Generalised Bargmann-Wigner classification:

Mixed-symmetry fields in Minkowski and $(A)dS_{d+1>3}$

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PLAN

1. Motivation

2. Bargmann - Wigner in Minkowski space

3. Bargmann - Wigner in AdS_{d+1} space

4. UIR's of $SO(1, d+1)$ & fields in dS_{d+1}

① A motivation for higher-spin fields: Quantum Field Theory

- At dawn of QFT. Majorana (1932), Dirac (1936), Fierz-Pauli (1939), and most notably Wigner's 1939 classification of UIR's of Poincaré group $ISO(3,1)$.

  . Relativistic, linear & covariant equations: Bargmann-Wigner (1948)

  $\longrightarrow$ massless, helicity particles characterized by

  - Mass $m = 0$; • helicity $s \in \{0, 1/2, 1, 3/2 \dots\}$

[∘ Rem: In the $m = 0$ case, there are also the "continuous" or "infinite" spin UIR's $\rightsquigarrow \vec{\mu} \neq \vec{0}$ in $\mathbb{R}^{D-2}$.]

• Some comments about interactions

↳ Problems with .

- Minimal $u(1)$ coupling for $s \geq 3/2$ (1961)

- Minimal Lorentz coupling for $s \geq 5/2$ (1964)

- Infinite-component Majorana-like equations (1968)

  (has tachyons)

↳ Together with the observation of high-spin hadronic resonances

Belief that consistent high-spin interactions require infinitely-many fields of unbounded spin .

Once the HS representations have been seen to exist in the sense of UIR's of spacetime isometry algebra, i.e. first quantization, then standard second quantization naturally requires a covariant Lagrangian

Fierz-Pauli program

Associate a quadratic, local and covariant Lagrangian to every UIR of maximally-symmetric spacetime-isometry algebra.

- Initiated by F.P in 1939 for massive, spin-2 particle in $\mathbb{R}^{1,3}$. Then, notably [Chang (67), Schwinger (70), Singh-Hagen (74)]

- In 1978, Fronsdal and Fang gave Lagrangian for $m=0$ helicity-s field around $\mathbb{R}^{1,3}$ and $(A)dS_4$ by taking the $m \to 0$ limit of Singh-Hagen's $\mathcal{L}$

Questions: Can we generalize this program to arbitrary spacetime dimensions $D > 4$? Interactions?

(Rem : $D = 2+1$ very interesting too. Not in this talk.)

- The BW program in $\mathbb{R}^{1,d}$ was achieved in late 80's [W. Siegel & B. Zwiebach] and in minimal form in [Labastida 89, X. Bekaert & N.B. 2001]

- The BW program in AdS_{d+1} in the late nineties by R. Metsaev.

Before attacking the ambitious problem of introducing consistent interactions among the various fields, one may want to fill a gap: the Bargmann-Wigner program in dS_{d+1}: establish the dictionary in dS_{d+1} between UIR's of $SO(1,d+1)$ and covariant linear wave equations in dS_{d+1}. First: review $\Lambda \leqslant 0$ cases.

② Wigner's classification of UIR's of $ISO(1,d)$

↳ One-to-one with the SO(1,d) orbits $\mathcal{O}_p$ of $p \in (\mathbb{R}^{1,d})^*$ together with UIR of little group $G_p \subseteq SO(1,d)$ stabilizing p.

1) $\forall g \in G_p \qquad g \cdot p = p$

2) Given a repres. R of G_p, induce a UIR $\mathcal{T}$ of $ISO(1,d)$ on the Hilbert space of functions on $\mathcal{O}_p$ valued in R.

$$\mathcal{T}(\Lambda, a) \cdot \tilde{\varphi}(q) = \sqrt{\rho_{\Lambda^{-1}}(q)}\; e^{i(q,a)}\, R\left(g_q^{-1} \circ \Lambda \circ g_{\Lambda^{-1}q}\right) \cdot \tilde{\varphi}(\Lambda^{-1} q)$$

[$\rho_{g_1 g_2}(q) = \rho_{g_1}(g_2 \cdot q)\, \rho_{g_2}(q)$, $d\mu_g(q) = \rho_g(q)\, d\mu(q)$, $g \in G$, $q \in X$ t.p. space with measure μ ; Radon-Nikodym derivative]

where $g_q \in SO(1,d)$ standard boost for p : $g_q \cdot p = q \in \mathcal{O}_p$

s.t. $g_q^{-1} \circ \Lambda \circ g_{\Lambda^{-1}q}$: $p \xrightarrow[g_{\Lambda^{-1}q}]{} \Lambda^{-1} q \xrightarrow[\Lambda]{} q \xrightarrow[g_q^{-1}]{} p$

The various orbits $\{\mathcal{O}_p\}$ correspond to p being

1) Timelike $\rightsquigarrow$ Massive particle $p^\mu = (m, 0, \dots, 0)$

$$p^2 = -m^2 \quad , \quad G_p \simeq SO(d) \quad [\, E := p^0 \ \& \ \eta = \mathrm{diag}(-,+,\dots,+) \,]$$

2) Light-like $\rightsquigarrow$ Massless particle

$$p^2 = 0 \quad \& \quad p \neq 0 \qquad p_\mu = (-E, 0, \dots, 0, E)$$

In Light frame $\quad x^\pm := \dfrac{x^d \pm x^0}{\sqrt{2}} \quad , \quad p_\mu = (p_-, \overset{p_+}{0}, \overbrace{0, \dots, 0}^{p_i})$

Little group $\quad G_p \cong ISO(d-1) \cong T_{d-1} \rtimes SO(d-1)$

M_{i-} & M_{-+} rejected $\Rightarrow \{M_{i+} =: \pi_i\} \cup \{M_{ij}\}$

$\hookrightarrow$ Take π_i trivial $\rightsquigarrow$ helicity UIR's : $G_p \cong SO(d-1)$.

3) Spacelike $\rightsquigarrow$ tachyons : $G_p \cong SO(1, d-1)$

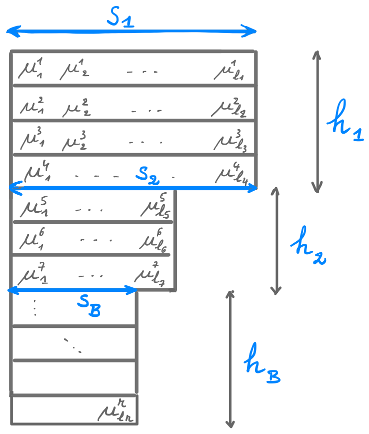
4) Nul $\quad p = (0, \dots, 0)$ in $(\mathbb{R}^{1,d})^*$: $G_0 \cong SO(1,d)$

As for covariant, linear wave equations

. Take $\varphi_{\mathbb{Y}}(x)$ valued in $GL(d+1)$ irrep $\rightsquigarrow \mathbb{Y}$

Young tableau

$\mathbb{Y}$ =



with $\boxed{c_1 + c_2 \leq d-1}$

$$\vec{s} = (\underbrace{\ell_1, \dots, \ell_4}_{\text{all equal to } s_1}, \underbrace{\ell_5, \dots, \ell_7}_{\text{all equal to } s_2}, \dots, \overset{s_B}{\overset{\shortparallel}{\ell_r}})$$

• Symmetrizing the indices of any row with any index of a lower row gives zero identically.

• Antisymmetrizing the indices of a column with any index of a column at its right gives zero identically. Define $p_B = \sum_{I=1}^{B} h_I$ the height of $\mathbb{Y}$.

