Linearized 3D higher-spin dynamics from unfolding

Nicolas Boulanger

Service de Physique de l'Univers, Champs et Gravitation

Université de Mons – UMONS

From works in collaboration with D. Ponomarev, E. Sezgin & P. Sundell [1412.8209]

and with Th. Basile & R. Bonezzi [1701.08645]

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① Motivations

- 3D spacetimes admit a rich variety of fundamental systems, either topological (Chern-Simons) or locally propagating.

- AdS_3/CFT_2 Gaberdiel-Gopakumar HS_3/CFT_2

  ↳ bulk non-linear dynamics believed to be Prokushkin-Vasiliev higher-spin gravity coupled to matter fields.

No fully non-linear higher-spin extension of TMG, the latter has 1 bulk massive graviton & interesting black-hole solutions.

⟹ In this 3D lecture, we review the classification of bosonic, propagating tensor fields in AdS_3, as well as a detailed presentation of free conformal higher-spin geometry

↪ First part with D. Ponomarev, E. Sezgin & P. Sundell

↪ Second part with Th. Basile & R. Bonezzi

extra motivation with 3D

<u>Note</u> : 2+1-dimensional system can also accommodate fractional spin fields. Interacting systems unifying 3D higher-spin gravity with internal U(N) gauge fields via fractional-spin "matter" fields have been constructed

↪ with P. Sundell & M. Valenzuela, not reviewed here

[1504.04286]

② Classification of propagating, 3D higher-spin tensor fields

Unfolding ⇝ Cartan formulation of dynamical system with local d.o.f. in ∞ towers of zero-forms Lorentz tensors that carry (U)IR of bkgd spacetime isometry algebra.

→ makes identification of local d.o.f. systematic

→ makes possible construction of fully non-linear HS-symmetric equations (M.A. Vasiliev).

Aim at

HS generalization of various higher-derivative extensions of gravity including critical versions with log. modes.

- In AdS_3 : propagating modes caracterized by labels of

$$\mathfrak{so}(2,2) \simeq \mathfrak{so}(1,2)_{(+)} \oplus \mathfrak{so}(1,2)_{(-)}$$ isometry algebra,

or equival. by eigenvalues of $M^2 := -P^a P_a$ mass-like operator

and by an IRREP of Lorentz $\mathfrak{so}(1,2)$.

Conventions • $\{M_{AB}\}$ Hermitian generators of $\mathfrak{so}(2,2)$

$$A, B = 0', 0, 1, 2$$

$\{P_a := \lambda M_{0'a}\}$ $a = 0, 1, 2$ AdS_3 transvections $[P_a, P_b] = i\lambda^2 M_{ab}$

$\{M_{ab}\}$ Lorentz, $(\eta_{ab}) = \mathrm{diag}(-,+,+)$ $\nabla := d - \frac{i}{2}\omega^{ab} e(M_{ab})$

$$J_a^{(\varepsilon)} := \tfrac{1}{2}\left(M_a + \tfrac{\varepsilon}{\lambda} P_a \right), \quad \varepsilon = \pm, \quad \text{generators of } \mathfrak{so}(1,2)_{(\varepsilon)}$$

$$M^a M_a = \tfrac{1}{4} \varepsilon^{abc} \varepsilon_{ade} M_{bc} M^{de} = -\tfrac{1}{4} \cdot 2 M_{bc} M^{de} = -\tfrac{1}{2} M_{ab} M^{ab} \Rightarrow C_2[\text{Lor.}] = -M^a M_a .$$

$$M^a := \tfrac{1}{2} \varepsilon^{abc} M_{bc}, \quad \varepsilon^{012} = +1 . \qquad C_2\left(\mathfrak{so}(1,2)_{(\varepsilon)}\right) := \eta^{ab} J_a^{(\varepsilon)} J_b^{(\varepsilon)}$$

$$\frac{M^2}{\lambda^2} \equiv C_2(\mathfrak{so}(2,2)) - C_2(\mathfrak{so}(1,2)) \equiv \tfrac{1}{2} M^{AB} M_{AB} - \tfrac{1}{2} M^{ab} M_{ab},$$

