ideal is $Ann^{R[\mathbf{X}]^E}(\mathbf{a}) := \{f \in R[\mathbf{X}]^E : \ f(\mathbf{a}) = 0\}$. (When the context is clear we will omit the superscript $R[\mathbf{X}]^E$.)

As usual the definition of E-ideal is set-up in such a way that if $I \subseteq R$ is an E-ideal, then on the quotient R/I, we have a well-defined exponential function given by:

$$E(r + I) := E(r) + I$$

for $r \in A(R)$. So $(R/I, E)$ is a again a partial E-ring.

We now recall a result from A. Macintyre on E-ideals closed under partial derivation. Note that the proof is purely algebraic, using that one can measure the complexity of exponential polynomials.

Fact 3.7. [20, Theorem 15 and Corollary] *Let R be a partial E-domain. Let $1 \leq i \leq n$. Let $I \subseteq R[\mathbf{X}]^E$ be an E-ideal closed under the E-derivation ∂_{X_i}. Then either $I = 0$ or I contains a non-zero element of $R[\mathbf{X}_{\hat{i}}]^E$. In particular, if $I \neq 0$ is closed under all E-derivations ∂_{X_i}, $1 \leq i \leq n$ and R is a field, then $I = R[\mathbf{X}]^E$.* {dpnul}

Let $K \subseteq L$ be partial E-fields. Fact 3.7 actually shows that ecl^L-independent elements over K do not satisfy any hidden exponential-algebraic relations over K.

{norel}

Corollary 3.8. *Let $\mathbf{a} := (a_1, \ldots, a_n) \in L^n$ be such that $a_1, \ldots, a_n$ are ecl^L-independent over K. Then there is no $g \in K[\mathbf{X}]^E \setminus \{0\}$ such that $g(\mathbf{a}) = 0$.*

Proof. By the way of contradiction assume there is $g \in K[\mathbf{X}]^E$ be such that $g(\mathbf{a}) = 0$. Then for $i = 1, \ldots, n$, $\partial_{X_i} g(\mathbf{a}) = 0$ otherwise $a_i \in ecl^L(K(\mathbf{a}_{\hat{i}}))$. (Indeed, letting $h(X) := g(\mathbf{a}_{\hat{i}}, X)$, we would have $H_h(a_i)$.) Hence the ideal $Ann(\mathbf{a})$ is an E-ideal, closed under all partial E-derivations ∂_{X_i}, $1 \leq i \leq n$. So by Fact 3.7, since $Ann(\mathbf{a}) \neq 0$, it is equal to $K[\mathbf{X}]^E$, a contradiction. $\square$

Let K_δ be an expansion of the partial E-field K by an E-derivation δ and let $\tilde{K}$ be an E-field extending K. Let $A \subseteq \tilde{K}^n$. Let $I_K(A) \subseteq K[\mathbf{X}]^E$ be the set of E-polynomials with coefficients in K which vanish on A, namely $I_K(A) = \bigcap_{\mathbf{a} \in A} Ann^{K[\mathbf{X}]^E}(\mathbf{a})$. Note that it is an E-ideal as an intersection of E-ideals.

{torsor}

Definition 3.9. *For $A \subseteq \tilde{K}^n$, let $\tau(A) \subseteq \tilde{K}^{2n}$ be the E-torsor of A (over K), namely:*

$$\tau(A) := \{(\mathbf{a}, \mathbf{b}) \in \tilde{K}^{2n} : \mathbf{a} \in A \textit{ and } \sum_{i=1}^{n} \partial_{X_i} f(\mathbf{a}).b_i + f^\delta(\mathbf{a}) = 0 \textit{ for all } f(\mathbf{X}) \in I_K(A)\}.$$

Note that if we can find $f_i(\mathbf{X}) \in Ann^{K[\mathbf{X}]^E}(\mathbf{a})$, $\mathbf{a} \in A$, $1 \leq i \leq m \leq n$ such that $\nabla f_1(\mathbf{a}), \ldots, \nabla f_m(\mathbf{a})$ are K-linearly independent, then setting

$$T_{\mathbf{a}} := \{\mathbf{b} \in \tilde{K}^n : \sum_{i=1}^{n} \partial_{X_i} f(\mathbf{a}).b_i = 0 \text{ for all } f(\mathbf{X}) \in Ann^{K[\mathbf{X}]^E}(\mathbf{a})\},$$

we have that $\ell\dim(T_{\mathbf{a}}) \leq n - m$.

{Lang-tor}

Lemma 3.10. *Let $K \subseteq \tilde{K}$ be partial E-fields, and let δ be an E-derivation on K. Let $\bar{f} = (f_1, \ldots, f_m) \subseteq K[\mathbf{X}]^E$. Suppose that there are $(\mathbf{a}, \mathbf{b}) \in \tilde{K}^{2n}$ such that $\mathbf{a} \subseteq \tilde{K}^n$ is a generic point of $V_n^{reg}(\bar{f})$ and $(\mathbf{a}, \mathbf{b}) \in \tau(V_n^{reg}(\bar{f}))$. Then there is an E-derivation δ^* on $\tilde{K}$ extending δ, uniquely determined on $\mathrm{ecl}^{\tilde{K}}(K(\mathbf{a}))$ and such that $\delta^*(a_i) = b_i$, for $i = 1, \ldots, n$.*

Proof. Since $\mathbf{a} \in V_n^{reg}(\bar{f})$, we have that $\nabla f_1(\mathbf{a}), \ldots, \nabla f_m(\mathbf{a})$ are $\tilde{K}$-linearly independent. By permuting the coordinates of $\mathbf{a}$, assume $\nabla_m f_1(\mathbf{a}), \ldots, \nabla_m f_m(\mathbf{a})$ are $\tilde{K}$-linearly independent. Set $\mathbf{a}_{[n-m]} := (a_1, \ldots, a_{n-m})$ and $\mathbf{a}_{[m]} := (a_{n-m+1}, \ldots, a_n)$. Note that $det(J_{\bar{f}_{\mathbf{a}_{[n-m]}}}(\mathbf{a}_{[m]})) \neq 0$. Since $n - m = \dim^{\tilde{K}}(\mathbf{a}/K)$, $a_1, \ldots, a_{n-m}$ are $\mathrm{ecl}^{\tilde{K}}$-independent.

By Corollary 2.19, there is an E-derivation $\tilde{\delta}$ on $\tilde{K}$ extending δ on K and such that $\tilde{\delta}(a_i) = b_i$, $1 \leq i \leq n - m$.

By assumption $(\mathbf{a}, \mathbf{b}) \in \tau(V_n^{reg}(\bar{f}))$. In particular $\bigwedge_{i=1}^{m} \sum_{j=1}^{n} \partial_{X_j} f_i(\mathbf{a}) b_j + f_i^\delta(\mathbf{a}) = 0$ (†). We break the sum $\sum_{j=1}^{n} \partial_{X_j} f_i(\mathbf{a}) b_j$ in two parts: $\sum_{j=1}^{n-m} \partial_{X_j} f_i(\mathbf{a}) b_j$, $\sum_{j=n-m+1}^{n} \partial_{X_j} f_i(\mathbf{a}) b_j$. By assumption $det(J_{\bar{f}_{\mathbf{a}_{[n-m]}}}(\mathbf{a}_{[m]})) \neq 0$, so the fact that $\mathbf{b}$ satisfies (†) is equivalent to the fact that the subtuple $\mathbf{b}_{[m]}$ satisfies the equation (8) below:

$$\text{(8)} \qquad \begin{pmatrix} b_{n-m+1} \\ \vdots \\ b_n \end{pmatrix} = -J_{\bar{f}_{\mathbf{a}_{[n-m]}}}(\mathbf{a}_{[m]})^{-1} \cdot \begin{pmatrix} f_1^\delta(\mathbf{a}) - \sum_{j=1}^{n-m} \partial_{X_j} f_1(\mathbf{a}) b_j \\ \vdots \\ f_m^\delta(\mathbf{a}) - \sum_{j=1}^{n-m} \partial_{X_j} f_m(\mathbf{a}) b_j \end{pmatrix}$$

So there is only one such E-derivation satisfying $\delta^*(a_i) = b_i$ for $i = n-m+1, \ldots, n$ on $\mathrm{ecl}^{\tilde{K}}(K(\mathbf{a}))$ by Lemma 2.15. Note that by using Lemma 2.7, we explicitly define a mapping δ^* on $K(\mathbf{a})_E$ as follows. Let $p(\mathbf{X}) \in K[\mathbf{X}]^E$, define $\delta^*(p(\mathbf{a})) := \sum_{j=1}^n \delta^*(a_i)\partial_{X_i} p(\mathbf{a}) + p^\delta(\mathbf{a})$ (note that $p^\delta \in K[\mathbf{X}]^E$). Furthermore, by Corollary 3.8, since $a_{n-m+1}, \ldots, a_n$ are $ecl^{\tilde{K}}$-independent, $Ann(\mathbf{a}_{[n-m+1]}) \cap K[\mathbf{X}]^E = \{0\}$ and so given $q_1(\mathbf{X})$, $q_2(\mathbf{X})$ in $K[\mathbf{X}]^E \setminus \{0\}$, we can define

$$\delta^*(q_1(\mathbf{a}_{[n-m]})/q_2(\mathbf{a}_{[n-m]})) := \frac{q_1(\mathbf{a}_{[n-m]})\delta^*(q_2(\mathbf{a}_{[n-m]})) - q_2(\mathbf{a}_{[n-m]})\delta^*(q_1(\mathbf{a}_{[n-m]}))}{q_2(\mathbf{a}_{[n-m]})^2}.$$

□

{top}

3.4. **Topological E-fields.** In section 2, we introduced the notion of a partial E-field F as a two-sorted structure $(F, A(F), E)$ where F is a field and $A(F)$ is an additive group and $E : A(F) \to F^*$, a morphism from $A(F)$ to the multiplicative group of F. (Further we identified $A(F)$ with a subgroup of the additive group of F.)

In this section we will revert to a one-sorted setting. Let $\mathcal{L}$ be a relational extension of $\mathcal{L}_E \cup \{^{-1}\}$ and let $\mathcal{L}^-$ be the reduct of $\mathcal{L}$ when taking off the inverse function. Starting with a two-sorted structure $(K, A(K), E)$ which is a partial E-field, we will consider $\mathcal{L}$-structures $\mathcal{K}$ with domain K and with the convention that the domain of E is $A(K)$, an additive subgroup of K. Classically one requires that the functions are defined everywhere and so for instance, one extends $^{-1}$ by the rule $0^{-1} = 0$. But in the following we will assume that K is a topological field and that an implicit function theorem holds for exponential $\mathcal{L}$-terms. So, we instead proceed as follows. We use the fact that one can associate with any $\mathcal{L}$-term $t(x_1, \ldots, x_n)$ a quantifier-free formula $D_t(x_1, \ldots, x_n)$ which exactly holds on the domain of definition (also denoted by D_t) of t (see for instance [34, Section 2]). (One works by induction on the complexity of terms in a similar way as we did in section 2.) Furthermore letting t' be the formal derivative of t (with the rule $E' = E$), $D_t(x_1, \ldots, x_n) \leftrightarrow D_{t'}(x_1, \ldots, x_n)$. The advantage of proceeding in this way, instead of extending the functions when undefined, is that one can then require that the terms induce continuous functions (or continuously differentiable, i.e. C^1-functions, or C^∞, or analytic functions) on their domains of definition, in case K is also a topological field.

Let $\mathcal{V}$ denote a basis of neighbourhoods of 0. Then $(\mathcal{K}, \mathcal{V})$ is a topological $\mathcal{L}$-field if $\mathcal{V}$ induces an Hausdorff (non-discrete) topology such that the functions of $\mathcal{L}$ are interpreted by C^1-functions on their domains of definition and that each relation and its complement is a union of an open set and the zero-set of a finite system of E-polynomials. So w.l.o.g. we may assume that every quantifier-free $\mathcal{L}$-formula is a bboolean combination of atomic formulas of the form $t_1(\mathbf{x}) = 0$ or a conjunction of basic formulas expressing that $\mathbf{x}$ belongs to an open set. This notion of topological $\mathcal{L}$-fields extends the one given in [14, section 2.1]. We will say that $\mathcal{K}$ is endowed with a definable topology if there is an $\mathcal{L}$-formula $\chi(x, \mathbf{y})$ such that a basis of neighbourhoods of 0 in K is given by $\chi(K, \mathbf{d})$, where $\mathbf{d} \in K^n$, $n = |\mathbf{y}|$. Note that if $\mathcal{K}$ is endowed with a definable topology, then any field $\mathcal{K}_0$ elementary equivalent to $\mathcal{K}$ can be endowed with a definable topology using the same formula $\chi(x, \mathbf{y})$. Moreover if $\mathcal{K}$ is endowed with a definable topology with corresponding formula $\chi(x, \bar{y})$ and $\tilde{\mathcal{K}}$ an elementary extension of $\mathcal{K}$ endowed with a topology induced by χ, then $\tilde{\mathcal{K}}$ is a topological extension of $\mathcal{K}$. As usual, the cartesian products of K are endowed with the product topology. Let $\mathbf{x}$ be a m-tuple, we will denote by $\bar{\chi}(\mathbf{x}, \mathbf{y})$ the formula $\bigwedge_{i=1}^m \chi(x_i, \mathbf{y})$.

{Ksmall}

Notation 3.11. Let $(\mathcal{K},\mathcal{V}) \subseteq (\tilde{\mathcal{K}},\mathcal{W})$ be two topological $\mathcal{L}$-fields with $(\tilde{\mathcal{K}},\mathcal{W})$ be a topological extension of $(\mathcal{K},\mathcal{V})$ [14, Definition 2.3], namely $\mathcal{K}$ is an $\mathcal{L}$-substructure of $\tilde{\mathcal{K}}$ and for any $V \in \mathcal{V}$ there exists $W \in \mathcal{W}$ such that $V = W \cap K$. Let $\mathcal{W}_K := \{W \in \mathcal{W} : W \cap K \in \mathcal{V}\}$.

On elements $a, b \in \tilde{K}$ we have the equivalence relation $a \sim_{\mathcal{W}_K} b$ which means that $a - b$ belongs to every element of $\mathcal{W}_K$. (We will also use the notation $a \sim_K b$.)