• Build the curvature

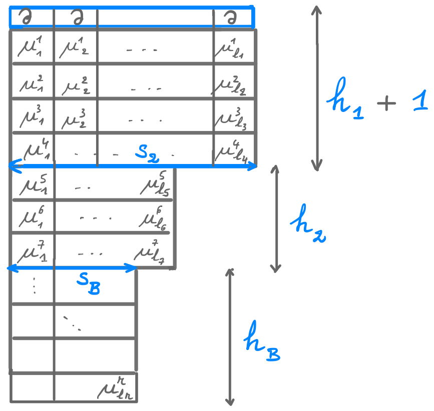
$$\boxed{K_{\overline{\mathbb{Y}}} := d^{(1)} \dots d^{(s_1)} \varphi_{\mathbb{Y}}}$$

by acting on $\varphi_{\mathbb{Y}}$ with s_1 curls

and impose the wave equation

$$\boxed{\mathrm{Tr}\, K_{\overline{\mathbb{Y}}} \approx 0} \; .$$

$\overline{\mathbb{Y}} =$



From Bianchi identity $\quad d^{(i)} K_{\overline{\mathbb{Y}}} \equiv 0 \qquad \forall\, i \in \{1, \dots, s_1\}$

Deduce that $\quad d^{\dagger}_{(i)} K \approx 0 \quad \forall\, i \quad$ where $\quad \boxed{d^{\dagger}_{(i)} := *_i\, d^{(i)} *_i}$ divergence.

Hence $\quad \{ d^{(i)}, d^{\dagger}_{(i)} \} K_{\overline{\mathbb{Y}}} \equiv \Box K_{\overline{\mathbb{Y}}} \approx 0 \quad \Rightarrow \quad K_{\overline{\mathbb{Y}}}$ massless field.

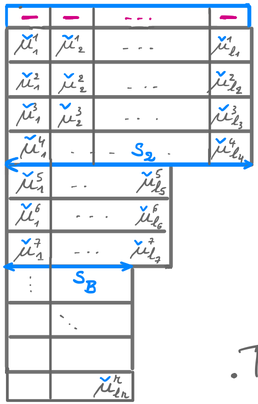
Fourier modes $\tilde{K}_{\overline{\mathbb{Y}}}(p)$ on $p^2 = 0$

light-cone : mass shell for light-like particles.

$$\boxed{\begin{gathered} [d^{(i)}, d^{\dagger}_{(j)}]_{\mathbb{Z}_2} = \delta^i_j\, \Box \\ [d^{(i)}, d^{(j)}]_{\mathbb{Z}_2} = 0 = [d^{\dagger}_{(i)}, d^{\dagger}_{(j)}]_{\mathbb{Z}_2} \end{gathered}}$$

Bianchi Id.

↳ $d^{(i)}K = 0 \quad \forall i$, $\boxed{P_{[-}\tilde{K}_{\mu_1^1\mu_1^2\dots\mu_1^{p_B}]|\dots} \equiv 0}$ $\iff$ $\tilde{K}_{\underline{\mathbb{Y}}} \rightsquigarrow \mathbb{Y}$ of $GL(d,\mathbb{R})$ (*)

$\tilde{K} =$ 

$$\check{\mu} \in \{+, \overbrace{1, \dots, d-1}^{\text{transverse directions}}\}$$

$d^{\dagger}_{(i)} K \approx 0$ Divergenceless : $p^{+}\tilde{K}_{+\dots} \approx 0$

Means that $\tilde{K}$ valued in $\mathbb{Y}$ of $GL_{(d-1)}$

. Tracelessness in $so(1,d)$ $\Longrightarrow$ Tracelessness in $so(d-1)$

$\boxed{\begin{array}{l} \text{Ccl} : \tilde{K}_{\underline{\mathbb{Y}}} \text{ reduces on-shell to } \tilde{K} \text{ in UIR } R_{\mathbb{Y}} \text{ of } G_p \\ G_p \cong SO(d-1) \text{ little group for helicity particles} \xrightarrow[\text{induce}]{} T(\Lambda, a) \text{ UIR of } ISO(1,d+1) \end{array}}$

(*)

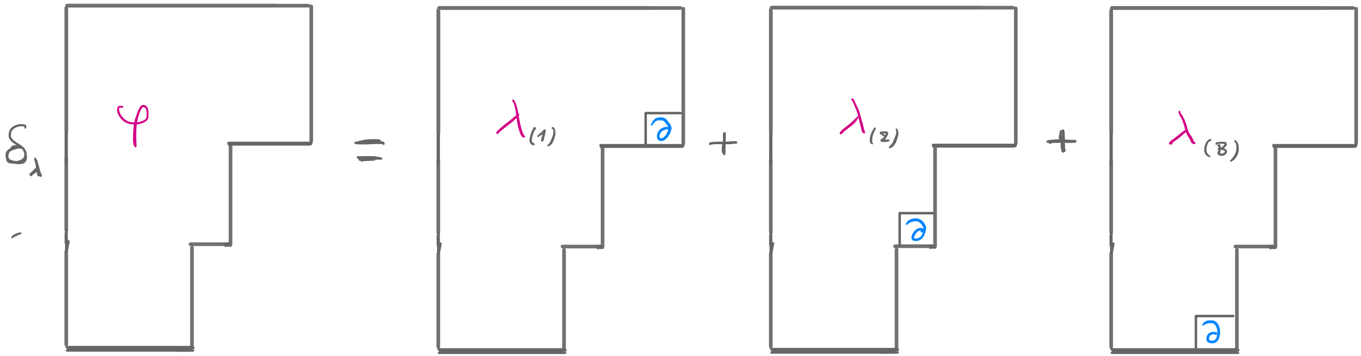
$K_{-\check{\mu}\check{\nu}|-\check{e}}$ $\quad 0 \equiv K_{[-\check{\mu}\check{\nu}|\check{e}]-} = 3\tilde{K}_{-[\check{\mu}\check{\nu}|\check{e}]-} - \cancelto{0}{\tilde{K}_{\check{\mu}\check{\nu}\check{e}|--}} \iff \tilde{K}_{-[\check{\mu}\check{\nu}|\check{e}]-} \equiv 0$

Gauge invariance

$$K_{\bar{Y}} = d^{(1)} \dots d^{(s_1)} \varphi_Y$$

Wave equation $\mathrm{Tr}\, K_{\bar{Y}} \approx 0$ is PDE order s_1 for φ

Invariant under



On-shell, fixing gauge $\tilde{K}$ reduces to $\tilde{K} \approx (p_-)^{s_1} \varphi_{i_1 \dots j \dots}$

X.B. & N.B. Partial gauge fixing of $\boxed{\mathrm{Tr}\, \tilde{K} = 0}$ $SO(d-1)$

to $\boxed{\left(\Box - \sum_{i=1}^{s_1} d^{(i)} d^{\dagger}_{(i)} + \frac{1}{2} \sum_{i,j=1}^{s_1} d^{(i)} d^{(j)} \mathrm{Tr}_{ij}\right) \varphi \approx 0}$: Labastida 89.

Now : turn to AdS_{d+1}

3 WAVE EQUATIONS in $(A)dS_{d+1}$

Conventions and notation

Lie algebra $so\left(1+\frac{1+\sigma}{2},\, d+\frac{1-\sigma}{2}\right)$

with generators $M_{AB} = M_{AB}^{\dagger}$

$$\left(\eta^{(\sigma)}_{AB}\right) = \mathrm{diag}\big(-\sigma, \underbrace{-, \overbrace{+, \cdots, +}^{(\delta_{ij})}}_{(\eta_{ab})}\big) \longrightarrow \begin{cases} \sigma = +1 & AdS_{d+1} \\ \sigma = -1 & dS_{d+1} \end{cases}$$

$A, B, \ldots = 0_\sigma, 0, 1, \ldots, d$

$a, b, \ldots = 0, 1, \ldots, d$

$$\eta_{ab} = \mathrm{diag}(\underset{0}{-}, \underset{1}{+}, \underset{\cdot}{\cdots}, \underset{d}{+}) \qquad so(1,d)$$

$$[M_{AB}, M_{CD}] = i\left(\eta^{(\sigma)}_{BC} M_{AD} - \eta^{(\sigma)}_{AC} M_{BD} - \eta^{(\sigma)}_{BD} M_{AC} + \eta^{(\sigma)}_{AD} M_{BC}\right)$$