$$L_j^{\pm} := M_{0j} \mp i M_{0'j} \quad (j = 1,2), \qquad E := \lambda M_{0'0} \equiv P_0 \equiv \lambda\left(J_0^{(+)} - J_0^{(-)}\right)$$

$$S := M_{12} = -\left(J_0^{(+)} + J_0^{(-)}\right), \qquad [E, L_i^{\pm}] = \pm \lambda L_i^{\pm},$$

$$[L_i^+, L_j^-] = 2i M_{ij} - \tfrac{2}{\lambda} \delta_{ij} E$$

• Local degrees of freedom captured by $\mathcal{C}_{(s)} = \{ \phi_{a(s)}, \phi_{a(s+1)}, \ldots \}$

$\left[\phi_{a(s)} \equiv \phi_{a_1 \ldots a_s} \right.$ totally symmetric & traceless $so(1,2)^{\text{Lo.}}$ tensor $\left.\right]$

obeying

$$\nabla_b \phi_{a(n)} = \phi_{ba(n)} + \frac{n\lambda^2}{2n+1} \left(\frac{M_0^2}{\lambda^2} + 1 - n^2 \right) \eta_{b\{a} \phi_{a(n-1)\}} \, , \quad n \geqslant 0$$

$\rightsquigarrow$ SCALAR

$$\boxed{\begin{aligned} \nabla_b \phi_{a(n)} = \phi_{ba(n)} &+ \frac{-\mu}{n+1} \varepsilon_{b\,a}{}^{c} \phi_{a(n-1)c} \\ &+ \frac{n^2 - s^2}{n(2n+1)} \left(\frac{\mu^2}{s^2} - \lambda^2 n^2 \right) \eta_{b\{a} \phi_{a(n-1)\}} \end{aligned}}$$

for $n \geqslant s > 0$

$\rightsquigarrow$ massive spin-s field

For $\mu = 0$, linearized Prokushkin–Vasiliev equation at critical point

$\nu = -(2s+1)$ where ν deformation parameter $\rightsquigarrow hs(\lambda)$ where $\lambda = \frac{1-\nu}{2}$

Indecomposable structure

- $\nabla_b \phi_{a(n)} = \phi_{ba(n)} + \overbrace{\frac{\mu}{n+1}}^{\mu_n} \varepsilon_{ba}{}^{c} \phi_{a(n-1)c} + \overbrace{\frac{n^2-s^2}{n(2n+1)}\left(\frac{\mu^2}{s^2} - \lambda^2 n^2\right)}^{\lambda_n} \eta_{b\{a}\phi_{a(n-1)\}}$

$n \geq s > 0$.

At $\mu = \pm \lambda s s'$ for $s' \in \{s+1, s+2, \dots\}$, the coefficient $\lambda_{s'} = 0$,

there appears an (∞-dim.) ideal $I_{s,s'} = \{ \phi_{a(s')}, \phi_{a(s'+1)}, \dots \}$

What is left after quotient : $R_{s'} = \{ \phi_{a(s)}, \phi_{a(s+1)}, \dots, \phi_{a(s'-1)} \}$

$$\boxed{\mathcal{V} \simeq I_{s,s'} \ni R_{s'}} \qquad R_{s'} \ni I_{s',s}$$

For example $\mu = \lambda s(s+1)$ i.e. $s' = s+1$: $I = \{ \phi_{a(s+1)}, \phi_{a(s+2)}, \dots \}$

$\nabla\phi_s = \phi_{s+1} + \mu_s \phi_s + 0$, $\nabla\phi_{s+1} = \phi_{s+2} + \mu_{s+1}\phi_{s+1} + \cancel{\lambda_{s+1}}^{0}\,\phi_s$, $\nabla\phi_{s+2} = \phi_{s+3} + \mu_{s+2}\phi_{s+2} + \lambda_{s+2}\phi_{s+1}$, ...

$\hookrightarrow$ can start here, forget about ϕ_s.

$\{ \phi_{s+1}, \phi_{s+2}, \dots \}$ closed under cov. derivative.