We will say that a non zero element $a \in \tilde{K}$ is K-small if $a \sim_{\mathcal{W}_K} 0$ (that we abbreviate by $a \sim_K 0$).

Recall that a topological field K has a V-topology if whenever $X, Y \subseteq K$ are bounded away from 0, then XY is bounded away from 0 (a subset Z is bounded away from 0 if 0 does not belong to the closure of Z). One calls such fields V-topological fields [27, Section 3]. By results of Kowalsky-Dürbaum, and Fleischer if K is a V-topological field then its topology is either induced by an archimedean absolute value or by a non-trivial valuation [27, Theorem 3.1]. One can define a notion of topological henselianity (t-henselianity) for V-topological fields [27, Theorem 7.2]. One can show that one can embed any V-topological field in a t-henselian field [15, Lemma 2.2] and a t-henselian field satisfies the implicit function theorem for polynomial maps [27, Theorem 7.4], [15, Fact 2.4].

{sec:imp} 3.5. **Implicit function theorem.** From now on, we will assume that $\mathcal{K}$ is a topological $\mathcal{L}$-field where the topology is a V-topology and it is definable with corresponding formula χ.

Notation 3.12. [33, Definition 4.4] Let $\mathcal{S}$ be a neighbourhood system in K^n, namely a non-empty collection of open non-empty definable neighbourhoods closed under finite intersection. Let $\mathbf{a} \in K^n$, we will denote by $\mathcal{S}_{\mathbf{a}}$ the neighbourhood system consisting of all definable neighbourhoods of $\mathbf{a}$.

Denote by $\mathfrak{D}^n(\mathcal{S})^- := \{(f, U) \colon U \in \mathcal{S}, f : U \to K$ a C^∞-function, definable in $\mathcal{K}\}$. One defines on $\mathfrak{D}^n(\mathcal{S})^-$ an equivalence relation $\sim$ as follows: $(f_1, U_1) \sim (f_2, U_2)$ if there is $U \subseteq U_1 \cap U_2$ such that $f_1 \restriction U = f_2 \restriction U$. Let $\mathfrak{D}^n(\mathcal{S}) := \mathfrak{D}^n(\mathcal{S})^-/\sim$. We denote by $[f, U]$ the equivalence class containing (f, U).

Denote by $\mathfrak{D}^n_{an}(\mathcal{S})^- := \{(f, U) \colon U \in \mathcal{S}, f : U \to K$ an analytic function, definable in $\mathcal{K}$ $\}$ and by $\mathfrak{D}^n_{an}(\mathcal{S}) := \mathfrak{D}^n_{an}(\mathcal{S})^-/\sim$.

We now introduce the following *implicit function theorem* hypothesis (IFT) that we put on the class of fields under consideration. The implicit function theorem for C^1-functions, or C$^\infty$-functions, or analytic functions is classically proven in fields like $\mathbb{R}$, $\mathbb{Q}_p$ (or more generally complete (non-discrete) valued fields) [2, section 1.5]. A. Wilkie stated it for any field $\mathcal{K}$ elementary equivalent to an expansion of the field of reals [33, section 4.3], T. Servi recasted the results of Wilkie in definably complete expansions of ordered fields [31].

{imp} **Definition 3.13.** Let $n = \ell + m$, $n > 1$, $\ell, m > 0$, let $(\mathbf{a}, \mathbf{b}) \in K^{\ell+m}$ and let $\mathcal{S}_{(\mathbf{a},\mathbf{b})}$ be the corresponding neighbourhood system. Let $f_1(\mathbf{x}, \mathbf{y}), \ldots, f_m(\mathbf{x}, \mathbf{y})$ be definable C^∞-functions in $\mathcal{K}$, $|\mathbf{x}| = \ell$, $|\mathbf{y}| = m$, denote $\bar{f}(\mathbf{a}, \mathbf{y}) = (f_1(\mathbf{a}, \mathbf{y}), \ldots, f_m(\mathbf{a}, \mathbf{y}))$ by $\bar{f}_{\mathbf{a}}(\mathbf{y})$. Then $\mathcal{K}$ satisfies (IFT) if the following holds. Assume that $\bar{f}_{\mathbf{a}}(\mathbf{b}) = 0$ and that $det(J_{\bar{f}_{\mathbf{a}}}(\mathbf{b})) \neq 0$ (see Notation 2.10). Then there are neighbourhoods $O_{\mathbf{a}} \subseteq K^\ell$ of $\mathbf{a}$ (respectively $O_{b_i} \subseteq K$, $1 \leq i \leq m$, of b_i) and C^1-functions $g_i(\mathbf{x}) : O_{\mathbf{a}} \to O_{b_i}$, , $1 \leq i \leq m$, such that , setting $\bar{g} := (g_1, \ldots, g_m)$,

(9) $\qquad \bar{g}(\mathbf{a}) = \mathbf{b} \wedge$

(10) $\qquad \forall\, \mathbf{x} \in O_{\mathbf{a}} \ \big(\bar{f}(\mathbf{x}, \bar{g}(\mathbf{x})) = 0 \ \wedge J_{\bar{g}}(\mathbf{x}) = -(\nabla_m \bar{f}(\mathbf{x}, \bar{g}(\mathbf{x}))^{-1} \nabla_\ell \bar{f}(\mathbf{x}, \bar{g}(\mathbf{x})))\big) \wedge$

(11) $\qquad \forall\, \mathbf{x} \in O_{\mathbf{a}}\, \forall\, \mathbf{y} \in O_{\mathbf{b}}\, (\bar{f}(\mathbf{x}, \mathbf{y}) = 0 \leftrightarrow \mathbf{y} = \bar{g}(\mathbf{x}))$.

{IFTA}

Notation 3.14. As noted in [33, 4.3], when the topology on K is definable, this implies that whenever the functions f_i are definable (respectively C^∞), the g_i's are definable (respectively C^∞), using the above equations (9), (10), (11). If when the functions f_i, $1 \leq i \leq m$, are analytic functions in a neighbourhood of $(\mathbf{a}, \mathbf{b})$, the functions $\bar{g}$ in the scheme $(IFTA)_{an}$ are also analytic in a neighbourhood of $\mathbf{a}$, we will denote the corresponding scheme $(\mathrm{IFT})_{an}$.

Notation 3.15. [33, below Notation 4.6] Keeping the same notation as in Definition 3.13, and under the same hypothesis, we may define a map $\hat{}: \mathfrak{D}^n(\mathcal{S}_{(\mathbf{a},\mathbf{b})})^- \to \mathfrak{D}^\ell(\mathcal{S}_{(\mathbf{a},\mathbf{b})})^- : f \mapsto \hat{f}$ sending the function $f \upharpoonright O_{\mathbf{a}} \times O_{\mathbf{b}} \to K$, where $O_{\mathbf{b}} := O_{b_1} \times \ldots \times O_{b_m}$, to the function $\hat{f} : O_{\mathbf{a}} \to K : \mathbf{x} \mapsto f(\mathbf{x}, g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))$. It is convenient to introduce an $(\ell + m)$-tuple $(\overline{\tilde{g}})$ of functions defined as follows: $\tilde{g}_i(\mathbf{x}) = x_i$ for $1 \leq i \leq \ell$ and $\tilde{g}_{\ell+i} := g_i(\mathbf{x})$, $1 \leq i \leq m$. With this notation $\hat{f}(\mathbf{x}) = f(\overline{\tilde{g}}(\mathbf{x}))$.

{hat}

Lemma 3.16. *Let $\mathcal{K}$ satisfying* (IFT)*. Let $(\mathbf{a}, \mathbf{b}) \in K^{\ell+m}$ and let $f_1, \ldots, f_m, h \in \mathfrak{D}^{\ell+m}(\mathcal{S}_{(\mathbf{a},\mathbf{b})})$. Assume that $\bar{f}(\mathbf{a}, \mathbf{b}) = 0$ and assume that $det(J_{\bar{f}_{\mathbf{a}}}(\mathbf{b})) \neq 0$. Then, keeping the same notations as in Definition 3.13, the sequence of vectors $\nabla \bar{f}(\mathbf{a}, \mathbf{b})$, $\nabla h(\mathbf{a}, \mathbf{b})$ is K-linearly independent iff $\nabla \hat{h}(\mathbf{a}) \neq 0$.*

Proof: The proof is the same as the one of [33, Lemma 4.7] (and it was also used in [21] (see [21, Lemma 5.1.3])). □

We will need the following *lack of flat functions* (LFF) property [33, Lemma 4.5], [31, Lemma 25].

Notation 3.17. We say that $\mathcal{K}$ satisfies (LFF) if the following holds. Let $\mathcal{S}$ is a neighbourhood system in K^n and let M be a subring of $\mathfrak{D}^n(\mathcal{S})$ closed under differentiation. Let $I \subseteq M$ a finitely generated ideal closed under differentiation and let $[g_1, U_1], \ldots, [g_s, U_s]$ be generators for I. Let Z be the set of zeroes of g_i, $1 \leq i \leq s$, in $U_1 \cap \ldots \cap U_s$. Then there is $U \in \mathcal{S}$ such that $U \cap Z$ is an open subset of K^n.

If in $\mathcal{K}$, $(\mathrm{IFT})_{an}$ holds and if we restrict M to be a subring of $\mathfrak{D}^n_{an}(\mathcal{S})$, then (LFF) holds. In case K is either real-closed or an ordered field which is definably complete, then (LFF) holds in general. Indeed, in that last case, it follows from the following property of solutions of systems of linear differential equations:
given an open interval U of K and given a system of linear differential equations (of order 1) with coefficients in $\mathcal{F}$, there is a unique C^1-function, solution of that system on U [31, Theorem 8].

Observe that given a finite given number of elements of $K[\mathbf{x}]^E$, we can put them in a noetherian differential subring of $K[\mathbf{x}]^E$. Indeed, using the complexity function ord defined in $K[\mathbf{X}]^E$, this is always possible to find such a ring. An exponential polynomial corresponds to an $\mathcal{L}_E$-term and those are constructed by induction in finitely many steps. So we place ourselves in the ordinary polynomial ring generated by all the (finitely many) sub-terms appearing in the construction and this ring is closed under differentiation. Then in a noetherian subring, all ideals are finitely generated and the property (LFF) can be applied.

The next result was first observed for $(\bar{\mathbb{R}}, exp)$ by A. Wilkie [33] but note that it also holds without the assumption of noetherianity, for definably complete structures by G. Jones and A. Wilkie [17]. Then it was re-used in [21, Proposition 5.1.4] in the case of the valued field $(\mathbb{Q}_p, E_p)$; there one needs the version of the implicit function theorem for analytic functions.

{max}

Proposition 3.18. [33, Theorem 4.9] *Assume that $\mathcal{K}$ satisfies* (IFT) *and* (LFF).

Let $\mathbf{r} \in K^n$ and let R_n be a noetherian subring of $\mathfrak{D}^n(\mathcal{S}_\mathbf{r})$ closed under differentiation.

Let $m \in \mathbb{N}$ and let $f_1, \ldots, f_m \in R_n$. and assume $\mathbf{r} \in V_n^{reg}(f_1, \ldots, f_m)$. Then, exactly one of the following is true:

(a) *$n = m$; or,*
(b) *$m < n$ and for all $h \in R_n$ with $h(\mathbf{r}) = 0$, h vanishes on $U \cap V_n^{reg}(f_1, \ldots, f_m)$ for some open neighbourhood U containing $\mathbf{r}$,*
(c) *$m < n$ and for some $h \in R_n$, $\mathbf{r} \in V^{reg}(f_1, \ldots, f_m, h)$.*

□

We will end the section by showing, that in case $\mathcal{K}$ satisfies (IFT), that we can find ecl-independent elements, which are K-small in an elementary extension of $\mathcal{K}$.

{zero-iso}

Remark 3.19. *Let $\mathcal{K}$ satisfying* (IFT). *Let $f_1, \ldots, f_m \in K[\mathbf{x}, \mathbf{y}]^E$, and $(\mathbf{a}, \mathbf{b}) \in K^{\ell+m}$, with $|\mathbf{x}| = \ell, |\mathbf{y}| = m$. Let $\mathbf{a} \in K^\ell$, $\mathbf{b} \in K^m$. Assume that $H_{\bar{f}_\mathbf{a}}(\mathbf{b})$ holds, namely $\bar{f}(\mathbf{a}, \mathbf{b}) = 0$ and $det(J_{\bar{f}_\mathbf{a}}(\mathbf{b})) \neq 0$ (see Definition 2.11). Then, $\mathbf{b}$ is an isolated zero of the system $\bar{f}_\mathbf{a}(\mathbf{y}) = 0$.*

{sat-ecl-petit}

Lemma 3.20. *Let $\mathcal{K}$ satisfy* (IFT). *Let $\mathcal{K}_1$ be a $|K|^+$-elementary extension of $\mathcal{K}$. Then there is an element $t \in K_1 \setminus \mathrm{ecl}^{K_1}(K)$ with $t \sim_K 0$. More generally for every $n \in \mathbb{N}^*$ there are n elements $t_1, \ldots, t_n \in K_1$* ecl*-independent over K and K-small.*

Proof. Consider the partial type $tp_K(x)$ consisting of $\mathcal{L}(K)$-formulas expressing that $x \sim_K 0$ and $x \notin ecl(K)$. The first property is expressed by the set of formulas $\chi(x, \bar{a})$, where $\bar{a}$ varies in K and the second property by $\neg \exists \bar{y}\ H_{\bar{f}}(x, \bar{y})$ where $\bar{f}$ varies in $K[X, \bar{Y}]^E$. By Remark 3.19, this set of formulas is finitely satisfiable. So $tp_K(x)$ is realized in a $|K|^+$-saturated extension of K (see for instance [23, Theorem 4.3.12]).