- $P_a := \lambda\, M_{0_\sigma a}$ transvections (A)dS$_{d+1}$

$$\boxed{\begin{aligned} &[M_{ab}, M_{cd}] = i\,\eta_{bc} M_{ad} + \dots \\ &[M_{ab}, P_c] = 2i\,\eta_{c[b} P_{a]} \qquad [P_a, P_b] = i\sigma \lambda^2 M_{ab} \end{aligned}}$$

- Another useful decomposition of M_{AB}, adapted to CFT :

$$D := i c_\sigma M_{0_\sigma 0}\ , \qquad P_i := M_{0i} + c_\sigma M_{0_\sigma i}\ , \qquad K_i := M_{0i} - c_\sigma M_{0_\sigma i}$$

where $c_\sigma = \begin{cases} i & \text{for } \sigma = +1 \\ 1 & \text{for } \sigma = -1 \end{cases}$, s.t. $c_\sigma^2 = -\sigma$.

$$\boxed{\begin{aligned} &[M_{ij}, M_{kl}] = i\,\delta_{jk} M_{il} + \dots \qquad && [K_i, P_j] = 2\left(iM_{ij} + \delta_{ij} D\right) \\ &[M_{ij}, P_k] = 2i\,\delta_{k[j} P_{i]} && [M_{ij}, K_k] = 2i\,\delta_{k[j} K_{i]} \\ &[D, P_i] = P_i && [D, K_i] = -K_i \end{aligned}}$$

<u>Note</u> : $\sigma = +1$: $D = -M_{0'0} \equiv E$

Quadratic Casimir $C_2\left[so\left(1+\frac{1+\sigma}{2},\ d+\frac{1-\sigma}{2}\right)\right] = \frac{1}{2} M^{AB} M_{AB}$.

Using $\quad M_{0i} = \frac{1}{2}(P_i + K_i)\ , \qquad M_{0_\sigma i} = \frac{1}{2c_\sigma}(P_i - K_i)$

$$\frac{1}{2} M^{AB} M_{AB} = D(D-d) - P^i K_i + C_2[so(d)]$$

$\Rightarrow$ On a lowest-weight state $|\Delta, \vec{s}\rangle$ annihilated by ladder op. K_i ,

s.t. $\quad (D-\Delta)\,|\Delta, \vec{s}\rangle = 0\ , \qquad K_i\,|\Delta,\vec{s}\rangle = 0$,

one finds

$$\boxed{C_2\left[so\left(1+\frac{1+\sigma}{2},\ d+\frac{1-\sigma}{2}\right)\right] = \Delta(\Delta-d) + \sum_{\ell=1}^{r} s_\ell\,(s_\ell + d - 2\ell)}$$

In the $so(1+\frac{1+\sigma}{2}, d+\frac{1-\sigma}{2})$ - covariant basis where $P_a := \lambda M_{0_\sigma a}$,

represent $\boxed{P_a = -i\nabla_a}$ as a diff. operator, ∇ the Lorentz-covariant deriv.

$$\Rightarrow C_2 = \frac{1}{2} M^{AB} M_{AB} \equiv C_2[so(1,d)] - \sigma \eta^{ab} M_{0_\sigma a} M_{0_\sigma b}$$

$$= C_2[so(1,d)] - \frac{\sigma}{\lambda^2} P^a P_a$$

- Set $$\frac{\sigma}{\lambda^2} \nabla^a \nabla_a = -\frac{\sigma}{\lambda^2} P^2 = \underbrace{\frac{1}{2} M^{AB} M_{AB}}_{\Delta(\Delta-d)+\sum_{\ell=1}^{r} s_\ell(s_\ell+d-2\ell)} - \frac{1}{2} M^{ab} M_{ab} \overset{!}{=} \sigma m^2_{\mathbb{Y}} \qquad (*)$$

$\Rightarrow$ Gives a relation between wave equation (linear, relativistic) and (abstract) UIR of $so(1+\frac{1+\sigma}{2}, d+\frac{1-\sigma}{2})$.

$$\boxed{(\Box - \lambda^2 m^2_{\mathbb{Y}})\varphi_{\mathbb{Y}} = 0}$$

• Asking gauge invariance of wave equation

$(\Box - \lambda^2 m^2_{\mathbb{Y}})\,\varphi_{\mathbb{Y}} = 0\ , \quad \mathrm{Tr}\,\varphi_{\mathbb{Y}} = 0 = \nabla\cdot\varphi_{\mathbb{Y}}$ on all indices

under $\delta_\lambda \varphi_{\mathbb{Y}} = \sum_{I=1}^{B} (\nabla^{(I)})^t \lambda_{(I)}$

gives [Metsaev '95, $t=1$] a set of possibilities for fixed I

$$\sigma\ m^2_I \in \left\{ (s_I - p_I - t)(s_I - p_I + d - t) - \sum_{k=1}^{r} s_k \right\}_{I=1,\dots,B}$$

where $p_I := \sum_{J=1}^{I} h_J$

together with similar conditions on the gauge para. $\lambda_{(I)}$

and the gauge-for-gauge parameters $\{\lambda^i_{(I)}\}_{i=2,\dots,p_I}$

Note: In (A)dS, at most 1 gauge parameter! Different from Minkowski!

Group theoretical description in AdS_{d+1}

Generalized Verma module

$$\mathcal{V} = \{ P_{i_1} \dots P_{i_n} \; |e_0, \vec{s}\,\rangle_{j\dots k\dots} \}_{n=0,1,\dots}$$

$so(2) \oplus so(d) \subset so(2,d)$

Recall $\quad C_2[so(2,d)] = e_0(e_0 - d) + C_2[so(d)]$

with

$$\begin{cases} e_0 > s_1 - h_1 + d - 1 & \text{Massive unitary field} \\ e_0 = e_t^I := s_I - p_I + d - t & \text{partially-massless (gauge) fields} \\ e_0 \neq e_t^I \ \& \ e_0 < e_0^1 & \text{Massive non-unitary} \end{cases}$$

Observe $\quad \sigma m_I^2 \in \left\{ e_0^I(e_0^I - d) - \sum_{k=1}^{n} s_k \right\}_{I=1,\dots,B}$

In accordance with $\quad \dfrac{\sigma}{\lambda^2} \Box = \dfrac{1}{2} M^{AB} M_{AB} - \dfrac{1}{2} M^{ab} M_{ab}$

$$e_0(e_0 - d) + \sum_{\ell=1}^{r} s_\ell(s_\ell + d - 2\ell) - \sum_{\ell=1}^{r} s_\ell(s_\ell + d + 1 - 2\ell)$$

$$\frac{\sigma}{\lambda^2} \Box \varphi = -e_0(-e_0 + d) - \sum_{\ell=1}^{n} s_\ell$$

Gauge invariance of Fierz-Pauli-type wave equation

reflected by

Gauge field
Irr. module

$$\mathcal{D}(e_t^I, \mathbb{Y}) \cong \frac{\mathcal{V}(e_0^I, \mathbb{Y})}{\mathcal{D}(\underbrace{e_0^I + t}, \mathbb{Y}_{(I)})}$$

Generalized Verma m.