Mass operator : The primary tensor $\phi_{a(s)}$ obeys

- $\left(\Box - \left[\frac{\mu^2}{s^2} - \lambda^2(s+1)\right]\right)\phi_{a(s)} = 0$ For $s>0$, $\Box$ Laplace – Beltrami on AdS_3.
- $(\Box - M_0^2)\,\phi = 0$ For $s=0$ $\rightsquigarrow$ Massive Klein – Gordon.
- $C_2[so(2,2)]_{\tilde{\mathcal{D}}_s} = s^2 - 1 + \frac{\mu^2}{\lambda^2 s^2}$ $\qquad C_2[so(2,2)]_{\tilde{\mathcal{D}}_0} = \frac{M_0^2}{\lambda^2}$

Equation $n=s$: $\nabla_b \phi_{a(s)} = \phi_{ba(s)} + \frac{-\mu}{s+1}\,\varepsilon_{b\,a}{}^{c}\,\phi_{a(s-1)c}$ $(s>0)$

ε^{bad} $\qquad$ η^{ba}

$$\boxed{\nabla^\mu \phi_{\mu\nu(s-1)} = 0}$$

$$\boxed{\phi_{\nu(s)} + \frac{s}{\mu}\,\varepsilon_\nu{}^{e\sigma}\,\nabla_e \phi_{\sigma\nu(s-1)} = 0} \quad (*)$$

on top of

$$\boxed{\bar{g}^{\mu\nu}\phi_{\mu\nu e(s-2)} = 0}$$

$\Longrightarrow$ For $s=2$: Linearized TMG

Rem : Due to $\nabla\cdot\phi = 0$, no need to symmetrize over ν's in $(*)$.

• The system of equations written above $(s>0)$ can be generalized to

$$\nabla_b \phi_{a(n)} = \left[\frac{(n+1)^2 - s^2}{(2n+1)(2n+3)}\right]^{\alpha} \left[\frac{\mu^2}{s^2} - \lambda^2 (n+1)^2\right]^{\beta} \phi_{ba(n)} + \frac{\mu}{n+1}\, \varepsilon_{ba}{}^{c}\, \phi_{a(n-1)c}$$

$$+ \Theta(n-s)\, \frac{2n-1}{n} \left[\frac{n^2 - s^2}{4n^2-1}\right]^{1-\alpha} \left[\frac{\mu^2}{s^2} - \lambda^2 n^2\right]^{1-\beta} \eta_{b\{a}\, \phi_{a(n-1)\}}$$

$\Theta(x) = 1 \quad x > 0$
$\quad\;\; = 0 \quad$ otherwise

where $\alpha, \beta \in [0,1]$. The above system $\rightsquigarrow \alpha = 0 = \beta$

• If $\mu = \pm \lambda\, s\, s'$ for some $s' > s$ $\longrightarrow$ indecomposable structure

$\underline{\beta = 0}$: ideal closed under ∇ is $\{\phi_{a(n)}\}_{n = s', s'+1, s'+2, \ldots}$

$\underline{\beta = 1}$: ideal closed under ∇ is $\{\phi_{a(n)}\}_{n = s, s+1, \ldots, s'-1}$

$(\lambda_{s'} = 0)$

Spectrum and unitarity

- Lowest-energy UIR of $so(2,2) \simeq so(1,2)_{(+)} \oplus so(1,2)_{(-)}$

$\hookrightarrow$ $\mathcal{D}(e_0, s_0)$ where $e_0 \rightsquigarrow E$ of lowest-energy state

$s_0 \rightsquigarrow so(2)$ spin of the lowest-energy state

- In case of scalar field, $\mathcal{D}(1 \pm \sqrt{1 + M_0^2}, 0)$ $\quad M_0^2 \geqslant -1$

(E Hermit.)

$\rightsquigarrow$ unitary $\begin{cases} \text{for } + \text{ sign } \forall \; M_0^2 \geqslant -1 \\ \text{for } - \text{ sign for } -1 \leqslant M_0^2 \leqslant 0 \end{cases}$

AdS_{d+1}

- Quadric Casimir of $so(2,2)$:

$\hookrightarrow$ $\boxed{C_2[so(2,2)] \, |e_0, s_0\rangle = \Big(-e_0(-e_0 + 2) + C_2[so(2)]\Big) |e_0, s_0\rangle}$