Then by induction on n, assume we found n elements $t_1, \ldots, t_n$ ecl-independent over K and K-small. Consider the partial type $tp_{K(t_1,\ldots,t_n)}(x)$ consisting of $\mathcal{L}(K(t_1, \ldots, t_n))$-formulas expressing that $x \sim_K 0$ and $x \notin \mathrm{ecl}^{K_1}(K(t_1, \ldots, t_n))$. Again by Remark 3.19, it is finitely satisfiable and so it is realized in $\mathcal{K}_1$ by an element t_{n+1} such that $t_1, \ldots, t_{n+1}$ are ecl-independent over K and K-small. □

{generegalter}

Proposition 3.21. *Let $\mathcal{K}$ satisfy* (IFT). *Let $\bar{f} = (f_1, \ldots, f_m) \subseteq K[\mathbf{X}]^E$, $|\mathbf{X}| = n > m$. Suppose that there is $\mathbf{a} \in V_n^{reg}(\bar{f}) \cap K^n$.*

Then there is an elementary $\mathcal{L}$-extension $\tilde{\mathcal{K}}$ of $\mathcal{K}$ and $\mathbf{b} \in V_n^{reg}(\bar{f}) \cap \tilde{K}^n$ with $\mathbf{b} - \mathbf{a} \sim_K \bar{0}$ and $\dim^{\tilde{K}}(\mathbf{b}/K) = n - m$. In particular, $\mathbf{b}$ is a generic point of $V_n^{reg}(\bar{f}) \cap \tilde{K}^n$.

Proof. Let $\mathbf{a} \in V_n^{reg}(\bar{f})$, then $\nabla f_1(\mathbf{a}), \ldots, \nabla f_m(\mathbf{a})$ are linearly independent over K. By permuting the variables $X_1, \ldots, X_n$, assume that $\nabla_m f_1(\mathbf{a}), \ldots, \nabla_m f_m(\mathbf{a})$ are K-linearly independent (see Notation 3.2). So we have $det(J_{\bar{f}_{\mathbf{a}_{[n-m]}}}(\mathbf{a}_{[m]})) \neq 0$, with $\mathbf{a} := (\mathbf{a}_{[n-m]}, \mathbf{a}_{[m]})$ (see Notation 3.4). By (IFT), there are definable neighbourhoods $O \subseteq K^{n-m}$ of $\mathbf{a}_{[n-m]}$, $O' \subseteq K^m$ of $\mathbf{a}_{[m]}$ and definable functions $g_1, \ldots, g_m$ from $O \to O'$ such that $\mathbf{a}_{[m]} = g(\mathbf{a}_{[n-m]})$ and such that for all $\mathbf{x} \in O$, $\bigwedge_{i=1}^m f_i(\mathbf{x}, g_1(\mathbf{x}), \ldots, g_m(\mathbf{x})) = 0$. By Lemma 3.20, there is an elementary $\mathcal{L}_E$-extension $\tilde{\mathcal{K}}$ of $\mathcal{K}$ containing $n - m$ K-small elements $t_1, \ldots, t_{n-m}$ which are ecl-independent over K.

Let $\mathbf{t}_{[n-m]} := (t_1, \ldots, t_{n-m})$ and $\mathbf{b} := \mathbf{a}_{[n-m]} + \mathbf{t}_{[n-m]} \in K_{n-m}$. Then $\mathbf{b} \in \tilde{K}$ are ecl-independent over K, $\mathbf{a} - \mathbf{b} \sim_K 0$ and $\bigwedge_{i=1}^n f_i(\mathbf{b}, g_1(\mathbf{b}), \ldots, g_m(\mathbf{b})) = 0$. □

{sec:ec}

4. Topological differential exponential fields

4.1. **Differential fields expansions.** Throughout this section, we will place ourselves in the same setting as in subsection 3.4; in particular the language $\mathcal{L}$ is a relational expansion of $\mathcal{L}_E \cup \{^{-1}\}$. Again, we assume that the topological $\mathcal{L}$-field $\mathcal{K}$ is endowed with a definable field topology with corresponding formula χ and that this topology is a V-topology.

Let $\mathcal{L}_\delta$ be the expansion of $\mathcal{L}$ by an E-derivation δ and given $\mathcal{K}$, let $\mathcal{K}_\delta$ denotes the expansion of $\mathcal{K}$ by an E-derivation δ.

Given an $\mathcal{L}$-theory of topological $\mathcal{L}$-fields, we denote by T_δ the theory T together with the axioms of E-derivation (see Definition 2.4). In particular if $\mathcal{K} \models T$, then $\mathcal{K}_\delta$ is a model of T_δ.

Any $\mathcal{L}_\delta$-term $t(\mathbf{x})$ with $\mathbf{x} = (x_1, \ldots, x_n)$ is equivalent, modulo the theory of differential E-fields, to an $\mathcal{L}_\delta$-term $t^*(\bar{\delta}^{m_1}(x_1), \ldots, \bar{\delta}^{m_n}(x_n))$ where t^* is an $\mathcal{L}$-term, for some $(m_1, \ldots, m_n) \in \mathbb{N}^n$. Recall that we associated with any $\mathcal{L}$-term t^* a quantifier-free formula D_{t^*} and its domain of definition.

By possibly adding tautological conjunctions like $\delta^k(x_i) = \delta^k(x_i)$ if needed, we may assume that all the m_i's are equal. We use the following notation $\bar{\delta}^m(\mathbf{x}) := (\mathbf{x}, \delta(\mathbf{x}), \ldots, \delta^m(\mathbf{x}))$, with $\delta^i(\mathbf{x}) := (\delta^i(x_1), \ldots, \delta^i(x_n))$, $1 \leq i \leq m$. Therefore, we may associate with any quantifier-free $\mathcal{L}_\delta$-formula $\varphi(\mathbf{x})$ an equivalent $\mathcal{L}_\delta$-formula, modulo the theory of differential E-fields, of the form $\varphi^{*,m}(\bar{\delta}^m(\mathbf{x}))$, $m \in \mathbb{N}$, where $\varphi^{*,m}$ is an $\mathcal{L}$-quantifier-free formula which arises by uniformly replacing every occurrence of $\delta^m(x_i)$ by a new variable x_i^m in φ with the following choice for the order of variables $\varphi^{*,m}(\mathbf{x}^0, \ldots, \mathbf{x}^m)$, where $\mathbf{x}^i = (x_1^i, \ldots, x_n^i)$, $0 \leq i \leq m$; furthermore since we made the convention that the functions are not everywhere defined, we assume in addition that the formula $\varphi^{*,m}(\mathbf{x}^0, \ldots, \mathbf{x}^m)$ contains for each term $t^*(\mathbf{x}^0, \ldots, \mathbf{x}^m)$ the quantifier-free $\mathcal{L}$-formula $D_t^*(\mathbf{x}^0, \ldots, \mathbf{x}^m)$. Let $\mathcal{T}_\varphi$ be the set of $\mathcal{L}_\delta$-terms occurring in φ.

Furthermore since we are only interested in existentially closed models, we will add new variables (that we will quantify existentially) and we replace in the formula $\varphi^{*,m}(\mathbf{x}^0, \ldots, \mathbf{x}^m)$, each occurrence of an $\mathcal{L}$-subterm of the form s^{-1} by a new variable u together with the existential formula $\exists u\ us = 1$, in order to transform atomic $\mathcal{L}$-formulas into atomic $\mathcal{L}^-$-formulas in variables $\mathbf{x}^0, \ldots, \mathbf{x}^m, \bar{u}$. Note that $\delta(u)$ is expressed in terms of s, $\delta(s)$. So we get

$$\varphi(\mathbf{x}) \wedge \bigwedge_{t \in \mathcal{T}_\varphi} D_{t^*}(\bar{\delta}^m(\mathbf{x})) \Leftrightarrow \exists \bar{u}\, \varphi_-^{*,m}(\bar{\delta}^m(\mathbf{x}), \bar{u}) \wedge \bigwedge_{t \in \mathcal{T}_\varphi} D_{t^*}(\bar{\delta}^m(\mathbf{x})),$$

where now $\varphi_-^{*,m}$ is a quantifier-free $\mathcal{L}^-$-formula. We will call the least such m, the order of the quantifier-free $\mathcal{L}_\delta$-formula φ. We will call an atomic formula of the form $s(\mathbf{y}) = 0$ an $\mathcal{L}_-$-equation (or $\mathcal{L}_E$-equation), where $s(\mathbf{y})$ is an $\mathcal{L}_-$-term. We will usually drop the superscript m in the formula $\varphi_-^{*,m}$. We will make the following notational simplifications: we will no longer specify that we work on the domains of definitions of our terms.

4.2. **Scheme** $(\mathrm{DL})_E$**.** Given a model-complete theory T of topological $\mathcal{L}$-fields, we consider the class of existentially closed differential expansions of models of T and under additional assumptions on the class of models of T, we will show that this class is elementary and produce an axiomatisation. Namely, by a scheme of first-order axioms, we will express that certain systems of differential exponential equations have a solution. In order to determine which ones, we first associate, using the process explained above, to a quantifier-free $\mathcal{L}_\delta$-formula $\varphi(\mathbf{x})$ of order m, a quantifier-free $\mathcal{L}_-$-formula $\varphi_-^{*,m}(\bar{\delta}^m(\mathbf{x}), \bar{u})$. From now on we will make the additional hypothesis that $\varphi(\mathbf{x})$ is a finite conjunction of basic formulas (namely either an atomic formula or the negation of an atomic formula), and one can easily check that the associated formula $\varphi_-^{*,m}(\bar\delta^m(\mathbf{x}),\bar u)$ is also a finite conjunction of basic formulas. Then we express all possible ecl-relations among the variables $\mathbf{x}$ (the new variables that we added are the subfield generated by $\mathbf{x}$). Since the derivation extends in a unique way to the ecl-closure, we enumerate partitions of the variables into two subsets: a first one where we impose no conditions and the other one where we express that there are regular solutions of an E-variety over this first subset of variables.

{const}

Definition 4.1. Let $\mathcal{K}_\delta$ be a differential topological $\mathcal{L}$-field. Let $\varphi(\mathbf{x})$ be a quantifier-free $\mathcal{L}_\delta(K)$-formula of order m, of the form a finite conjunction of basic formulas. Denote by $\mathbf{x}^i := (x_1^i,\dots,x_n^i)$, $0\le i\le m$ with $\mathbf{x}^0=\mathbf{x}=(x_1,\dots,x_n)$.

We will associate with $\varphi(\mathbf{x})$ a Khovanskii formula $H(\mathbf{x}^0,\dots,\mathbf{x}^{m-1},\bar z)$ with extra variables $\bar z$ that we define below. Let $\bar c$ be new constant symbols that will be interpreted by the parameters coming from K.

Let $n\ge \ell_0\ge \ell_1\ge\dots\ge \ell_{m-1}\ge 0$ and $\mathbf{x}^i_{[\ell_i]} := (x_1^i,\dots,x_{\ell_i}^i)$, $0\le i\le m-1$. We are going to enumerate all possible Khovanskii systems expressing that each element x_j^i, $\ell_i+1\le j\le n$, of the subtuple $(x^i_{\ell_i+1},\dots,x_n^i)$ of $\mathbf{x}^i$ is in the ecl^K-closure of $\mathbf{x}^0_{[\ell_0]},\dots,\mathbf{x}^i_{[\ell_i]}$. For $0\le i\le m-1$, $\ell_i<j\le n$, let $\bar f_{j,i}$, be a tuple of E-polynomials with coefficients in $\mathbb{Q}(\bar c,\mathbf{x}^i_{[\ell_i]},\dots,\mathbf{x}^0_{[\ell_0]})$, $\bar c\in K$, and consider the Khovanskii systems $H_{\bar f_{j,i}}(x_j^i,\mathbf{z}_{j,i})$ with $\mathbf{z}_{j,i}$ a tuple of new variables expressing this ecl^K-dependence (see Definition 2.11).

A **Khovanskii formula** is an $\mathcal{L}(\bar c)$-formula of the form:

$$H(\mathbf{x}^0,\dots,\mathbf{x}^{m-1},\bar z) := \bigwedge_{i=0}^{m-1}\bigwedge_{j=\ell_i+1}^{n} H_{\bar f_{j,i}}(x_j^i,\mathbf{z}_{j,i})$$

where the tuple $\bar z := (\mathbf{z}_{(\ell_i+1),i},\dots,\mathbf{z}_{n,i})_{0\le i\le m-1}$.

Recall that whenever $H_{\bar f_{j,i}}(x_j^i,\mathbf{z}_{j,i})$ holds, it implies that $\delta(x_j^i)$ is uniquely determined, for $\ell_i+1\le j\le n$. We take it into account in the following way. We have that $\delta(x_j^i,\mathbf{z}_{j,i}) = \mathbf{t}^{1,*}_{\bar f_j}(x_j^i,\mathbf{z}_{j,i})$, where $\mathbf{t}^{1,*}_{\bar f_j}$ is a tuple of E-rational functions with coefficients in $\mathbb{Q}(\bar\delta^1(\bar c),\mathbf{x}^{i+1}_{[\ell_i]},\dots,\mathbf{x}^1_{[\ell_0]},\mathbf{x}^0_{[\ell_0]})$ (see Notation 2.16).

Now the $\mathcal{L}(\bar c)$-formula $\varphi_H^*(\mathbf{x}^0,\dots,\mathbf{x}^m,\bar z,\bar u)$ is constructed by adding to the $\mathcal{L}_-$-formula $\varphi_-^{*,m}(\mathbf{x}^0,\dots,\mathbf{x}^m,\bar u)$ for each $0\le k\le m-1$:

• the atomic formula $(x_j^{k+1},\mathbf{z}_{j,k+1}) = \mathbf{t}^{1,*}_{\bar f_j}(x_j^k,\mathbf{z}_{j,k})$, $\ell_k+1\le j\le n$, where $\mathbf{t}^{1,*}_{\bar f_j}$ is a tuple of E-rational functions with coefficients in $\mathbb{Q}(\bar\delta(\bar c),\mathbf{x}^{k+1}_{[\ell_k]},\dots,\mathbf{x}^1_{[\ell_0]},\mathbf{x}^0_{[\ell_0]})$,

• a formula expressing that the determinants of the Jacobian matrices occurring in these Khovanskii systems are non-zero.

Furthermore we will assume that clearing denominators, we put φ_H^* in the following equivalent form: a finite conjunction of E-polynomials equations (that we will denote by $V_{\varphi_H^*}$) and an atomic $\mathcal{L}_-$-formula expressing that a tuple belongs to an open set.