Gauge param. module, itself a quotient in general (gauge for gauge)

minimal energy of states in the module

$$\delta\varphi = \nabla^t \lambda_{(I)}$$

AdS_{d+1}	dS_{d+1}
Vacuum $so(2) \oplus so(d)$ module	Vacuum $so(1,1) \oplus so(d)$ module
$\mathbb{V}(e_0, \mathbb{Y})$	$\mathbb{V}(\Delta_c, \mathbb{Y})$
• Casimir	• Casimir
$C_2 = e_0(e_0 - d) + C_2[so(d)]$	$C_2 = \Delta_c(\Delta_c - d) + C_2[so(d)]$
• Critical mass	
$m^2_{\mathbb{Y}} = e_0(e_0 - d) - \sum_{k=1}^{r} s_k$	$-m^2_{\mathbb{Y}} = \Delta_c(\Delta_c - d) - \sum_{k=1}^{r} s_k$
• massless for $e_0 = e^I_t$	• massless for $\Delta_c = ?$
unitarity known $(L^-_i)^\dagger = L^+_i$	unitarity ?

$$\boxed{(\nabla^2 - \lambda^2 m^2_{\mathbb{Y}})\varphi_{\mathbb{Y}} = 0}$$

④ $\boxed{\text{UIR's of } SO(1, d+1)}$

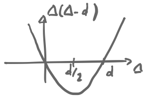
$\mathbb{R}$

- Principal series : $\Delta_c = \frac{d}{2} + i\rho$, $\mathbb{Y}$ & ρ arbitrary

[Rem : $\nabla^2 \varphi_0 = (-\lambda^2)\Delta_c(\Delta_c - d)\varphi_0$ where $\Delta_c(\Delta_c - d) = (\frac{d}{2} + i\rho)(i\rho - \frac{d}{2}) = -\rho^2 - \frac{d^2}{4} \Rightarrow \nabla^2 \geq 0$ in dS_{d+1}]

- Complementary series : $p < \Delta_c < d - p$, $p \in \{0, 1, \dots, r-1\}$

$P = P_B$



$\ell_k = 0$ for $k = p+1, \dots, r$.

- Exceptional series : $\Delta_c = d - p$ (or $\Delta_c = p$) , $p \in \{1, \dots, r-1\}$

$\ell_k = 0$ for $k = p+1, \dots, r$. (no scalar)

- $(d = 2r + 1)$ Discrete series : $\Delta_c = \frac{d}{2} + k$, $k \in \frac{\mathbb{N}}{2}$

i.e. $\Delta_c = \frac{d-1}{2} + k'$, $k' \in \mathbb{N}$ maximal height $0 < k' \leq \ell_r$

[For $so(1, 2r+2)$, rank$[so(1, d+1)]$ = rank$[so(d+1)] = r+1$.

the Cartan subgroup is $SO(d+1)$, compact. The $r+1$ (commuting) generators of the Cartan subgroup are <u>compact</u>.

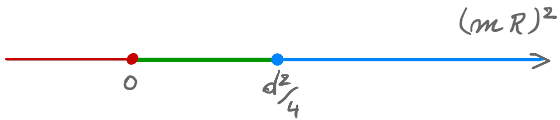
For $d = 2r$, no compact Cartan subgroup. There is a $so(1,1)$ generator among the $r+1$.]

Dictionary

Computing de $so(d+2)$ characters of Generalized Verma modules [using Bernstein – Gel'fand – Gel'fand resolution] and comparing with characters of $SO(1,d+1)$ UIR's from the math literature, we obtained the dictionary

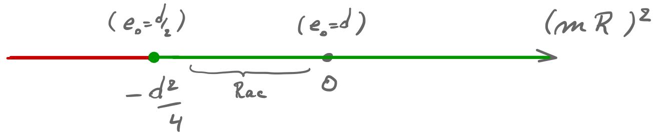
- Principal & complementary : Massive fields, e.g.

dS_{d+1}



compl. princip.

AdS_{d+1}



$R^2 m^2 = e_0(e_0 - d)$ $\quad e_0 = \frac{d-2}{2} \Rightarrow$ singleton

$e_0 \geq s - p + d - 1$

$m^2_{Rac} = -\frac{1}{4}(d^2 - 4)$

$e_0^{Rac} = \frac{d-2}{2} < \frac{d}{2}$, $\quad m^2_{Rac} = -\frac{d^2}{4} + 4$, $\quad m^2_{d/2} = -d^2/4$

$e_0 \geq d - 1$ & $e_0 = \frac{d-2}{2}$

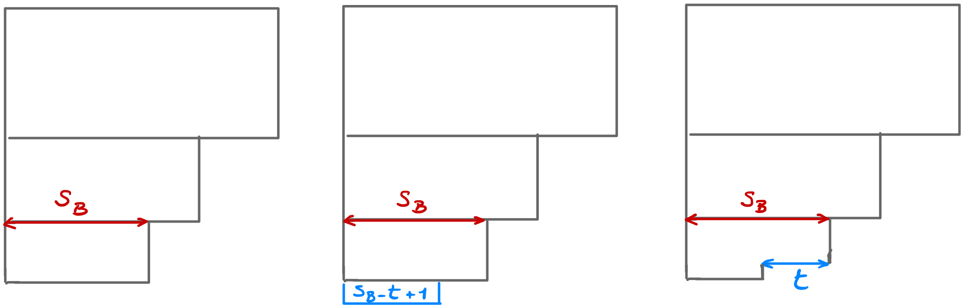
- **Exceptional series** : (partially) massless fields with less-than-maximal height

Unitarity: only the last block must be activated

$\Delta_c = s_B - p + d - t$ $p \equiv p_B$ contrary to the first one in AdS.

Rem: The weights $(\Delta_c, \mathbb{Y})$ labelling the UIR $\rightsquigarrow$ curvature and not φ potential

- **Discrete series** : massless field φ with maximal height



φ potential K curvature λ gauge parameter

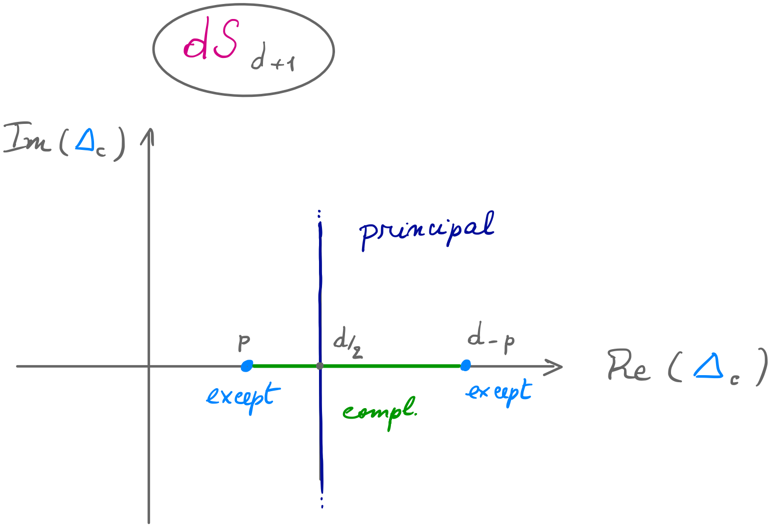
$\Delta_c = \ell_r - r + d - t$

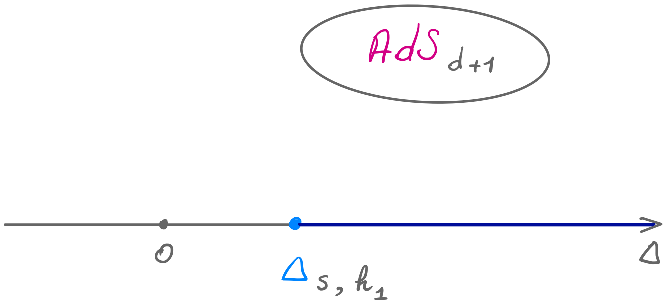
$P_B \equiv p = r$

$s_B \equiv \ell_r$

Massless cases : $t = 1$; PM : $1 < t \leq s_B$.

Summary : Unitary fields.





• In the scalar case $s = 0$, the field $\phi(x)$ obeys

$$\left[(\Box - \lambda^2 m^2_{\mathbb{Y}}) \varphi_{\mathbb{Y}} = 0 \, , \quad m^2_{\mathbb{Y}} = e_0(e_0 - d) - \sum_{i=1}^{r} s_i \right]$$

$$(\Box + 2\lambda^2)\phi(x) = 0 \quad \text{in } AdS_4 \, , \quad \text{with}$$

$$m_0^2 = -2 = C_2\left[so(2,3) \,|\, \mathcal{D}(e_0, \vec{0})\right] = -e_0(-e_0 + 3)$$

leaving 2 possibilities compatible with unitarity :

↳ $e_0 = 1$ (Dirichlet) or 2 (Neuman) BC's .