- From the equation $\phi_{\nu(s)} + \frac{s}{\mu} \varepsilon_{\nu}{}^{e\sigma} \nabla_e \phi_{\sigma\nu(s-1)} = 0$ at $\mu > 0$

  we get $(e_0, s_0) = (\frac{\mu}{s} + 1, s)$ or $(e_0, s_0) = (-\frac{\mu}{s} + 1, -s)$ for $\mu < 0$

- Unitarity requires $\mu \geq s(s-1)$ or $\mu \leq -s(s-1)$ for $\mu < 0$

  ↳ Unitarity bound saturated for singleton $D(s,s)$ or $D(s,-s)$

  [$e_0 \geq |s_0|$ for $D(e_0, s_0)$, $e_0 \geq s + d - 2$ for $so(2,d)$ spin-s field in AdS_{d+1}]

- For $\mu = 0$, unitarity requires $D(1, \pm 1)$ i.e. $s = \pm 1$.

↳ In terms of labels of $so(1,2)_{(\varepsilon)}$, $j^{(\varepsilon)}$ eigenvalue of $J_0^{(\varepsilon)}$,

$$D\left(1 + \frac{\mu}{s}, s\right) \longleftrightarrow \left(j^{(+)}, j^{(-)}\right) = \left(\frac{1}{2}\left[1 - s + \frac{\mu}{s}\right], \frac{1}{2}\left[-1 - s - \frac{\mu}{s}\right]\right)$$

$$D\left(1 - \frac{\mu}{s}, -s\right) \longleftrightarrow \left(j^{(+)}, j^{(-)}\right) = \left(\frac{1}{2}\left[1 + s - \frac{\mu}{s}\right], \frac{1}{2}\left[-1 + s + \frac{\mu}{s}\right]\right)$$

- Lowest-energy rep. ⇝ (Lowest-weight rep. $so(1,2)_{(+)}$) $\otimes$ (Highest-weight rep. $so(1,2)_{(-)}$)

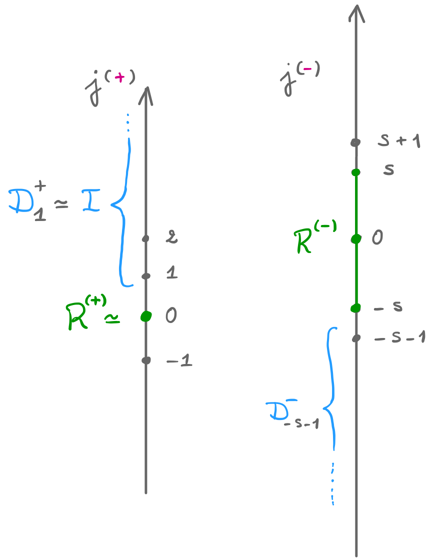
• Critical points $\mu = \pm\, s\, s'$, $s' \geqslant s+1$. Unitarity: $\mu \geqslant s(s-1)$

• Finite-dim. rep. $R_{s,s'}$ appear in $\mathcal{V}(e_0, s_0)$, $(e_0, s_0) = \left(\frac{\mu}{s} + 1, s\right)$ for $\mu = s s'$

From e.g.

$s' = s+1$: $\left(j_-^{(+)}, j_-^{(-)}\right) = \left(\frac{1}{2}\left[1 - s + \frac{\mu}{s}\right], \frac{1}{2}\left[-1-s-\frac{\mu}{s}\right]\right) \xrightarrow{\mu = s(s+1)} (1, -s-1)$

LWR HWR



• $R_{s,\, s'=s+1} \simeq R^{(+)} \otimes R^{(-)}$, $\dim = s'^2 - s^2$

↳ All components in $\{\phi_{a(s)}, \dots, \phi_{a(s'-1)}\}$

→ $so(1,2)_{(+)}$ $j_-^{(+)} = \frac{1}{2}(1 + s - s')$ positive lowest w.

→ $so(1,2)_{(-)}$ $j_-^{(-)} = \frac{1}{2}(-1 + s + s')$ negative highest w.

• $\mathcal{D}(s+2, s) \simeq \mathcal{D}_1^+ \otimes \mathcal{D}^-_{-s-1}$

Singletons

- For $\mu \geq s(s-1)$ saturated, $\mathcal{D}(e_0 = \frac{\mu}{s} + 1, s) = \mathcal{D}(s, s)$ UIR

$\hookrightarrow$ spin-s singletons.