Note that varying over all possible ecl-dependence relations (with coefficients in $\bar c\subseteq K$) among the variables in the tuple $\mathbf{x}^0,\dots,\mathbf{x}^{m-1}$, we get the following equivalence where the right hand side is an infinite disjunction over the Khovanskii formulas $H := H(\mathbf{x}^0,\dots,\mathbf{x}^{m-1},\bar z)$

$$\varphi \leftrightarrow \exists \bar u \bigvee_H \exists \bar z\ H(\mathbf{x}^0,\dots,\mathbf{x}^{m-1},\bar z)\wedge \varphi_H^*(\mathbf{x}^0,\dots,\mathbf{x}^m,\bar z,\bar u).$$

Note that in case we do have a non-trivial relation between the $\mathbf{x}^i$, $0 \leq i \leq m-1$, with coefficients in $\mathbb{Q}(\bar{c})$, they cannot be all ecl-independent over $\mathbb{Q}(\bar{c})$ by Corollary 3.8.

In the scheme below, since the extra-variables $\bar{u}$ that we added in order to only consider $\mathcal{L}_{-}$-terms, are in the dcl^K-closure (and in particular in the ecl^K-closure) of $\mathbf{x}^0, \ldots, \mathbf{x}^m$, we will assume that they occur within $\mathbf{x}^0$.

{DL}

Definition 4.2. The scheme $(\mathrm{DL})_E$ has the following form: for each $\mathcal{L}_\delta(\bar{c})$-formula $\varphi(\mathbf{x})$ which is a finite conjunction of $\mathcal{L}_\delta(\bar{c})$-equations of order m, for each Khovanskii $\mathcal{L}$-formula $H(\mathbf{x}^0, \ldots, \mathbf{x}^{m-1}, \bar{z})$, we have: $\forall \bar{d}\, \forall \mathbf{x}^0 \ \ldots \ \forall \mathbf{x}^m$

$$(\exists \bar{z} H(\mathbf{x}^0, \ldots, \mathbf{x}^{m-1}, \bar{z}) \wedge \varphi_H^*(\mathbf{x}^0, \ldots, \mathbf{x}^m, \bar{z})) \rightarrow (\exists \alpha\ \varphi(\alpha) \ \wedge\ \chi(\bar{\delta}^m(\alpha) - (\mathbf{x}^0, \ldots, \mathbf{x}^m), \bar{d})),$$

where φ_H^* is an $\mathcal{L}(\bar{\delta}^m(\bar{c}))$-formula as in Definition 4.1.

Note that by quantifying over the coefficients $\bar{c}$, this scheme is first-order.

{dense}

Remark 4.3. In a model $\mathcal{K}_\delta \models T_\delta$ of the scheme $(\mathrm{DL})_E$, the differential points are dense in all cartesian products of K. Let $O \subseteq K^{m+1}$ and $(a_0, \ldots, a_m) \in O$. Consider the $\mathcal{L}_\delta$-formula $\varphi(x) := \delta^m(x) = a_m$. The formula $\varphi^*(x_0, \ldots, x_m) := x_m = a_m$. Let $\bigwedge_{i=0}^{m-1} H_i(x_i) := x_i - a_i = 0$, we find a differential solution b such that $\delta^m(b) = a_m$ and $\bar{\delta}^{m-1}(b)$ is close to $(a_0, \ldots, a_{m-1})$. This is analogous to [14, Lemma 3.12].

The same argument shows that the subfield of constants C_K is dense in K (and recall that since δ is an E-derivation, C_K is an E-subfield of K which is relatively algebraically closed in K). We even have that $\mathrm{ecl}^K(C_K) = C_K$ by Lemmas 2.7, 2.15.

The main result of this section is:

{ec}

Theorem 4.4. *Let T be a model-complete complete theory of topological $\mathcal{L}$-fields endowed with a V-topology which is definable with corresponding formula χ. Assume that the models of T satisfy the schemes* (IFT) *and* (LFF)*. Then the class of existentially closed models of T_δ is axiomatized by $T_\delta \cup (\mathrm{DL})_E$.*

The above theorem will follow from Theorems 4.6 and 4.7.

The strategy of the proof is the following. First show that a model $\mathcal{K}_\delta \models T_\delta$ satisfying (IFT) and (LFF) can be embedded in $\tilde{K}_\delta \models T_\delta$ satisfying this scheme $(\mathrm{DL})_E$ (Theorem 4.6). Second show that if T is model-complete, then we may choose $\tilde{K} \models T$. Finally show that if $T_\delta \cup (\mathrm{DL})_E$ is consistent, then it gives an axiomatization of the existentially closed models of T_δ (Theorem 4.7). (We only showed the consistency under the hypotheses (IFT) and (LFF) and there is the question whether the scheme (LFF) is elementary).

We begin by realizing one instance of the scheme $(\mathrm{DL})_E$ in a differential extension of $\mathcal{K}_\delta$.

{iter1}

Lemma 4.5. *Let $\mathcal{K}_\delta \models T_\delta$ be a topological $\mathcal{L}$-field endowed with a V-topology which is definable with corresponding formula χ. Suppose $\mathcal{K}$ satisfies* (IFT) *and* (LFF)*. Let $\mathcal{M}$ be a $|K|^+$- saturated elementary $\mathcal{L}$-extension of $\mathcal{K}$. Let $\varphi(\mathbf{x})$ be a finite conjunction of $\mathcal{L}_\delta(K)$-equations of order m, let $H(\mathbf{x}^0, \ldots, \mathbf{x}^{m-1}, \bar{z})$ be a Khovanskii formula with $\mathbf{x}^0 = \mathbf{x}, |\mathbf{x}^i| = n$, $0 \leq i \leq m-1$. Assuming that for some $\mathbf{a} := (\mathbf{a}^0 \ \ldots \ , \mathbf{a}^m) \in K$, $|\mathbf{a}^i| = n$, $0 \leq i \leq m$, we have:*

$$\mathcal{K} \models \exists \bar{z}\, (H(\mathbf{a}^0, \ldots, \mathbf{a}^{m-1}, \bar{z}) \wedge \varphi_H^*(\mathbf{a}^0, \ldots, \mathbf{a}^m, \bar{z})),$$

then for any $\bar{d} \in K$, we can find a tuple of elements $\bar{\gamma} \in M$ and we can extend δ on $\mathrm{ecl}^M(K, \bar{\gamma})$ such that for some $\alpha \in \mathrm{ecl}^M(K, \gamma)$

$$\mathrm{ecl}^M(K, \gamma) \models \varphi(\alpha) \ \wedge \ \bar{\chi}(\bar{\delta}^m(\alpha) - \mathbf{a}, \bar{d}).$$

Proof. First let us observe that the saturation hypothesis on M is only used in order to find K-small elements which are ecl-independent over K.

For sake of clarity, suppose first that $m = 1$. Let $\mathbf{a} := (\mathbf{a}^0, \mathbf{a}^1)$. Let $\bar{\delta}^m(\bar{c})$ be the parameters from K occurring in the $\mathcal{L}_\delta$-formula φ, in the Khovanskii formula H and in the formula φ^*_H. Suppose H is of the form $\bigwedge_{i=1}^{n-\ell} H_i(a^0_{\ell+i}, \mathbf{z}_{\ell+i})$, $0 < \ell < n$. Let $\bar{u} := (\mathbf{u}_{\ell+1}, \ldots, \mathbf{u}_n) \in K$ be such that $\mathcal{K} \models \bigwedge_{i=1}^{n-\ell} H_i(a^0_{\ell+i}, \mathbf{u}_{\ell+i})$. Let $n_i := |\mathbf{u}_{\ell+i}|$, $1 \leq i \leq n - \ell$ and N be the length of $(\mathbf{a}, \bar{u})$.

By Lemma 3.20, we can find $t_1, \ldots, t_\ell \in M$ which are K-small and ecl-independent over K. Let $\mathbf{t}_{[\ell]} := (t_1, \ldots, t_\ell)$.

Let $V_{\varphi^*_H}$ be the system of E-polynomial equations occurring in φ^*_H, in unknowns $\mathbf{x}^1_{[\ell]} := (x^1_1, \ldots, x^1_\ell)$ over $\mathbb{Q}\langle\bar{c}\rangle_\delta$, $\mathbf{x}^0$ and $\bar{z} := (\mathbf{z}_{\ell+1}, \ldots, \mathbf{z}_n)$. We denote the corresponding tuple of E-polynomials by $\bar{f} := \bar{f}(\mathbf{x}^0, \bar{z}, \mathbf{x}^1_{[\ell]})$ and, as in Definition 3.13, we will also use the notation $\bar{f}_{\mathbf{x}^0,\bar{z}}(\mathbf{x}^1_{[\ell]})$, in order to stress which are the variables we consider as unknowns. (Recall that the variables x^1_j, $\ell + 1 \leq j \leq n$, have been replaced by rational functions $t^{1,*}_j$ depending on $\mathbf{x}^0_{[\ell]}, \mathbf{x}^1_{[\ell]}, x^0_j, \mathbf{z}_j$.)

First assume that $|\bar{f}| = \ell$ and $\mathbf{a}^1_{[\ell]}$ is a regular zero of $V_\ell(\bar{f}_{\mathbf{a}^0,\bar{u}})$. The second part of the proof (in the case $m = 1$) will consist in desingularizing $V_{\varphi^*_H}$.

Set $\bar{y} = (\mathbf{x}^0, \bar{z})$ and let $(f_{\bar{y},i})_{i=1}^{\ell}$ enumerate the tuple of E-polynomials $\bar{f}_{\mathbf{x}^0,\bar{z}}$. Then we apply directly hypothesis (IFT). There exist O_1 be a definable neighbourhood of $(\mathbf{a}^0, \bar{u})$ and O_2 be a definable neighbourhood of $\mathbf{a}^1_{[\ell]}$ and definable C^∞ functions g_i from O_1 to O_2, $1 \leq i \leq \ell$, such that

$$\bigwedge_{i=1}^{\ell} g_i(\mathbf{a}^0, \bar{u}) = a^1_i \quad \wedge \quad \forall\, \bar{y} \in O_1 \ \ (\bigwedge_{i=1}^{\ell} f_{\bar{y},i}(g_1(\bar{y}), \ldots, g_\ell(\bar{y})) = 0).$$

Recall that we put the product topology on M^N with N be the length of $(\mathbf{a}^0, \bar{u})$. Let π be the projection sending a tuple $(\mathbf{a}^0, \mathbf{u})$ of M^N to the subtuple $\mathbf{a}^0_{[\ell]} \in M^\ell$ and π_i the projection sending $(\mathbf{a}^0, \mathbf{u})$ to the subtuple $(a^0_{\ell+i}, \mathbf{u}_{\ell+i}) \in M^{n_i+1}$, $1 \leq i \leq n - \ell$.

Let $(a^0_{\ell+i}, \mathbf{u}_{\ell+i})$ be regular zeroes of each system $H_i(x^0_{\ell+i}, \mathbf{z}_{\ell+i})$, $1 \leq i \leq n - \ell$, over $\mathbb{Q}(\bar{c}, \mathbf{a}^0_{[\ell]})$. For each $1 \leq i \leq n - \ell$, we apply (IFT) in $\mathcal{M}$ and find a neighbourhood $O_{1,1}$ of $\mathbf{a}^0_{[\ell]}$ with $O_{1,1} \subseteq \pi(O_1)$ and a neighbourhood $O_{1,\ell+i}$ of $(a^0_{\ell+i}, \mathbf{u}_{\ell+i})$ with $O_{1,\ell+i} \subseteq \pi_i(O_1)$ and definable functions $h_{i,0}, \ldots, h_{i,n_i}$ from $O_{1,1}$ to $O_{1,\ell+i}$ such that

$$(12)\quad \bigwedge_{i=1}^{n-\ell} h_{i,0}(\mathbf{a}^0_{[\ell]}) = a^0_{\ell+i} \wedge \bigwedge_{j=1}^{n_i} h_{i,j}(\mathbf{a}^0_{[\ell]}) = u_{\ell+i,j} \wedge \forall\, \bar{y} \in O_{1,1}\, (\bigwedge_{i=1}^{n-\ell} H_i(h_{i,0}(\bar{y}), \ldots, h_{i,n_i}(\bar{y}))).$$

Let $\bar{h}_i := (h_{i,0}(\bar{w}), \ldots, h_{i,n_i}(\bar{w}))$ with $\bar{w} = (w_1, \ldots, w_\ell)$. Applying $\bar{h}_i$ to $(\mathbf{a}^0_{[\ell]} + \mathbf{t}_{[\ell]})$, we get a solution to each system $H_i(x^0_{\ell+i}, \mathbf{z}_{\ell+i})$, close to $(a^0_{\ell+i}, \mathbf{u}_{\ell+i})$, $1 \leq i \leq n - \ell$. Denote this solution by $(a'_{\ell+i}, \mathbf{u}'_{\ell+i})$, $1 \leq i \leq n - \ell$. Let

$$(\widetilde{\mathbf{a}}, \widetilde{\mathbf{u}}) := (\mathbf{a}^0_{[\ell]} + \mathbf{t}_{[\ell]}, a'_{\ell+1}, \ldots, a'_n, \mathbf{u}'_{\ell+1}, \ldots, \mathbf{u}'_n).$$

Since $(\widetilde{\mathbf{a}},\widetilde{\mathbf{u}})$ belongs to $O_1(M)$, we may apply the functions $g_1,\dots,g_\ell$ in order to obtain $(g_1(\widetilde{\mathbf{a}},\widetilde{\mathbf{u}}),\dots,g_\ell(\widetilde{\mathbf{a}},\widetilde{\mathbf{u}}))\in V_\ell(\bar f_{\widetilde{\mathbf{a}},\widetilde{\mathbf{u}}})$. Set $(\tilde b_1,\dots,\tilde b_\ell):=(g_1(\widetilde{\mathbf{a}},\widetilde{\mathbf{u}}),\dots,g_\ell(\widetilde{\mathbf{a}},\widetilde{\mathbf{u}}))$.