• So, in the zoology of "massless" UIR's

⇝ (bosonic) fields propagating in AdS_4 , we have

$\mathcal{D}(s+1, s)$ $s = 0, 1, 2, \ldots$ and $\mathcal{D}(2, 0)$. Fronsdal on-shell fields

Dirac singletons and Flato-Fronsdal

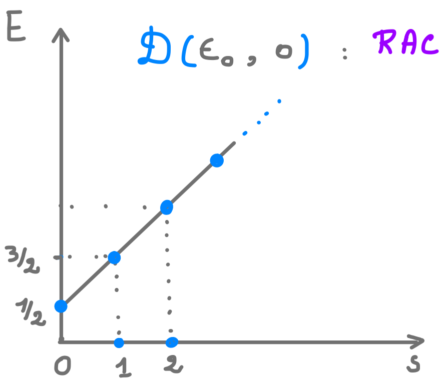
$[\epsilon_0 = \frac{d-2}{2}]$.

Two remarkable $so(2,d)$-UIRs : $\mathcal{D}(\epsilon_0, 0)$ & $\mathcal{D}(\epsilon_0 + \frac{1}{2}, \frac{1}{2})$

Not propagating inside AdS_{d+1} but at ∂AdS_{d+1} .

↳ Single line in compact weight space

$\mathcal{D}(\epsilon_0, 0)$: RAC & $\mathcal{D}(\epsilon_0 + \frac{1}{2}, \frac{1}{2})$: DI



Flato - Fronsdal theorem ($d=3$)

- $\mathcal{D}(\tfrac{1}{2},0) \otimes \mathcal{D}(\tfrac{1}{2},0) \simeq \bigoplus_{s=0}^{\infty} \mathcal{D}(s+1,s)$

- $\mathcal{D}(1,\tfrac{1}{2}) \otimes \mathcal{D}(1,\tfrac{1}{2}) \simeq \mathcal{D}(2,0) \oplus \bigoplus_{s=1}^{\infty} \mathcal{D}(s+1,s)$

Consequence : Compositeness of massless particles in AdS_4

RAC : $\Box_3 \, \phi(x) = 0 \quad (*)$ with $\dim(\phi) = \frac{1}{2}$ $\left(\int d^3x \, \partial\phi . \partial\phi\right)$

conformal scalar

- Symmetries of $(*)$: $\dfrac{\mathcal{U}(so(2,d))}{\mathrm{Annih}(\mathrm{RAC})} \simeq \mathcal{A}$ associative algebra

$\downarrow [\cdot , \cdot]$

$hs(d+1)$

Module $\mathcal{D}(e_0,\vec{0})$ for RAC in $so(2,d)$

- From $E = -M_{0'0} = D$ and $L^-_i = K_i$, $L^+_i = P_i$, $[E, L^\pm_i] = \pm L^\pm_i$

and $[L^-_i, L^+_j] = 2(iM_{ij} + \delta_{ij}E)$,

$$\mathcal{V}(e_0;\vec{0}) = \{ L^+_{i_1}\dots L^+_{i_n}|e_0,0\rangle \}_{n\in\mathbb{N}} \quad \text{where} \quad L^-_i|e_0,0\rangle = 0 ,$$

one searches for nul vectors in $\mathcal{V}$, level by level:

Level 1: $$L^-_i L^+_j|e_0,0\rangle = L^+_j \cancel{L^-_i|e_0,0\rangle}^{\,0} + 2(iM_{ij} + \delta_{ij}E)|e_0,0\rangle$$

$$= 2\delta_{ij}e_0|e_0,0\rangle \implies \text{cannot vanish.}$$

$\hookrightarrow$ Level 2: $$L^-_i L^+_k L^+_j|e_0,0\rangle = \left(L^+_k L^-_i L^+_j + [L^-_i, L^+_k]L^+_j\right)|e_0,0\rangle =$$

$$= 2e_0 L^+_k\delta_{ji}|e_0,\vec{0}\rangle + 2(iM_{ik} + \delta_{ik}\underbrace{E)L^+_j}_{(e_0+1)L^+_j|e_0,0\rangle}|e_0,0\rangle$$

$$= 4e_0 L^+_{(k}\delta_{j)i}|e_0,0\rangle + 2\delta_{ik}L^+_j|e_0,0\rangle + 2i\cdot i\left(\delta_{jk}L^+_i - \delta_{ji}L^+_k\right)|e_0,0\rangle$$

$$= 2\left[2(e_0+1)L^+_{(k}\delta_{j)i} - \delta_{jk}L^+_i\right]|e_0,\vec{0}\rangle$$

Hence $L^-_i L^+_k L^{+k}|e_0,\vec{0}\rangle = 2[2(e_0+1) - d]L^+_i|e_0,\vec{0}\rangle = 0$ for $e_0 = \frac{d-2}{2} = \epsilon_0$

• It turns out that there is no other null vector that are not descendant of

$|N\rangle = L^+_i L^{+i} | \frac{d-2}{2}, 0\rangle$, therefore $\mathcal{D}(\frac{d-2}{2}, \vec{0}) = \dfrac{\mathcal{V}(\frac{d-2}{2}, 0)}{\mathcal{D}(\frac{d}{2}+1, 0)}$

Rac Verma module

• Another example : $\vec{s} = \boxed{\quad s \quad}$ of $so(d)$, $|e_0, \vec{s}\rangle_{i(s)}$ vacuum.

$L^-_j |e_0, \vec{s}\rangle_{i(s)} \stackrel{!}{=} 0$ by assumption, $\mathcal{V}(e_0 \vec{s}) = \{ L^+_{j_1} \dots L^+_{j_n} | e_0, \vec{s}\rangle_{i(s)} \}$

↳ Level 1 : Define $|e_0+1, s-1\rangle_{i(s-1)} := L^{+j} | e_0, s\rangle_{j\, i(s-1)}$. Then,

$$L^-_k |e_0+1, s-1\rangle_{i(s-1)} = [L^-_k, L^+_j]\, \delta^{i_1 j} |e_0, s\rangle_{i_1 i_2 \dots i_s} = 2\, \delta^{i_1 j} (i M_{kj} + \delta_{kj} E) |e_0, s\rangle_{i(s)} =$$

$$= 2 i\, i\, \delta^{i_1 j} \Big[\delta_{i_1 j} |e_0, s\rangle_{k i_2 \dots i_s} - \delta_{i_1 k} |e_0, s\rangle_{j i_2 \dots i_s} +$$

$$+ (s-1) \big(\delta_{i_2 j} |e_0, s\rangle_{i_1 k i_3 \dots i_s} - \delta_{i_2 k} |e_0, s\rangle_{i_1 j i_3 \dots i_s} \big) \Big] + 2\, \delta^{i_1}_k\, e_0 |e_0, s\rangle_{i(s)}$$

$\div 2$

$$\frac{1}{2} L^-_k |e_0+1, s-1\rangle_{i(s-1)} = [-(d-1+s-1-0) + e_0] |e_0, s\rangle_{k\, i(s-1)} \Rightarrow \text{Null vector at level 1}$$

if $e_0 = s+d-2$.

Again, no higher independant null vector : $\mathcal{D}(s+d-2, s)$ the Fronsdal spin-s module.

$\Rightarrow$ Fronsdal module $\mathcal{D}(s+d-2, s) = \dfrac{\mathcal{V}(s+d-2, s)}{\mathcal{D}(s+d-1, s-1)}$ UIR of $so(2,d)$.