Since now $a_1^0+t_1,\dots,a_\ell^0+t_\ell$ are ecl^K-independent, we may define

$$\delta(a_1^0+t_1):=\tilde b_1,\dots,\delta(a_\ell^0+t_\ell)=\tilde b_\ell. \tag{13}$$

(13) {star}

Note that the values of the successive derivatives of $\tilde b_1,\dots,\tilde b_\ell$ are determined since we can express $\delta(\tilde b_1),\dots,\delta(\tilde b_\ell)$ using that $(\tilde b_1,\dots,\tilde b_\ell)$ is a regular zero of $V_\ell(\bar f_{\widetilde{\mathbf{a}},\widetilde{\mathbf{u}}})$. Note that $\tilde b_1,\dots,\tilde b_\ell\in \mathrm{ecl}^M(K,\mathbf{a}^0_{[\ell]}+\mathbf{t}_{[\ell]})$. By equation (12), $a'_{\ell+1},\dots,a'_n\in\mathrm{ecl}^M(\bar c,\mathbf{a}^0_{[\ell]}+\mathbf{t}_{[\ell]})$, we can also express their derivatives in terms of $\mathbf{a}^0_{[\ell]}+\mathbf{t}_{[\ell]},a'_{\ell+1},\dots,a'_n$, the witnesses $\mathbf{u}'_{\ell+1},\dots,\mathbf{u}'_n$ and the derivatives of $\mathbf{a}^0_{[\ell]}+\mathbf{t}_{[\ell]}$, namely $\tilde b_1,\dots,\tilde b_\ell$. So first we extend δ on $\mathrm{ecl}^M(K,\mathbf{a}^0_{[\ell]}+\mathbf{t}_{[\ell]})$ sending the tuple $\mathbf{a}^0_{[\ell]}+\mathbf{t}_{[\ell]}$ to $(\tilde b_1,\dots,\tilde b_\ell)$ and then by Corollary 2.19 to M. This extension is uniquely determined on the subfield of M generated by K, $\mathbf{a}^0_{[\ell]}+\mathbf{t}_{[\ell]},a'_{\ell+1},\dots,a'_n,\mathbf{u}'_{\ell+1},\dots,\mathbf{u}'_n$ and $\tilde b_1,\dots,\tilde b_\ell$.

Now assume that either the tuple of E-polynomials $\bar f_{\mathbf{x}^0,\bar z}$ has length $<\ell$ or that $\mathbf{a}^1_{[\ell]}$ is not a regular zero of $V_\ell(\bar f_{\mathbf{a}^0,\bar u})$. Let $\mathbf{s}'_1:=(s'_{1,1},\dots,s'_{1,n})$, $\mathbf{s}'_2:=(\mathbf{s}'_{2,\ell+1},\dots,\mathbf{s}'_{2,n})$,

$$S_{(\mathbf{a}^0,\bar u,\mathbf{a}^1_{[\ell]})}:=\{(\mathbf{s}'_1,\mathbf{s}'_2,\mathbf{r}')\in K^{N+\ell}:|\mathbf{s}'_1|=|\mathbf{x}^0|,|\mathbf{s}'_2|=|\bar z|,|\mathbf{r}'|=\ell\ \&$$
$$\mathbf{r}'\in V_\ell(\bar f_{\mathbf{s}'_1,\mathbf{s}'_2})\ \&\ \bigwedge_{i=1}^{n-\ell}H_i(s'_{1,\ell+i},\mathbf{s}'_{2,\ell+i})\}\cap(\mathbf{a}^0,\bar u,\mathbf{a}^1_{[\ell]})+\bar\chi(K,\bar d).$$

The set $S_{(\mathbf{a}^0,\bar u,\mathbf{a}^1_{[\ell]})}$ is non-empty since it contains $(\mathbf{a}^0,\bar u,\mathbf{a}^1_{[\ell]})$. Let $\mathcal{S}_{(\mathbf{a}^0,\bar u,\mathbf{a}^1_{[\ell]})}$ be the neighbourhood system containing $(\mathbf{a}^0,\bar u,\mathbf{a}^1_{[\ell]})$. Let $R_{N+\ell}$ be a noetherian subring of $\mathfrak{D}^{N+\ell}(\mathcal{S}_{(\mathbf{a}^0,\bar u,\mathbf{a}^1_{[\ell]})})$ closed under differentiation and containing the maps induced by the E-polynomials $\bar f(\mathbf{x}^0,\bar z,\mathbf{x}^1_{[\ell]})$. Denote by $R^{<\omega}_{N+\ell}$ the set of all finite tuples of elements of $R_{N+\ell}$. For $(\mathbf{s}'_1,\mathbf{s}'_2,\mathbf{r}')\in S_{(\mathbf{a}^0,\bar u,\mathbf{a}^1_{[\ell]})}$, consider the tuple $((\mathbf{s}'_1,\mathbf{s}'_2,\mathbf{r}'),\bar q)$ where $\bar q\in R^{<\omega}_{N+\ell}$, $(\mathbf{s}'_1,\mathbf{s}'_2,\mathbf{r}')\in V(\bar q)$ & $\det(J_{\bar q_{\mathbf{s}'_1,\mathbf{s}'_2}}(\mathbf{r}')=0$ whenever $|\bar q|=\ell$. Denote this set of tuples by Ann. By assumption the tuple $((\mathbf{a}^0,\bar u,\mathbf{a}^1_{[\ell]}),\bar f)$ does belong to Ann.

Suppose that we have $((\mathbf{s}_{1,n},\mathbf{s}_{2,n},\mathbf{r}_n),\bar q_n)\in Ann$, $n\in\mathcal{N}$, with the ideals $\langle\bar q_n\rangle$ forming an increasing chain, by noetherianity of $R_{N+\ell}$ we can assume such a chain is finite and there is m_0 such that for all $m\geq m_0$, $\langle\bar q_{m_0}\rangle=\langle\bar q_m\rangle$, for all $m\geq m_0$.

So we may choose among $(\mathbf{s}'_1,\mathbf{s}'_2,\mathbf{r}')\in S_{(\mathbf{a}^0,\bar u,\mathbf{a}^1_{[\ell]})}$, those such that there is $\bar q\in R^{<\omega}_{N+\ell}$ such that $((\mathbf{s}'_1,\mathbf{s}'_2,\mathbf{r}'),\bar q)\in Ann$ and $\langle\bar q\rangle$ maximal in $R^{N+\ell}$ whenever $\bar q$ has also the property that letting $\bar q=(q_1,\dots,q_k)$, $\nabla q_1(\mathbf{s}'_1,\mathbf{s}'_2),\mathbf{r}'),\dots,\nabla q_k(\mathbf{s}'_1,\mathbf{s}'_2,\mathbf{r}')$ are K-linearly independent.

$I_{(\mathbf{s}'_1,\mathbf{s}'_2,\mathbf{r}')}:=\{\bar q\in R^{<\omega}_{N+\ell}\colon(\mathbf{s}'_1,\mathbf{s}'_2,\mathbf{r}')\in V(\bar q)\,\&\,\det(J_{\bar q_{\mathbf{s}_1,\mathbf{s}_2}}(\mathbf{r}')=0$ whenever $|\bar q|=\ell\}$. Suppose that we have $\bar q_n\in I_{(\mathbf{s}_{1,n},\mathbf{s}_{2,n},\mathbf{r}_n)}$, $n\in\mathcal{N}$, with the ideals $\langle\bar q_n\rangle$ forming an increasing chain, by noetherianity of $R_{N+\ell}$ we can assume such a chain is finite. So among the ideals $Ann^{R_{N+\ell}}(\mathbf{s}'_1,\mathbf{s}'_2,\mathbf{r}')$, namely $I_{(\mathbf{s}'_1,\mathbf{s}'_2,\mathbf{r}')}$. Since $R_{N+\ell}$ is noetherian, there is $(\mathbf{s}_1,\mathbf{s}_2,\mathbf{r})\in S_{(\mathbf{a}^0,\bar u,\mathbf{a}^1_{[\ell]})}$ with $|\mathbf{r}|=\ell$, $|\mathbf{s}_1|=n$ and $|(\mathbf{s}_1,\mathbf{s}_2)|=N$ such that $I_{(\mathbf{s}_1,\mathbf{s}_2,\mathbf{r})}$ is maximal and we can find $h_1(\mathbf{x}^0,\bar z,\mathbf{x}^1_{[\ell]}),\dots,h_p(\mathbf{x}^0,\bar z,\mathbf{x}^1_{[\ell]})\in R_{N+\ell}$ with p maximal (†) such that $\bigwedge_{i=1}^p h_i(\mathbf{s}_1,\mathbf{s}_2,\mathbf{r})=0$ and for $k=\min\{p,\ell\}$, $\nabla h_{1\,(\mathbf{s}_1,\mathbf{s}_2)}(\mathbf{r}),\dots,\nabla h_{k\,(\mathbf{s}_1,\mathbf{s}_2)}(\mathbf{r})$ are K-linearly independent and $\nabla h_1(\mathbf{s}_1,\mathbf{s}_2,\mathbf{r}),\dots,\nabla h_p(\mathbf{s}_1,\mathbf{s}_2,\mathbf{r})$ are K-linearly independent.

Note that $p \geq 1$ since E-polynomials whose all partial derivatives are equal to 0 is itself 0 and the map sending an E-polynomial to the corresponding function is injective in this case.

If $p \geq \ell$, we consider the map $\Lambda : (y_1, \ldots, y_N, x_1, \ldots, x_\ell) \mapsto det(\partial_{x_j} h_i)_{1\leq i,j\leq \ell})$. By construction $\Lambda(\mathbf{s}_1, \mathbf{s}_2, \mathbf{r}) \neq 0$ so it doesn't vanish on a neighbourhood of $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{r})$. So, there is a neighbourhood U_0 of $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{r})$ where $\Lambda \restriction U_0$ is invertible. We have a map $\hat{}: \mathfrak{D}^{N+\ell}(\mathcal{S}_{(\mathbf{s}_1,\mathbf{s}_2,\mathbf{r})}) \to \mathfrak{D}^{\ell}(\mathcal{S}_{(\mathbf{s}_1,\mathbf{s}_2)})$ (applying (IFT) to $(h_1, \ldots, h_\ell)$. Consider $R_{N+\ell}[\Lambda^{-1}]$ and let $M \subseteq \mathfrak{D}(\mathcal{S}_{(\mathbf{s}_1,\mathbf{s}_2)})$ be its image by the map $\hat{}$. Let $I := \{h \in M : h(\mathbf{s}_1, \mathbf{s}_2) = 0\}$ be an ideal in M.

If $I = \{0\}$, then since $\hat{f}_i \in I$, f_i vanishes in a neighbourhood of $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{r})$. So when we modify $\mathbf{s}_1, \mathbf{s}_2, \mathbf{r})$ as in the first part of the proof, using that $\mathbf{r}$ is a regular zero of $(h_1, \ldots, h_\ell)$, we still get that $\bar{f}$ is zero on the modified tuple. If $I \neq \{0\}$, let us show that we contradict the choice of $h_1, \ldots, h_p$. By (LFF), I is not closed under differentiation. So there is $h \in I$ with $h \neq 0$ and so we can add to $h_1, \ldots, h_p$ contradicting maximality (using Lemma 3.16)

If $p < \ell$, first note since for $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{r}) \in S_{(\mathbf{a},\bar{u},\mathbf{a}^1_{[\ell]})}$, $\mathbf{r} \in V_\ell(\bar{f}_{\mathbf{s}_1,\mathbf{s}_2})$. Then let $1 \leq i_1 < \ldots < i_p \leq \ell$ be strictly increasing indices such that the determinant of the matrix $(\nabla_{\bar{i}} h_{1\,(\mathbf{s}_1,\mathbf{s}_2)}(\mathbf{r}), \ldots, \nabla_{\bar{i}} h_{p\,(\mathbf{s}_1,\mathbf{s}_2)}(\mathbf{r}))$ is nonzero, with $\bar{i} := (i_1, \ldots, i_p)$. Decompose $\mathbf{r}$ into two subtuples: $\mathbf{r}_{[p]}$ and $\mathbf{r}_{[\ell-p]}$ (see Notation 3.4). We will add $\mathbf{r}_{[\ell-p]}$ to the parameters $(\mathbf{s}_1, \mathbf{s}_2)$ and apply the hypothesis (IFT) to the corresponding square system. So we can find in a neighbourhood of $\mathbf{r}_{[p]}$ a point satisfying that system with coefficients close to $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{r}_{[\ell-p]})$ and still get that this point belongs to $V(\bar{f})$, using Proposition 3.18 (b), since we assumed p maximal (†). (This is where we use the hypothesis (LFF).)

Assume now that $m > 1$. Then we replace in the above discussion $\mathbf{a}^1_{[\ell]}$ by $\mathbf{a}^m_{[\ell_{m-1}]}$ and we proceed as before. □

{emb_delta} **Theorem 4.6.** *Let T be a model-complete theory of topological $\mathcal{L}$-fields. Let $\mathcal{K} \models T$ be a topological $\mathcal{L}$-field endowed with a V-topology which is definable with corresponding formula χ. Suppose $\mathcal{K}$ satisfies* (IFT) *and* (LFF). *Then the differential expansion $\mathcal{K}_\delta$ can be embedded in a model $\tilde{\mathcal{K}}_\delta$ of $T_\delta \cup (\mathrm{DL})_E$.*

Proof: We adapt [14, Lemma 3.7] and [14, Proposition 3.9] to this exponential setting. The differential extension $\tilde{\mathcal{K}}_\delta$ will be built as the union of a chain of differential extensions of $\mathcal{K}_\delta$ which will be in addition $\mathcal{L}$-elementary extensions of $\mathcal{K}$. In particular, we get that $\tilde{\mathcal{K}}$ is an $\mathcal{L}$-elementary extension of $\mathcal{K}$. We first construct such extension $\tilde{\mathcal{K}}_\delta$ where all the instances of the scheme $(\mathrm{DL})_E$ with coefficients in K are satisfied using transfinite induction and then we repeat the construction replacing in the previous argument $\mathcal{K}_\delta$ by $\tilde{\mathcal{K}}_\delta$ and we do it ω times. The union of this chain of extensions will be a model of the scheme $(\mathrm{DL})_E$ and an elementary extension of $\mathcal{K}$ (since T is model-complete).

It suffices to show that given an instance of the scheme $(\mathrm{DL})_E$, we can find an $\mathcal{L}_\delta$ extension $\mathcal{K}_1$ of $\mathcal{K}_\delta$ where it is satisfied, with $\mathcal{K} \preceq \mathcal{K}_1$.

Let $\mathbf{x} = (x_1, \ldots, x_n)$, let $\varphi(\mathbf{x})$ be an $\mathcal{L}_\delta(K)$-formula which is a conjunction of $\mathcal{L}_\delta(K)$-equations of order m, and let $H(\mathbf{x}^0, \ldots, \mathbf{x}^m, \bar{z})$ be a Khovanskii $\mathcal{L}$-formula with $\mathbf{x}^0 = \mathbf{x}$. Let $\bar{\chi}(K, \bar{d})$ be a definable neighbourhood of 0 (in $K^{n(m+1)}$) with $\bar{d} \in K$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in K$ and $\bar{a} = (\mathbf{a}^0, \ldots, \mathbf{a}^m) \in K$, where $\mathbf{a}^0 := \mathbf{a}$ be such that

$$\exists \bar{z} H(\mathbf{a}^0, \ldots, \mathbf{a}^{m-1}, \bar{z}) \wedge \varphi^*_H(\bar{a}, \bar{z}),$$

holds in $\mathcal{K}$.

In Lemma 4.5, we constructed a differential extension $\mathcal{K}_1$ of $\mathcal{K}$ containing an element α such that $\varphi(\alpha)$ holds and such that $\bar{\delta}^m(\alpha)$ is close to $\bar{a}$, with respect to a given neighbourhood $\bar{\chi}(\cdot, \bar{d})$ of 0. □

Recall that $\mathcal{L}$ is a first-order language satisfying the assumptions of section 3.4.

{DLE}

Theorem 4.7. *Let T be a model-complete theory of topological $\mathcal{L}$-fields. Assume that $\mathcal{K} \models T$ and that the differential expansion $\mathcal{K}_\delta$ is a model of $T_\delta \cup (\mathrm{DL})_E$. Then $\mathcal{K}_\delta$ is existentially closed in the class of models of T_δ. In particular if the theory $T_\delta \cup (\mathrm{DL})_E$ is consistent, then it is model-complete.*

Proof: Let $\mathcal{K}_\delta \models T_\delta \cup (\mathrm{DL})_E$ and suppose that $\mathcal{K}_\delta \subseteq \tilde{\mathcal{K}}_\delta$ with $\tilde{\mathcal{K}}_\delta \models T_\delta$.

Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\xi(\mathbf{x})$ be a quantifier-free $\mathcal{L}_\delta(K)$-formula of order m and assume that for some tuple $\mathbf{a} \in \tilde{K}$, $\tilde{\mathcal{K}} \models \xi(\mathbf{a})$. Since T is model-complete and $\mathcal{K} \models T$, we may assume that we are in the case where $m \geq 1$. Furthermore we may assume that $\xi(\mathbf{x})$ is of the form $\varphi(\mathbf{x}) \wedge \bar{\delta}^m(\mathbf{x}) \in O$, where $\varphi(\mathbf{x})$ is a conjunction of $\mathcal{L}_\delta(K)$-equations and O is an $\mathcal{L}(K)$-definable open subset of some cartesian product of $\tilde{K}$.

If the formula is of the form $\bar{\delta}^m(\mathbf{x}) \in O$, then we may conclude using the density of differential points (see Remark 4.3). So, from now on, assume that there is a non-trivial $\mathcal{L}_\delta(K)$-equation occurring in $\varphi(\mathbf{x})$.

We consider all the ecl^K-relations that may occur within the tuple $\bar{\delta}^{m-1}(\mathbf{a})$. Set $\mathbf{a}^i := \delta^i(\mathbf{a})$, $0 \leq i \leq m$. If all $\mathbf{a}^i$, $0 \leq i \leq m-1$, are ecl^K-independent, then by the scheme (DL_E), we can find a differential solution in $\mathcal{K}$ close to $\mathbf{a}$. So from now on let us assume this is not the case. Let $\mathbf{a}^0_{[\ell]} = (a_1, \dots, a_\ell)$ be the longest sub-tuple of $\mathbf{a} = (a_1, \dots, a_n)$ which is ecl-independent over K (which we may assume by re-indexing to be an initial subtuple since ecl has the exchange property). (If there is no such ℓ, then $a_1, \dots, a_n \in \mathrm{ecl}^{\tilde{K}}(K)$ and their successive derivatives can be expressed in terms of $a_i, \bar{u}_i$ for some tuples of elements of $\tilde{K}$, $1 \leq i \leq n$, and elements from K. So we can transform the $\mathcal{L}_\delta$-formula φ into an $\mathcal{L}$-formula and use the fact that T is model-complete.) Then we consider the ecl-relations among $\mathbf{a}^1$ over K and $\mathbf{a}^0_{[\ell]}$. Note that we certainly have ecl-relations among $\mathbf{a}^1_{[n-\ell]}$ and $\mathbf{a}^0$. Again we possibly re-index the subtuple $\mathbf{a}^1_{[\ell]}$ such that these ecl-relations occur among the co-initial part of $\mathbf{a}^1_{[\ell]}$. We rename the corresponding subtuple $\widetilde{\mathbf{a}^1}$ and possibly permute the indices of $\widetilde{\mathbf{a}^0}$ to match indices. We proceed in this way getting successively $\widetilde{\mathbf{a}^2}, \dots, \widetilde{\mathbf{a}^{m-1}}$.

Namely, suppose we got $\widetilde{\mathbf{a}^i}$, $0 \leq i < m-1$. We consider the ecl-relations among $\mathbf{a}^{i+1}$ over K and $\widetilde{\mathbf{a}^0}, \dots \widetilde{\mathbf{a}^i}$. Again we re-index in order that the ecl-relations only occur in the co-initial part of $\mathbf{a}^{i+1}$ and we rename the corresponding subtuple $\widetilde{\mathbf{a}^{i+1}}$ as well as possibly permuting the indices of $\widetilde{\mathbf{a}^0}, \dots, \widetilde{\mathbf{a}^i}$ to match indices. Assume the length of $\widetilde{a^i}$ is equal to ℓ_i, $0 \leq i \leq m-1$ and by the way it was constructed $n \geq \ell = \ell_0 \geq \ell_1 \geq \dots \geq \ell_{m-1} \geq 0$.

For sake of simplicity let us assume that $m = 1$. Let $H_1(a_{\ell+1}, \bar{u}_{\ell+1}), \dots, H_{n-\ell}(a_n, \bar{u}_n)$ be $n - \ell$ Khovanskii systems over $K(\mathbf{a}_{[\ell]})$, setting $\mathbf{a}_{[\ell]} = \mathbf{a}^0_{[\ell]}$, witnessing that $a_{\ell+1}, \dots, a_n$ belong to $\mathrm{ecl}^L(K(\mathbf{a}_{[\ell]}))$.

Note that by Lemma 2.15 (and its proof), this implies that we can express $\delta(a_{\ell+i}), \delta(\bar{u}_{\ell+i})$ in terms of $a, \bar{u}_{\ell+i}, \delta(a_1), \dots, \delta(a_\ell)$, $1 \leq i \leq n-\ell$ and finitely many elements of K and their derivative occurring as coefficients of the E-polynomials appearing in the Khovanskii systems. Let $\bar{u} := (\bar{u}_{\ell+1}, \dots, \bar{u}_n)$.

Let φ^*_H be the $\mathcal{L}$-formula constructed from φ and these Khovanskii systems (see Definition 4.1).

Since T is model-complete, there exists $\bar{\gamma} \in O(K)$ and $\bar{z} \in K$ such that $\varphi_H^*(\bar{\gamma}, \bar{z})$ holds.
Then we apply the scheme $(\mathrm{DL})_E$ and get a differential solution $\bar{\delta}^m(\alpha) \in K$ satisfying φ_H^* and close to $\bar{\gamma}$. So $\mathcal{K}_\delta \models \xi(\alpha)$. □

4.3. **Geometric version of the scheme** $(\mathrm{DL})_E$**.** In this section we translate in geometric terms the scheme $(\mathrm{DL})_E$. It is similar in spirit to the differential lifting scheme introduced by Pierce and Pillay, which gave another axiomatization of the class of differentially closed fields of characteristic 0 [25].

For $n \leq m \in \mathbb{N}^*$, let $\pi_n^m : K^m \to K^n$ be the projection onto the first n coordinates and let $\pi_{(n,n)}^{2m} : K^m \times K^m \to K^n \times K^n : (x, y) \mapsto (\pi_n^m(x), \pi_n^m(y))$.

{schgeo} **Definition 4.8.** *Let $\mathcal{K}_\delta \models T_\delta$, then $\mathcal{K}_\delta$ satisfies the scheme $(\mathrm{DLg})_E$ if the following holds. Let $\tilde{\mathcal{K}}$ be a $|K|^+$-saturated $\mathcal{L}$-elementary extension of $\mathcal{K}$. Let $W := W(\bar{f}) \subseteq \tilde{\mathcal{K}}^{2n}$ be an E-variety defined over K and let $\bar{\chi}(K, \mathbf{d})$ be a neighbourhood of 0 in K^{2n} with $\mathbf{d}$ in K. Suppose that $0 \leq \dim^{\tilde{\mathcal{K}}}(\pi_n^{2n}(W)/K) = \ell < n$. Let $\mathbf{a}$ be a generic point of $\pi_n^{2n}(W)$ with $\mathbf{a}_{[\ell]}$ a subtuple of $\mathbf{a}$ of* ecl*-independent elements over K and let $(\mathbf{a}, \mathbf{b})$ be a generic point of W. Let $\mathbf{u}_{\ell+i}$ be tuples of elements in $\tilde{K}$, $1 \leq i \leq n - \ell$, witnessing that each component of $\mathbf{a}_{[n-\ell]}$ belongs to* $\mathrm{ecl}^{\tilde{\mathcal{K}}}(K, \mathbf{a}_{[\ell]})$*. Set $\bar{u} := (\mathbf{u}_{\ell+1}, \ldots, \mathbf{u}_n) \in \tilde{K}^m$ and assume that $(\mathbf{a}, \mathbf{b}) \in \pi_{(n,n)}^{2(n+m)}(\tau(Ann^{K[\mathbf{X}]^E}(\mathbf{a}, \bar{u})))$, $|\mathbf{X}| = m + n$, then we can find a differential point $(\alpha, \delta(\alpha)) \in W \cap K^{2n}$ with $\bar{\chi}((\alpha, \delta(\alpha)) - (\mathbf{a}, \mathbf{b}), \mathbf{d})$.*

The scheme $(\mathrm{DLg})_E$ as stated is not first-order. The first issue concerns expressing that a tuple is generic and the second is that a priori we have to consider all the E-polynomials in an annihilator. Concerning the second one, keeping the same notations as in Definition 4.8, one only needs the E-polynomials in $Ann^{K[\mathbf{X}]^E}(\mathbf{a}, \bar{u})$ occurring in the Khovanskii systems used to express that each component of $\mathbf{a}_{[n-\ell]}$ belongs to $\mathrm{ecl}(K, \mathbf{a}_{[\ell]})$.

Although $(\mathrm{DLg})_E$ is not first-order, one can easily see that if $\mathcal{K}_\delta$ satisfies the scheme $(\mathrm{DLg})_E$, then it satisfies the scheme $(\mathrm{DL})_E$ and conversely.
Suppose $\mathcal{K}_\delta$ satisfies the scheme $(\mathrm{DL})_E$ and that we are in the setting of $(\mathrm{DLg})_E$. Similarly to the construction in Definition 4.1, let φ_H^* be the formula obtained from $\bar{f}(\mathbf{x}^0, \mathbf{x}^1) = \bar{0}$, $|\mathbf{x}^i| = n$, together with a Khovanskii formula $H(\mathbf{x}^0, \bar{u})$ witnessing that each component of $\mathbf{a}_{[n-\ell]}$ belongs to $\mathrm{ecl}^{\tilde{\mathcal{K}}}(K, \mathbf{a}_{[\ell]})$. Because $(\mathbf{a}, \mathbf{b}) \in \pi_{(n,n)}^{2(n+m)}(\tau(Ann^{K[\mathbf{X}]^E}(\mathbf{a}, \bar{u}))$, one obtains $\varphi_H^*(\mathbf{a}, \mathbf{b})$. Hence by $(\mathrm{DL})_E$, $\exists \alpha\ \varphi(\alpha) \wedge \bar{\chi}((\alpha, \delta(\alpha)) - (\mathbf{a}, \mathbf{b}), \mathbf{d})$.
Conversely, suppose $\mathcal{K}_\delta$ satisfies the scheme $(\mathrm{DLg})_E$, and that we are in the setting of $(\mathrm{DL})_E$. Let $\varphi(\mathbf{x})$ be a $\mathcal{L}_\delta(\bar{c})$-formula which is a finite conjunction of $\mathcal{L}_\delta(\bar{c})$-equations of order m, and let $\varphi_H^*(\mathbf{x}^0, \ldots, \mathbf{x}^m, \bar{z})$ and $H(\mathbf{x}^0, \ldots, \mathbf{x}^{m-1}, \bar{z})$ be the associated $\mathcal{L}$-formula and Khovanskii $\mathcal{L}$-formula as constructed in Definition 4.1. Let

$$\theta_H^*(\mathbf{x}^0, \ldots, \mathbf{x}^{m-1}, \mathbf{y}^{m-1}, \bar{z}) := \varphi_H^*(\mathbf{x}^0, \ldots, \mathbf{x}^{m-1}, \mathbf{y}^{m-1}, \bar{z}) \wedge \mathbf{y}^0 = \mathbf{x}^1 \wedge \ldots \wedge \mathbf{y}^{m-1} = \mathbf{x}^m$$

$$\mathbf{u}^0 := (\mathbf{x}^0, \ldots, \mathbf{x}^{m-1})$$

$$\mathbf{u}^1 := (\mathbf{x}^1, \ldots, \mathbf{x}^m)$$

Suppose $|\mathbf{u}^i| = n$. Let $W(\bar{f}) = V_{\theta_H^*}$. By construction, $\mathbf{u}^0$ is a generic point of $\pi_n^{2n}(W)$, $(\mathbf{u}^0, \mathbf{u}^1)$ is a generic point of W and belongs to $\pi_{(n,n)}^{2(n+m)}(\tau(Ann^{K[\mathbf{X}]^E}(\mathbf{a}, \bar{z}))$. Hence by $(\mathrm{DLg})_E$ we can find a differential point $(\alpha, \delta(\alpha)) \in W \cap K^{2n}$ with $\bar{\chi}((\alpha, \delta(\alpha)) - (\mathbf{u}^0, \mathbf{u}^1), \mathbf{d})$. Writing α as $\beta^0, \ldots, \beta^{m-1}$, one obtains $\varphi(\beta^0)$.