Reminder on algebra and characters

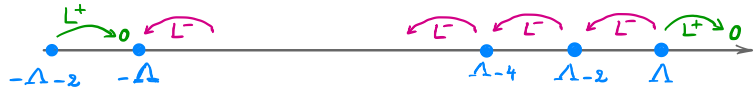
If V is a $\mathfrak{g}$-module, $\boxed{\chi_V = \sum_{\lambda \in \Phi_V} \mathrm{mult}_\lambda \, e^\lambda}$ function on weight space.

$$\hookrightarrow e^\lambda(\mu) := e^{(\lambda,\mu)} \quad \text{where} \quad (\lambda,\mu) = \underbrace{G^{ij}}_{\text{symmetric Cartan matrix}} \lambda_i \mu_j \quad \text{for} \quad \lambda = \sum_{i=1}^r \overbrace{\lambda^i}^{\text{Dynkin labels}} \underbrace{\Lambda_{(i)}}_{\text{fundam. weight}} = \sum_{i=1}^r \lambda_i \overbrace{\alpha^{(i)\vee}}^{\text{simple co-root}}$$

- $G^{ij} = (\overbrace{\alpha^{(i)\vee}}^{\text{simple co-root}}, \alpha^{(j)\vee}) = \dfrac{2}{(\alpha^{(i)},\alpha^{(i)})} A^{ij} = \dfrac{2(\alpha^{(i)},\alpha^{(j)})}{(\alpha^{(i)},\alpha^{(i)})(\alpha^{(j)},\alpha^{(j)})}$, dual to Killing form.
- G_{ij} inverse of G^{ij}.
- $\alpha^{(i)} = (\alpha^{(i)})^j \Lambda_{(j)} = (\alpha^{(i)})_k \alpha^{(k)\vee} \Rightarrow (\alpha^{(i)})^j = (\alpha^{(i)}, \alpha^{(j)\vee}) = (\alpha^{(i)})_k G^{kj} = A^{ij}$

$$\boxed{[H^i, E^{\alpha^{(j)}}] = (\alpha^{(j)})^i E^{\alpha^{(j)}} = A^{ij} E^{\alpha^{(j)}}.}$$

Example: $\mathfrak{sl}_2$, $A = 2$.



$$\boxed{\begin{aligned} &[L_i, L_j] = i\,\epsilon_{ijk} L_k \\ &L^\pm := L_1 \pm i L_2, \quad L_0 := 2L_3 \\ &[L^+, L^-] = L_0, \ [L_0, L^\pm] = \pm 2 L^\pm \end{aligned}}$$

$$\chi_\Lambda(\mu) = \sum_{n=0}^{\Lambda} 1 . \, e^{\Lambda - 2n}(\mu) = \sum_{n=0}^{\Lambda} \exp[(\Lambda-2n)^i \mu^j \overbrace{G_{ij}}^{1/2}] \quad (r=1)$$

$$= \sum_{n=0}^{\Lambda} \exp \frac{(\Lambda-2n)\mu}{2} = e^{\frac{\Lambda}{2}\mu} + e^{\frac{\Lambda-2}{2}\mu} + \dots + e^{\frac{-\Lambda+2}{2}\mu} + e^{-\frac{\Lambda}{2}\mu}$$

$$= e^{\Lambda\mu/2}\left(1 + e^{-\mu} + \dots + e^{-\Lambda\mu}\right) = e^{\frac{\Lambda\mu}{2}} \frac{1 - e^{-(\Lambda+1)\mu}}{1 - e^{-\mu}} = \cancel{e^{\frac{\Lambda\mu}{2}}} \frac{\cancel{e^{-\frac{(\Lambda+1)\mu}{2}}}}{\cancel{e^{-\mu/2}}} \frac{e^{\frac{(\Lambda+1)\mu}{2}} - e^{-\frac{(\Lambda+1)\mu}{2}}}{e^{\mu/2} - e^{-\mu/2}}$$

$$\Leftrightarrow \chi_\Lambda(\mu) = \frac{\sinh((\Lambda+1)\mu/2)}{\sinh(\mu/2)} \quad \boxed{\begin{aligned} &\beta := e^{\mu/2} \\ &\Lambda = 2s \end{aligned}} \quad \boxed{\chi_\Lambda(\beta) = \frac{\beta^{\Lambda+1} - \beta^{-\Lambda-1}}{\beta - \beta^{-1}} = \frac{\beta^{2s+1} - \beta^{-2s-1}}{\beta - \beta^{-1}}}$$

$$\boxed{\text{Flato-Fronsdal theorem for } d=3,\quad Rac \otimes Rac = \bigoplus_{s=0}^{\infty} \mathcal{D}(s+1,s),\ \text{using character formula:}}$$

1) Character of Rac : $\mathcal{D}(\tfrac{1}{2},0) = \{ |1/2,0\rangle, L^+_i |1/2,0\rangle, \underbrace{L^+_i L^+_j}_{\text{traceless}} |1/2,0\rangle, \dots \}$

$q := x_{\breve{0}} = (\exp e_{\breve{0}})(\mu)$, extra direction $e_{\breve{0}}$ in $\mathbb{E}^{r+1}$ associated with $E = so(2)$ generator.

Set $q = \alpha^2$ so that $\bullet\ \chi_{Rac}(\mu) = \sum_{j=0}^{\infty} (\alpha^2)^{j+1/2} \chi_j(\beta) = \alpha \sum_{j=0}^{\infty} \alpha^{2j} \dfrac{\beta^{2j+1} - \beta^{-2j-1}}{\beta - \beta^{-1}}$

$$\Leftrightarrow \chi_{Rac}(\mu) = \alpha \left(\frac{\beta}{\beta-\beta^{-1}} \frac{1}{1-(\alpha\beta)^2} - \frac{\beta^{-1}}{\beta-\beta^{-1}} \frac{1}{1-(\frac{\alpha}{\beta})^2} \right) = \boxed{\frac{1+\alpha^2}{\alpha(\alpha\beta - \frac{1}{\alpha\beta})(\frac{\alpha}{\beta} - \frac{\beta}{\alpha})} = \chi_{Rac}(\mu)}$$

We have seen $\mathcal{D}(s+1,s) = \{ L^+_{i_1} \dots L^+_{i_n} |s+1,s\rangle_{j_1 \dots j_s} \}$ s.t. $L^{+j} |s+1,s\rangle_{j i \dots} \sim 0$.

2) Character of Fronsdal $\mathcal{D}(s+1,s)$:

$\underline{k=0}$: $|s+1,s\rangle_{i_1 \dots i_s}$, $\underline{k=1}$: $\{ L^+_{\{i_1} |s+1,s\rangle_{i_2 \dots i_{s+1}\}}$, $\epsilon_{i_1}{}^{jk} L^+_j |s+1,s\rangle_{k i_2 \dots i_s}$, $\cancel{L^{+j} |s+1,s\rangle_{j i (s-1)}}$ $\}$ 0 in $\mathcal{D}$

$(\alpha^2)^{s+1} \chi_s(\beta)$ $\quad (\alpha^2)^{s+1+1} \chi_{s+1}(\beta)$, $(\alpha^2)^{s+1+1} \chi_s(\beta)$ $\quad \cancel{(\alpha^2)^{s+2} \chi_{s-1}(\beta)}$

$\underline{k=2}$: $L^+_{\{i} L^+_i |s+1,s\rangle_{i \dots i\}}$, $\epsilon_{\{i_1}{}^{jk} L^+_{|j_1} L^+_{i_2} |s+1,s\rangle_{i_3 \dots i_{s+1}\}k}$, $\cancel{L^+_{i_1} L^{+j} |s+1,s\rangle_{j i_2 \dots i_s}}$, $\cancel{L^{+j} L^{+k} |s+1,s\rangle}$,

$(\alpha^2)^{s+3} \chi_{s+2}(\beta)$ $\quad (\alpha^2)^{s+3} \chi_{s+1}(\beta)$ $\quad (\alpha^2)^{s+3} \chi_s(\beta)$ $\quad \cancel{(\alpha^2)^{s+3} \chi_{s-1}(\beta)}$

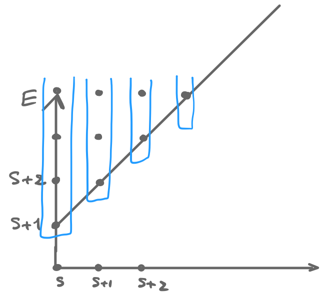
$(L^+ \cdot L^+) |s+1,s\rangle \}$

$\underline{k=3}$: $\Big\{ \underset{s+3}{L^+_{\{i}L^+_i L^+_i |s+1,s\rangle_{i\dots i}}\ ,\ \underset{s+2}{\epsilon_i^{\ jk} L^+_i L^+_j L^+_i |s+1,s\rangle_{k i\dots i}}\ ,\ \underset{s+1}{(L^+\cdot L^+) L^+_i |s+1,s\rangle_{i\dots i}}\ ,\ \underset{s}{(L^+\cdot L^+)\,\epsilon_i^{\ jk} L^+_j |\,..\rangle_k} \Big\}$

$\rightarrow$ Contributions to $\chi_s(\beta)$: $(\alpha^2)^{s+1}\sum_{k=0}^{\infty}\alpha^{2k}$

$\rightarrow$ Contributions to $\chi_{s+1}(\beta)$: $(\alpha^2)^{s+1}\sum_{k=0}^{\infty}\alpha^{2k+1}$

$\vdots$



$$\chi_{s+1,s}(\mu) = (\alpha^2)^{s+1}\sum_{k=0}^{\infty}\alpha^{2k}\left(\chi_s + \alpha^2\chi_{s+1} + (\alpha^2)^2\chi_{s+2} + \dots\right)$$

$$= (\alpha^2)^{s+1}\frac{1}{1-\alpha^2}\sum_{j=0}^{\infty}\alpha^{2j}\chi_{s+j}(\beta) = \frac{\alpha^{2s+2}}{1-\alpha^2}\sum_{j=0}^{\infty}\alpha^{2j}\,\frac{\beta^{2s+2j+1}-\beta^{-2s-2j-1}}{\beta-\beta^{-1}} =$$

$$= \frac{\alpha^{2s+2}}{1-\alpha^2}\,\frac{1}{\beta-\beta^{-1}}\left[\beta^{2s+1}\frac{1}{1-(\alpha\beta)^2} - \beta^{-2s-1}\frac{1}{1-(\frac{\alpha}{\beta})^2}\right]$$

$$\frac{1}{\cancel{\alpha\beta}(\frac{1}{\alpha\beta}-\alpha\beta)} \qquad \frac{1}{\frac{\alpha}{\cancel{\beta}}(\frac{\beta}{\alpha}-\frac{\alpha}{\beta})}$$

$$= \underbrace{\frac{\alpha^{2s+2}}{1-\alpha^2}\,\frac{1}{\beta-\beta^{-1}}}_{C}\ \frac{1}{\underbrace{(\frac{\alpha}{\beta}-\frac{\beta}{\alpha})(\alpha\beta-\frac{1}{\alpha\beta})}_{D}}\ \frac{1}{\alpha^2}\Big(\underbrace{\beta^{2s+1}-\alpha^2\beta^{2s-1}-\beta^{-2s-1}+\alpha^2\beta^{-2s+1}}_{(1-(\frac{\alpha}{\beta})^2)\beta^{2s+1}-\beta^{-2s-1}(1-(\alpha\beta)^2)}\Big)$$

$$= \frac{C}{D}\,(\beta-\beta^{-1})\left[\chi_s(\beta) - \alpha^2\chi_{s-1}(\beta)\right]$$

$$\Rightarrow \boxed{\chi_{s+1,s}(\mu) = \frac{\alpha^{2s}}{(1-\alpha^2)(\frac{\alpha}{\beta}-\frac{\beta}{\alpha})(\alpha\beta-\frac{1}{\alpha\beta})}\left(\chi_s(\beta)-\alpha^2\chi_{s-1}(\beta)\right)}$$

Difference of 2 characters of generalised Verma modules.

Finally, compute the sum

$$\Sigma := \chi_{(1,0)} + \sum_{s=1}^{\infty} \frac{\alpha^{2s}}{\underbrace{(1-\alpha^2)\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)\left(\alpha\beta-\frac{1}{\alpha\beta}\right)}_{\to\, D}} \left(\chi_s(\beta) - \alpha^2 \chi_{s-1}(\beta)\right)$$

$$\Leftrightarrow \quad \Sigma = \frac{1}{D}\left[\chi_0 + (\alpha^2\chi_1 - \alpha^4\chi_0) + (\alpha^4\chi_2 - \alpha^6\chi_1) + (\alpha^6\chi_3 - \alpha^8\chi_2) + \dots\right]$$

$$= \frac{1}{D}(1-\alpha^4)\left[\chi_0 + \alpha^2\chi_1 + \alpha^4\chi_2 + \dots\right]$$

$$= \frac{1+\alpha^2}{\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)\left(\alpha\beta-\frac{1}{\alpha\beta}\right)} \frac{1}{\beta-\beta^{-1}}\left[\beta-\beta^{-1} + \alpha^2(\beta^3-\beta^{-3}) + \alpha^4(\beta^5-\beta^{-5}) + \dots\right]$$

$$= \frac{1+\alpha^2}{\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)\left(\alpha\beta-\frac{1}{\alpha\beta}\right)} \frac{1}{\beta-\beta^{-1}}\left[\beta\left(1+\alpha^2\beta^2+\alpha^4\beta^4+\dots\right) - \beta^{-1}\left(1+\frac{\alpha^2}{\beta^2}+\frac{\alpha^4}{\beta^4}+\dots\right)\right]$$

$$= \frac{1+\alpha^2}{\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)\left(\alpha\beta-\frac{1}{\alpha\beta}\right)} \frac{1}{\beta-\beta^{-1}}\left[\frac{\beta}{1-(\alpha\beta)^2} - \frac{\beta^{-1}}{1-\frac{\alpha^2}{\beta^2}}\right] = \frac{1+\alpha^2}{\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)\left(\alpha\beta-\frac{1}{\alpha\beta}\right)} \frac{1}{\cancel{\beta-\beta^{-1}}} \frac{(1+\alpha^2)(\cancel{\beta-\beta^{-1}})}{\alpha^2\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)\left(\alpha\beta-\frac{1}{\alpha\beta}\right)}$$

$$= \frac{(1+\alpha^2)^2}{\alpha^2\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2\left(\alpha\beta-\frac{1}{\alpha\beta}\right)^2} \equiv (\chi_{Rac})^2 .$$

$$\Rightarrow \quad \boxed{\chi_{Rac}^2 = \chi_{(1,0)} + \sum_{s=1}^{\infty} \chi_{(s+1,s)}} \quad \Longleftrightarrow \quad \boxed{D(\tfrac{1}{2},0) \otimes D(\tfrac{1}{2},0) \cong \bigoplus_{s=0}^{\infty} D(s+1,s)}$$

↪ Holography HS_4/CFT_3

cfr other talks!

In the orthonormal basis for $B_r = so(2r+1)$ and $D_r = so(2r)$, $\{e_i\}_{i=1,\dots,r}$ $\mathbb{E}^r$ basis Cartesian

	simple roots	positive roots
B_r	$e_i - e_{i+1}$ $\quad 1 \le i \le r-1$ e_r	$e_i \pm e_j$ $\quad 1 \le i < j \le r$, e_i $\quad 1 \le i \le r$
D_r	$e_i - e_{i+1}$ $\quad 1 \le i \le r-1$ $e_{r-1} + e_r$	$e_i \pm e_j$ $\quad 1 \le i < j \le r$

- $\lambda = \sum_{i=1}^{r} \ell_i e_i$,
- $x_i := e^{e_i}(\mu) = e^{\vec{\mu}.\vec{e_i}} = e^{\mu_i}$.
- $\lambda_{\{n_\alpha\}} := \lambda + \sum_{\alpha \in \Phi_-} n_\alpha . \alpha$

for $\{n_\alpha\}_{\alpha \in \Phi_-}$ set of non-negative integers.