5. Model-complete theories of (partial) exponential fields

{exa}

In this section, we apply our previous results to theories of topological fields $\mathcal{K}$ where the topology is either induced by an ordering $<$ or by a valuation map v. In the case of valued field $\mathcal{K} := (K, v)$ we will replace the valuation map by a binary relation div defined as follows:

$$v(a) \leq v(b) \text{ iff } a \text{ div } b.$$

Denote by $\mathcal{O}_K$ be the valuation ring of K and $\mathcal{M}_K$ the maximal ideal of $\mathcal{O}_K$. Let D be a binary function symbol for division in the valuation ring $\mathcal{O}_K$, defined as follows: $D(x,y) := \begin{cases} \frac{x}{y} & \text{if } v(x) \geq v(y) \text{ and } y \neq 0, \\ 0 & \text{otherwise,} \end{cases}$

5.1. **The real numbers.** A. Wilkie showed that the theory of $(\bar{\mathbb{R}}, exp)$ where $\bar{\mathbb{R}}$ is the ordered field of real numbers is model-complete [33, Second MainTheorem]. So setting $T := Th(\bar{\mathbb{R}}, exp)$, Theorem 4.4 holds for models of T since they also satisfy $(\text{IFT})_E$ and $(\text{LFF})_{\mathcal{E}}$.

5.2. **The p-adic numbers.** Let $(\mathbb{Q}_p, v)$ be the valued field of p-adic numbers. A. Macintyre showed that the theory of $\mathbb{Q}_p$ admits quantifier elimination in the language of fields together with the binary relation symbol div and for each $n \geq 2$, the predicates P_n defined by $P_n(x)$ iff $\exists y\ y^n = x$.

Then J. Denef and L. van den Dries showed that the theory of the valuation ring $\mathbb{Z}_p$ of $\mathbb{Q}_p$ (or the theory of $\mathbb{Q}_p$) enriched by all restricted power series with coefficients in $\mathbb{Z}_p$ together with the predicates P_n, $n \geq 2$ and the binary function $D : \mathbb{Z}_p^2 \to \mathbb{Z}_p$ for division in $\mathbb{Z}_p$, admits quantifier elimination [5, Theorem (1.1)]. N. Mariaule showed that the theory of the valuation ring $\mathbb{Z}_p$ of $\mathbb{Q}_p$ expanded by the exponential function $E_p(x)$ (see Examples 2.1 (5)) together with for each $n \geq 2$ the so-called decomposition functions for $E_p(x)$ is model-complete [21, Theorem 4.4.5]. We will recall below precisely what are these decomposition functions [21, Chapter 4].

From that one can easily deduce that the theory of the partial exponential valued field $(\mathbb{Q}_p, E_p)$ is model-complete in the language of fields together with the predicates P_n, $n \geq 2$, the binary function div, the exponential function $E_p(x)$ and the decomposition functions. (Note that N. Mariaule proves *strong model-completeness* [7, section 2 (2.2)]). So again Theorem 4.4 holds for $T = Th(\mathbb{Q}_p, E_p)$.

Now let us recall what are these decomposition functions. They are the analog of the functions *sin* and *cos* in the real case, but their definition is more complicated since $\mathbb{Q}_p$ has infinitely many proper algebraic extensions.

The field $\mathbb{Q}_p$ is bounded, namely for each fixed $d \geq 2$ it has only finitely many algebraic extensions of degree d. So one may define a chain of finite algebraic extensions K_n of $\mathbb{Q}_p$ with the following properties:

(1) K_n contains any extension of degree n of $\mathbb{Q}_p$,
(2) K_n is the splitting field of an irreducible polynomial $q_n \in \mathbb{Q}[X]$ of degree N_n.

One may further assume that $q_n \in \mathbb{Z}_p[X]$. Let β_n be a root of q_n and let $K_n = \mathbb{Q}_p(\beta_n)$, $\mathcal{O}_{K_n} = \mathbb{Z}_p[\beta_n]$. Then $\mathcal{O}_{K_n}$ is a $\mathbb{Z}_p$-module with basis $1, \beta_n, \ldots, \beta_n^{N_n-1}$. Let $y \in \mathcal{O}_{K_n}$ and write it as $\sum_{i=0}^{N_n-1} x_i \beta_n^i$. Then $E_p(y) = \prod_{i=0}^{N_n-1} E_p(x_i \beta_n^i)$, with $x_i \in \mathbb{Z}_p$ and one adds the decomposition functions for each $E_p(x\beta_n^i)$, namely functions from $\mathbb{Z}_p$ to $\mathbb{Z}_p$ which allows to express $E_p(x\beta_n^i)$ in $\mathcal{O}_{K_n}$. Namely, write $E_p(x\beta_n^i) = \sum_{j=0}^{N_n-1} \tilde{c}_{i,j,n}(x)\beta_n^i$. Conversely, one has: $(\tilde{c}_{i,j,n}(x))_{i<N_n} = V^{-1}(E_p((\beta_n^j)^\sigma x))_{\sigma\in Gal(K_n/\mathbb{Q}_p)}$, where V is the Vandermonde matrix associated to the roots of q_n.

Finally since $det(V)$ might be of strictly positive valuation, one has to multiply the $\tilde{c}_{i,j,n}(x)$ by the norm $N_{K_n/\mathbb{Q}_p}(det(V))$ in order to obtain the decomposition functions $c_{i,j,n}(x)$ [21, page 66].

Let $\mathcal{L}_{pEC}$ be the language $\mathcal{L}_E$ together with the predicates P_n, $n>1$, and the decomposition functions $c_{i,j,n}$, $0\leq j\leq N_n$, $i,n\in\mathbb{N}^*$. Then the $\mathcal{L}_{pEC}$-theory T of $(\mathbb{Q}_p,E_p)$ is model-complete [21, Theorem 4.4.5]. Since $\mathbb{Q}_p$ satisfies the analytic version of the implicit function theorem, we may apply Theorem 4.4. Note that we made a slight formal extension of our former result since we not only use the exponential function E_p but also the decomposition functions, but in view of the relationships described above between the decomposition functions and the exponential function E_p, there is no problem in doing so. The key point being able to transform an $\mathcal{L}_{pEC,\delta}$-term $t(x_1,\ldots,x_n)$ into an $\mathcal{L}_{pEC}$-term t^* in $\bar{\delta}^{m_1}(x_1),\ldots,\bar{\delta}^{m_n}(x_n)$.

5.3. **The completion of the algebraic closure of the p-adic numbers.** Let $\mathbb{C}_p$ be the completion of the algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers. As a valued field, $\mathbb{C}_p$ is a model of the theory $\mathrm{ACVF}_{0,p}$ of algebraically closed valued fields of characteristic 0 and residue characteristic p. It admits quantifier elimination in the language $\{+,-,\cdot,0,1,\mathrm{div}\}$ [29]. (Note that A. Robinson only proved model-completeness of the theory but the quantifier elimination result is easily deduced.) N. Mariaule showed that the theory of the valuation ring $\mathcal{O}_p$ of $\mathbb{C}_p$ endowed with the exponential function $E_p(x)$ is model-complete [21, Theorem 6.2.11]. From that one can easily deduce that the theory T of the partial exponential valued field $(\mathbb{C}_p,\mathrm{div},E_p)$ is model-complete. Since $\mathbb{C}_p$ also satisfies the analytic version of the implicit function theorem, we may apply Theorem 4.4. (Note that in this case since $\mathbb{C}_p$ is algebraically closed, one does not need to add additional functions such as the decomposition functions).

5.4. **Non-standard extensions of $\mathbb{Q}_p$.** Let (K,v) be a valued field extending $(\mathbb{Q}_p,v)$. Let $\mathcal{O}_K$ be the valuation ring of K and let $\mathcal{O}_K\langle\xi\rangle$ be the ring of strictly convergent power series over $\mathcal{O}_K$ in $\xi:=(\xi_1,\ldots,\xi_m)$. An element $f(\xi)$ is given by $\sum_{\nu\in\mathbb{N}^n}a_\nu\xi^\nu$, where $\xi^\nu=\xi_1^{\nu_1}\ldots\xi_n^{\nu_n}$ and $v(a_\nu)\mapsto+\infty$, when $|\nu|=\nu_1+\ldots+\nu_n\mapsto+\infty$. Such f defines a function from ${\mathcal{O}_K}^n$ to $\mathcal{O}_K$ defined by $f(u)=\begin{cases}\sum_{\nu\in\mathbb{N}^n}a_\nu u^\nu & \text{for } u\in\mathcal{O}_K^n,\\ 0 & \text{otherwise}\end{cases}$

The language $\mathcal{L}_{an}$ is the language of rings augmented by a n-ary function symbol for each $f\in\mathcal{O}_K\langle\xi\rangle$ and $n\geq1$. Let D be a binary function symbol for division restricted to the valuation ring as defined above. Let $\mathcal{L}_{an,\mathrm{div}}:=\mathcal{L}_{an}\cup\{\mathrm{div}\}\cup\{P_n:n\geq2\}$. and $\mathcal{L}^D_{an,\mathrm{div}}:=\mathcal{L}_{an,\mathrm{div}}\cup\{D\}$. Let $\mathcal{K}$ denote the $\mathcal{L}_{an,\mathrm{div}}$-structure with domain K and the above interpretation of the symbols of the language. In view of the way the functions f are interpreted in both $\mathbb{Q}_p$ and $\mathcal{K}$, we have that $\mathbb{Q}_p$ is an $\mathcal{L}_{an,\mathrm{div}}$-substructure of K. Then using the quantifier elimination theorem of J. Denef and L. van den Dries, that if $\mathcal{K}$ is a model of $Th_{\mathcal{L}_{an,\mathrm{div}}}(\mathbb{Q}_p)$, then $\mathcal{K}$ is an elementary $\mathcal{L}_{an,\mathrm{div}}$-extension of $\mathbb{Q}_p$. Now if we restrict the language $\mathcal{L}_{an,\mathrm{div}}$ to the language $\mathcal{L}_{pEC}$, we get that the theory T of $\mathcal{K}$ in this restricted language is also model-complete (and in fact equal to the theory of $(\mathbb{Q}_p,E_p)$. In order to apply Theorem 4.4 to $\mathcal{K}_\delta$, we need to check that $\mathcal{K}$ satisfies IFT_E^{an}. A way to do this is to get a universal axiomatisation of $Th_{\mathcal{L}_{an,\mathrm{div}}}(\mathbb{Q}_p)$. It will imply that any definable function from $\mathcal{O}_K^n$ to $\mathcal{O}_K$ is piecewise given by $\mathcal{L}_{an,\mathrm{div}}$-terms and so analytic functions. (This argument was used for $\mathbb{R}_{an}$ in [9].)

We express that $K^*/(K^*)^n \cong \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^n$ and that cosets representative of the subgroup of n^{th} powers can be found in $\mathbb{N}$, namely for every $x \in K^*$ there exist $\lambda, r \in \mathbb{N}$ with $0 \leq r < n$, $0 \leq \lambda < p^{\beta(n)}$ and $\beta(n) = 2v(n) + 1$ and $P_n(x\lambda p^r)$ [1, Lemma 4.2]. This can be expressed by a finite disjunction and translates the fact that $v(K^*)$ is a $\mathbb{Z}$-group

Then we express that K is henselian in the following way. Let $p(X) \in \mathcal{O}_K[X]$ be an ordinary polynomial of degree n. Then one defines a function $h_n : \mathcal{O}_K^{n+1} \to \mathcal{O}_K$ sending $(a_0, \ldots, a_n, b) \mapsto u$ with $a_nb^n + \ldots + a_1b + a_0 = 0$, $v(p(b)) > 0$, $v(\partial_X p(b)) = 0$ and $v(u-b) > 0$ and to 0 otherwise [4, Definition 3.2.10].

So this gives us a non-standard model of T to which we may apply Theorem 4.4.

6. Construction of models of the scheme $(\mathrm{DL})_E$

{construction}

In this section we will place ourselves in the same setting as in section 4.1. We show how to endow certain exponential topological fields K endowed with a V-topology, satisfying (IFT) and (LFF) with a derivation in such a way they become a model of the scheme $(\mathrm{DL})_E$. One can follow a similar strategy as in [3], [28] to endow certain (ordered) fields with a derivation in such a way they become a model of the scheme (DL) introduced in [14], generalizing for certain differential topological fields the axiomatization CODF of closed ordered differential fields given by M. Singer in [32].

In the proposition below, we will assume that the field K, as a topological space is separable and first-countable and so its cardinality is at most $2^{\aleph_0}$.

{external}

Proposition 6.1. *Let $\mathcal{L}$ be a countable language and $\mathcal{K}$ be a topological $\mathcal{L}$-field of cardinality $\aleph_1$, endowed with a V-topology, which is definable with corresponding formula χ. Assume that K as topological space, is first-countable and separable. Suppose $\mathcal{K}$ satisfies* (IFT) *and* (LFF)*. Then we can endow K with a derivation δ such that $\mathcal{K}_\delta$ is a model of the scheme $(\mathrm{DL})_E$.*

Proof: Let $\{\chi(K, \bar{d}_i) : \bar{d}_i \in K, i \in \omega\}$ be a countable basis of neighbourhoods of 0 and further assume, setting $W_i := \chi(K, \bar{d}_i)$ that $W_{i+1} + W_{i+1} \subseteq W_i$. Let D be a countable dense subset of K. Let $\mathcal{K}_0$ be the (countable, dense) $\mathcal{L}$-substructure of $\mathcal{K}$ generated by $(\bar{d}_i)_{i\in\omega}$ and D. Moreover, we may assume, by Lowenheim-Skolem theorem, that $\mathcal{K}_0$ is an elementary substructure of $\mathcal{K}$. Express K as $K_0(B)$ with B a subset of elements of K which are ecl-independent over K_0 (so $|B| = \aleph_1$). Set $B := (t_\alpha)_{\alpha<\aleph_1}$.

{small}

Claim 6.2. *For each W_i, $i \in \omega$, and each $\ell \in \omega$, there are elements $s_1, \ldots, s_\ell \in W_i$ that are* ecl*-independent over K_0 and with the property that $s_j - t_j \in K_0$, $1 \leq j \leq \ell$.*

Proof of Claim:

Fix W_i a neighbourhood of 0 in K and choose $t_0, \ldots, t_\ell \in B$, $\ell \in \omega$. Since K_0 is dense in K, there are for each $0 \leq j \leq \ell$, $r_{ji} \in K_0$ such that $t_j - r_{ji} \in W_i$. Set $s_j := t_j - r_{ji}$, $0 \leq j \leq \ell$. The elements $s_1, \ldots, s_\ell \in K$, are ecl-independent over K_0 and belong to W_i. □

We will express $\mathcal{K}$ as the union of an elementary chain of countable subfields $\mathcal{K}_0 \preceq \mathcal{K}_\alpha$ endowed with a derivation δ_α, $\alpha < \aleph_1$, starting by putting on K_0 the trivial derivation δ_0.

The subfields $\mathcal{K}_\alpha$ have the following property. Given a neighbourhood of zero W_j and a quantifier-free $\mathcal{L}_\delta(K_\alpha)$-formula $\varphi(\mathbf{x})$ of order m and any Khovanskii system H (with parameters in K_α) and associated $\mathcal{L}$-formula φ_H^* (see Definition 4.1) such that $H(\mathbf{a}^0, \ldots, \mathbf{a}^{m-1}, \bar{b}) \wedge \varphi_H^*(\bar{a}, \bar{b})$ holds in K_α with $\bar{a} := (\mathbf{a}^0, \ldots, \mathbf{a}^m)$ and $\bar{b} \in K_\alpha$, we can find $\beta \in K_{\alpha+1}$ such that $\varphi(\beta)$ holds and $\bar{\delta}_{\alpha+1}^m(\beta) - \bar{a} \in W_j$.

By induction on α, assume we have constructed $\mathcal{K}_0 \subseteq \mathcal{K}_\alpha \preceq \mathcal{K}$ a countable elementary substructure of $\mathcal{K}$ and suppose $\mathcal{K}_\alpha$ is endowed with a derivation δ_α. Let $\bar{x} := (\mathbf{x}^0, \dots, \mathbf{x}^m)$, $\mathbf{x} := \mathbf{x}^0$, $|\mathbf{x}| = n$ and, keeping the notations of Definition 4.1, set

$$\begin{aligned}\mathcal{F}_\alpha :=& \{\exists \bar{z} H(\mathbf{x}^0, \dots, \mathbf{x}^{m-1}, \bar{z}) \wedge \varphi_H^*(\bar{x}, \bar{z}) : \mathcal{K} \models \exists \bar{x}\, \exists \bar{z}\, (H(\mathbf{x}^0, \dots, \mathbf{x}^{m-1}, \bar{z}) \wedge \varphi_H^*(\bar{x}, \bar{z}))\\ &\text{with } \varphi \text{ varying over all the } \mathcal{L}_\delta(K_\alpha)-\text{formulas of order } m \geq 1\}.\end{aligned}$$

We will construct a differential extension $\mathcal{K}_{\alpha+1}$ of $\mathcal{K}_\alpha$ containing t_α, satisfying the scheme $(\mathrm{DL})_E$ relative to $\mathcal{F}_\alpha$.

Let $\varphi(x)$ be an $\mathcal{L}_\delta(K_\alpha)$-formula of order $m \geq 1$ and consider the formula

$$\exists \bar{z} H(\mathbf{x}^0, \dots, \mathbf{x}^{m-1}, \bar{z}) \wedge \varphi_H^*(\bar{x}, \bar{z}) \in \mathcal{F}_\alpha$$

with $|\mathbf{x}| = n$. Assume H is of the form $\bigwedge_{i=1}^{n-\ell} H(u_{\ell+i}, \bar{z}_i)$, with $\bar{z} := (\bar{z}_1, \dots, \bar{z}_{n-\ell})$. Let $t_1, \dots, t_\ell \in B$. Let $\mathbf{u} := (u_1, \dots, u_n), \bar{b} \in K$ be such that $\varphi_H^*(\bar{u}, \bar{b})$ holds, where $\bar{u} := (\mathbf{u}^0, \dots, \mathbf{u}^m)$.

Let W_j be a fixed neighbourhood of zero. Then by Claim 6.2 and Lemma 4.5, there is an elementary extension of $\mathcal{K}_\alpha$ inside $\mathcal{K}$, an element $\beta \in K$ and a derivation $\tilde{\delta}_\alpha$ extending δ on K_α and β such that $\varphi(\beta)$ holds and $\bar{\tilde{\delta}}_\alpha^m(\beta) - \bar{u} \in W_j$. Furthermore we may assume that this extension is countable and ecl^K-closed. (Note that in Lemma 4.5, we had a hypothesis of saturation but it was only to ensure the existence of ecl-independent elements (over K_0). The property that they were K_0-small is replaced by finding elements $s_1, \dots, s_\ell \in W_j$, ecl-independent and congruent to $t_1, \dots, t_\ell$ modulo K_0.)

We consider $\mathrm{ecl}(K_\alpha(\bar{\tilde{\delta}}_\alpha^m(\beta))$. In case t_α does not belong to this subfield, we define $\tilde{\delta}_\alpha(t_\alpha) = 1$. Then let $K_{\alpha,1} = \mathrm{ecl}(K_\alpha(t_\alpha, \bar{\tilde{\delta}}_\alpha^m(\beta))$. We enumerate $\mathcal{F}_\alpha$ and the extension $\mathcal{K}_{\alpha,i}$ corresponds to where the i^{th} formula in $\mathcal{F}_\alpha$ has a differential solution close to the algebraic one in W_j. Set $\mathcal{K}_\alpha^{(1)} := \bigcup_i \mathcal{K}_{\alpha,i}$. Then we redo the construction with $\mathcal{K}_\alpha^{(1)}$ in place of $\mathcal{K}_\alpha$ with a smaller neighbourhood of zero, say W_{j+1}. Set $\mathcal{K}_{\alpha+1} := \bigcup_m \mathcal{K}_\alpha^{(m)}$. Note that $\mathcal{K}_{\alpha+1}$ is countable.

So we described what happens at successor ordinals and at limit ordinals we simply take the union of the subfields we have constructed so far. Finally we express $\mathcal{K}$ as the union of a chain of differential subfields and given any $\mathcal{L}_\delta$-formula $\varphi(x)$ of order $m \geq 1$, Khovanskii formula $H(\mathbf{x}^0, \dots, \mathbf{x}^{m-1}, \bar{z})$ and an associated $\mathcal{L}$-formula φ_H^* such that for some $\bar{u} = (\mathbf{u}^0, \dots, \mathbf{u}^m), \bar{b} \in K$ with $\mathbf{u} := (u_1, \dots, u_n)$ the formula $\varphi_H^*(\bar{u}, \bar{b})$ holds in K, we find an element of the chain $\mathcal{K}_\alpha$ such that $\varphi \in \mathcal{F}_\alpha$ and $\bar{u}, \bar{b} \in K_\alpha$. Therefore given a neighbourhood of zero W_i, we have $\beta \in K_{\alpha+1}$ such that $\varphi(\beta)$ holds and $\bar{\delta}^m(\beta) - \bar{u} \in W_i$. □

Denote by $\mathcal{L}_-$ the language $\mathcal{L}$ where we take off the exponential function and denote by T_- the theory of the $\mathcal{L}_-$-reducts of the models of T. Let us assume that T_- admits quantifier elimination. Then in [14], we showed that the class of existentially closed models of $T_{-,\delta}$ was elementary, assuming that the models of T satisfied Hypothesis (I). That last property is an analog for topological fields of the property of being large, property introduced by F. Pop [26]). Let us first recall the following notation. Given a differential polynomial $p(X) \in K\{X\}$ of order $m > 0$, with $|X| = 1$, the separant s_p of p is defined as $s_p := \frac{\partial}{\partial \delta^m(x)} p \in K\{X\}$.

Definition 6.3. [14, Definition 3.5] *The scheme of axioms* (DL) *is the following: given a model $\mathcal{K}$ of $T_{-,\delta}$, $\mathcal{K}$ satisfies* (DL) *if for every differential polynomial $p(X) \in K\{X\}$ with*

$|X| = 1$ *and* $ord_X(p) = m \geqslant 1$, *for variables* $\mathbf{y} = (y_0, \ldots, y_m)$ *it holds in* $\mathcal{K}$ *that*

$$\forall z\big((\exists y(p^*(\mathbf{y}) = 0 \wedge s_p^*(\mathbf{y}) \neq 0) \rightarrow \exists x\big(p(x) = 0 \wedge s_p(x) \neq 0 \wedge \chi_\tau(\bar{\delta}^m(x) - \mathbf{y}, z)\big)\big).$$

By quantifying over coefficients, the axiom scheme (DL) can be expressed in the language $\mathcal{L}_{-,\delta}$.

Corollary 6.4. *Let* $\mathcal{K}$ *be a topological* $\mathcal{L}$*-field satisfying* (IFT) *and* (LFF) *endowed with a* V*-topology which is definable with corresponding formula* χ. *Assume that* K *is of cardinality* $\aleph_1$ *with a countable dense subfield. Then we can endow* $\mathcal{K}$ *with a derivation* δ *such that* $\mathcal{K}_\delta$ *is a model of the schemes* $(\mathrm{DL})_E \cup (\mathrm{DL})$.

Proof. We modify the proof of proposition above by also considering the instances of the scheme (DL) and alternating between solving a formula from scheme $(\mathrm{DL})_E$ to solving a formula from scheme (DL). We observe that if $t_1, \ldots, t_n$ are ecl-independent, then they are also algebraically independent by Corollary 3.8. □

Corollary 6.5. *Let* $\mathcal{K}$ *be an ordered real-closed exponential field. Assume that* K *is of cardinality* $\aleph_1$ *with a countable dense subfield. Then we can endow* $\mathcal{K}$ *with a derivation* δ *such that* $\mathcal{K}_\delta$ *is a model of* CODF *together with the scheme* $(\mathrm{DL})_E$. □

Remark 6.6. Now let us record a few cases when K is separable and first-countable.

First, suppose that (K, v) is an henselian perfect valued field of equicharacteristic 0, with value group G. Denote by $\{t^g \in K\colon\ g \in G \ \&\ v(t^g) = g\}$, a family of elements of K whose set of values is G. Then by a result of Kaplansky, the residue field k isomorphically embeds in the valuation ring of K [11, Lemma 3.8]. Assume that k is countable and $|G| = \aleph_0$. Consider the subring of K generated by k and $\{t^g :\ g \in G_{>0}\}$, then it is dense in the valuation ring of K. Since the inverse operation is continuous, K has a dense countable subfield. A countable basis of neighbourhoods of 0 is given by the balls $W_g := \{x \in K\colon v(x) > g\}$, where $g \in G_{>0}$.

Second, suppose now that $(K, \leq)$ is an ordered real-closed field. Either K is archimedean and so it embeds into $\mathbb{R}$. So assume that the archimedean valuation on K is non trivial. The residue field with respect to this archimedean valuation embeds in $\mathbb{R}$. In case the value group G is countable, the subring of K generated by $\mathbb{Q}$ and $\{t^g :\ g \in G_{>0}\}$ is dense in the valuation ring of K. As a countable basis of neighbourhoods we may take the balls $B_{n,g} := \{x \in K\colon |x| < \frac{1}{n}t^g\}$, with $n \in \mathbb{N}^*$, $g \in G_{>0}$ and where t^g is a strictly positive element of K with archimedean valuation equal to g.

7. Pairs of models of T

Let $\mathcal{K}_\delta \models T_\delta$ and assume it satisfies the scheme $(DL)_E$. Assume also that T is a complete theory. By Remark 4.3, the subfield of constants C_K is an $\mathcal{L}$-substructure and $\mathrm{ecl}^K(C_K) = C_K$. In this section we want to examine the case when the class of elementary dense pairs of models of T is complete and contains the pair (K, C_K). Usually one adds to $\mathcal{L}$ a new unary predicate P and consider the expansion $\mathcal{L} \cup P$, interpreting P as the smaller model of T. Let T_P denote the $\mathcal{L}_P$-theory of pairs of models of T of the form (K, L) with $L \preceq K$ and L dense in K (dense in the sense of the topology on K). A. Fornasiero [12,] showed in case T admits an existential matroid, namely one has a closure relation cl (in models of T) satisfying certain additional properties that we will recall below and if $Cl^K(L) = L$, then T_P is complete, when using another notion of *dense*, defined using the dimension function induced by the closure relation cl. Fornasiero defined $X \subseteq K$ dense in K if it intersects non-trivially a subset of dimension 1.

In our setting, it amounts to find when $C_K \preceq K$ and when, using ecl as the closure relation, (K, ecl) is an existential matroid.

For the first property, we use Tarski-Vaught test, so given any formula $\varphi(x, \bar{y})$ and parameters in C_K, assuming that $K \models \exists x\ \varphi(x, \bar{d})$ with $\bar{d} \in C_K$, find $b \in C_K$ such that $K \models \varphi(b, \bar{d})$. In case $\varphi(K, \bar{d})$ is the union of an open subset of K and elements in $\mathrm{dcl}(\bar{d})$, by density of C_K we have that we can find an element $C_K \cap \varphi(K, \bar{d}) \neq \emptyset$.

In case of $\mathbb{Q}_p$ use p-minimality (by a result of L. van den Dries, D. Haskell, D. Macpherson): any definable subset of the field is semi-algebraic namely definable in the language of rings [16, Proposition 4.1].

In case of $\mathbb{C}_p$, use C-minimality (by a result of L. Lipschitz and Z. Robinson): any definable subset of the field is a finite union of isolated points and open balls [16, Theorem 2.1]. These isolated points are in $\mathrm{dcl}(C_K)$ but one has still to show that they belong to C_K.

For the second property, we already know that ecl is a (finitary) closure relation satisfying the exchange property, so it is a matroid [10, Section 3]. Then ecl is an existential matroid if ecl is a definable matroid, satisfies existence and is non-trivial [10, Definition 3.25].

We could take instead acl which satisfies the exchange in case of a C-minimal or p-minimal or an o-minimal field [16, Lemma 2.2].

Finally we have to check that the notion of topological density is the same as the notion of density introduced by Fornasiero.

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