- $$C_\lambda = \sum_{\forall \{n_\alpha\}} e^{\lambda_{\{n_\alpha\}}} = e^\lambda \sum_{n_\alpha = 0}^{\infty} \prod_{\alpha \in \Phi_-} (e^\alpha)^{n_\alpha} = e^\lambda \prod_{\alpha \in \Phi_-} \frac{1}{1 - e^\alpha}$$

character of generalised Verma module, for g semi-simple. (non-degeneracy in root system)

$$\Rightarrow C^{so(2r)}_{\vec{\ell}}(\vec{x}) = \left(\prod_{i=1}^{r} e^{\ell_i \mu_i} \right) \prod_{1 \le i < j \le r} \frac{1}{1 - e^{(-e_i + e_j, \mu)}} \quad \frac{1}{1 - e^{(-e_i - e_j, \mu)}}$$

$$= \prod_{i=1}^{r} (x_i)^{\ell_i} \prod_{1 \le i < j < r} \frac{1}{(1 - x_i^{-1} x_j)(1 - x_i^{-1} x_j^{-1})}$$

• Gauge symmetries : $\delta \varphi_{\mu(s)} = s \nabla_\mu \lambda_{\mu(s-1)}$.

↳ Minimal set of fields & gauge parameters ⇝ Fronsdal.

$$-2\mathcal{L}(\varphi, \nabla\varphi) = \nabla_\nu \varphi_{\mu(s)} \nabla^\nu \varphi^{\mu(s)} - \frac{s(s-1)}{2} \nabla_\nu \varphi'_{\mu(s-2)} \nabla^\nu \varphi'^{\mu(s-2)}$$

$$+ s(s-1) \nabla_\nu \varphi'_{\mu(s-2)} \nabla_\rho \varphi^{\rho\nu\mu(s-2)} - s \nabla_\nu \varphi^\nu{}_{\mu(s-1)} \nabla_\rho \varphi^{\rho\mu(s-1)}$$

AdS$_D$

$$- \frac{s(s-1)(s-2)}{2} \nabla_\nu \varphi'^\nu{}_{\mu(s-3)} \nabla_\lambda \varphi'^{\lambda\mu(s-3)}$$

$$+ m_c^2 \varphi^{\mu(s)} \varphi_{\mu(s)} + m_c'^2 \varphi'^{\mu(s-2)} \varphi'_{\mu(s-2)} .$$

where

$$\boxed{m_c = \lambda^2 \left(s^2 + (D-6)s - 2D + 6 \right)}$$

• $\bar{R}_{\mu\nu\rho\sigma} = -\lambda^2 (\bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \bar{g}_{\nu\rho} \bar{g}_{\mu\sigma})$, $\lambda^2 = \dfrac{-2\Lambda}{(D-1)(D-2)}$, $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$.

- Maxwell's theory : $A_\mu(x) := \varphi_\mu(x)$, $\delta_\epsilon A_\mu(x) = \partial_\mu \epsilon(x)$

  - $S[A_\mu] = -\frac{1}{4}\int d^4x \, F^{\mu\nu} F_{\mu\nu}$, $F_{\mu\nu} := 2\,\partial_{[\mu} A_{\nu]}$

  - $\delta_\epsilon S[A_\mu] = 0 \iff \partial^\mu F_{\mu\nu} \equiv 0$ (Noether id.)

- Fierz-Pauli in metric-like notation :

  $h_{\mu_1\mu_2}(x) =: \varphi_{\mu(2)}(x)$, $\delta_\epsilon \varphi_{\mu(2)} = 2\,\partial_\mu \epsilon_\mu$ $\left(\delta_\epsilon h_{\mu\nu} = 2\,\partial_{(\mu}\epsilon_{\nu)}\right)$

  - $S_0[\varphi_{\mu(2)}] = -\frac{1}{2}\int d^4x \left[\partial^\nu \varphi^{\mu(2)} \partial_\nu \varphi_{\mu(2)} + \dots\right]$

  - $\delta_\epsilon S_0[\varphi_{\mu(2)}] = 0 \iff \partial^\mu \overset{(1)}{G}_{\mu\nu}(x) \equiv 0$, $\overset{(1)}{G}_{\mu\nu} := \overset{(1)}{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overset{(1)}{R}$.

• Fronsdal's formulation

• $\varphi_{\mu_1\dots\mu_s} = \varphi_{(\mu_1\dots\mu_s)} = \varphi_{\mu(s)}$,

↳ Gauge transformation : $\delta_\epsilon \varphi_{\mu_1\dots\mu_s} = s\, \bar\nabla_{(\mu_1} \epsilon_{\mu_2\dots\mu_s)}$

Constr. : $\bar g^{\mu\nu} \bar g^{\rho\sigma} \varphi_{\mu\nu\rho\sigma\dots} \equiv 0 \quad (s \geqslant 4)$, $\quad \bar g^{\mu\nu} \epsilon_{\mu\nu\dots} \equiv 0 \quad (s \geqslant 3)$

• $S^{Fr}[\varphi] = \int \mathcal{L}(\varphi, \bar\nabla\varphi)$, $\qquad \dfrac{\delta S^{Fr}}{\delta \varphi_{\mu(s)}} =: G^{\mu(s)} \approx 0$

$\nabla^{\mu_1} G_{\mu_1\mu_2\dots\mu_s} \propto \bar g_{(\mu_2\mu_3} \nabla^\alpha G'_{\mu_4 \quad \mu_s)\alpha}$ Noether identity

AdS / CFT & open problems

$\lambda \sim \left(\frac{R^2}{\alpha'}\right)^2$

$\lambda \to 0$

$\boxed{HS_4 / CFT_3}$ [Sezgin-Sundell, Klebanov-Polyakov]

$R \ll l_s$, $N \to \infty$

$\Updownarrow$

$\lambda \ll 1$, $N \to \infty$

BC on φ	type A	type B
$\Delta = 1$	UV fixed-pt Free singlet theory CFT_3	Gross-Neveu model critical
$\Delta = 2$	critical O(N) model	Free Fermions CFT_3

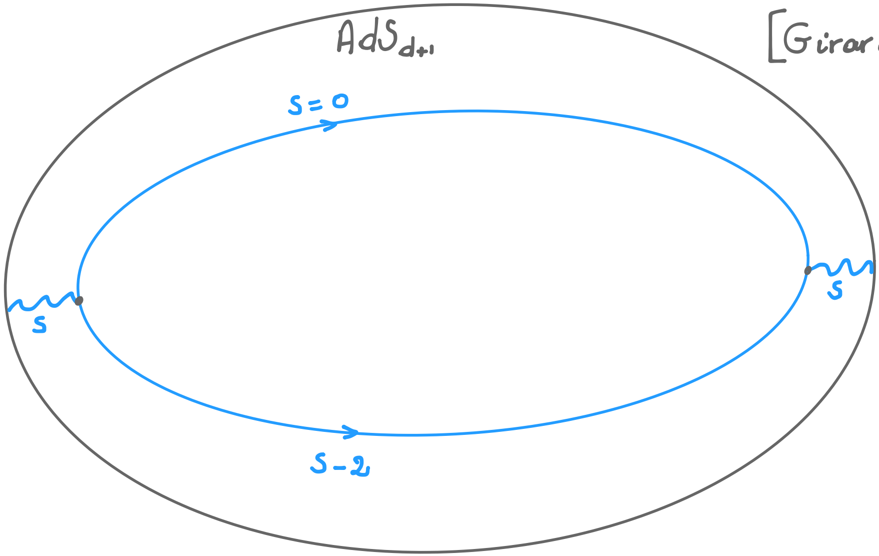
$\frac{G}{R^2} \sim \frac{1}{N}$

where G = Newton's.

$\boxed{HS_3 / CFT_2}$ Prokushkin-Vasiliev $\longleftrightarrow$ Minimal model CFT_2

[Gaberdiel-Gopakumar]

When bulk scalar field in $\Delta = 2$ BC,



[Girardello - Porrati - Zaffaroni]

- Boundary CFT: $\partial^\mu J^{(s)}_{\mu\nu(s-1)} = \frac{1}{\sqrt{N}} \partial_\nu J^{(0)} \cdot J^{(s-2)}_{\nu(s-2)}$

- Bulk: Gives mass to $s > 2$ fields, perturbatively. Spin fields $s \leq 2$ protected.