Critical mass : $\rho(-P^a P_a) = \frac{\mu^2}{s^2} - \lambda^2 (s+1) \xrightarrow{\mu = \lambda s(s-1)} \lambda^2 \left((s-1)^2 - s - 1\right)$

$$\longrightarrow \left(\Box - \lambda^2 s(s-3)\right) \Phi_{a(s)} = 0 \quad , \quad \nabla^a \Phi_{a\,b(s-1)} = 0$$

$$\eta^{aa} \Phi_{a(s)} = 0 \qquad s \geq 2 .$$

$\Rightarrow$ The modules $\mathcal{D}(1 + \frac{\mu}{s}, s)$ carry the d.o.f. of spin-s generalization of TMG at generic $\mu > 0$.

Several towers of zero-forms

- The systems studied so far : not most general !

$$\begin{pmatrix} \mu_n \\ \lambda_n \end{pmatrix} \longrightarrow \begin{pmatrix} (\mu_n)^i{}_j \\ (\lambda_n)^i{}_j \end{pmatrix}$$

matrices mixing several towers of

$$\{ \Phi^i{}_{\alpha(n)} \}_{n = s, s+1, \ldots}$$

- Cartan integrability : $[\mu_n, \lambda_n] \stackrel{!}{=} 0 \quad \forall\, n$

$\underline{s = 0}$:
$$\begin{cases} (\mu_n)^i{}_j = 0 \,, \\ (\lambda_n)^i{}_j = \dfrac{n\,\lambda^2}{2n+1} \left[\dfrac{(M_0^2)^i{}_j}{\lambda^2} + (1-n^2)\, \delta^i_j \right] , \qquad n \in \mathbb{N} \end{cases}$$

$\underline{s > 0}$:
$$\begin{cases} (\mu_n)^i{}_j = \dfrac{\mu^i{}_j}{n+1} \,, \\ (\lambda_n)^i{}_j = \dfrac{n^2 - s^2}{n(2n+1)} \left[\dfrac{(\mu^2)^i{}_j}{s^2} - \lambda^2 n^2 \delta^i_j \right] , \qquad n \geq s > 0 \end{cases}$$

Mass-squared

- $\ell(-P^a P_a)\,\phi^j{}_{a(s)} = \left(\dfrac{(\mu^2)^j{}_i}{s^2} - \lambda^2(s+1)\,\delta^j_i\right)\phi^i{}_{a(s)} \quad , \quad s > 0$

  ↳ not necessarily diagonal

- $\left(\Box\,\delta^i_j - (M_0^2)^i{}_j\right)\phi^j = 0 \quad , \quad s = 0$

- $\frac{1}{s}\,\mu^i{}_j\,\phi^j{}_{\nu(s)} + \varepsilon_\nu{}^{\varrho\sigma}\nabla_\varrho\,\phi^i{}_{\sigma\,\nu(s-1)} = 0 = \nabla^\nu\phi^i{}_{\nu\mu(s-1)} \quad , \quad s > 0 .$

New-massive spin-s theory (albeit linear)

Take $\mu = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$, $i, j \in \{1, 2\}$

$$\begin{cases} \frac{m}{s}\,\phi^2{}_{\nu(s)} + \varepsilon_\nu{}^{\varrho\sigma}\nabla_\varrho\,\phi^1{}_{\sigma\,\nu(s-1)} = 0 \\ \frac{m}{s}\,\phi^1{}_{\nu(s)} + \varepsilon_\nu{}^{\varrho\sigma}\nabla_\varrho\,\phi^2{}_{\sigma\,\nu(s-1)} = 0 \end{cases}$$

$$\left[\Box - \left(\frac{m^2}{s^2} - \lambda^2(s+1)\right)\right]\Phi_{a(s)} = 0$$

$\phi^2{}_{a(s)}$ expressed in terms of $\Phi_{a(s)}$:

$$\phi^1{}_{a(s)} = \Phi_{a(s)} ,$$

↳ For $s = 2$ ⟶ Linearized NMG

- Equation factorizes as

$$\left[D\left(\frac{\lambda s}{m}\right) D\left(-\frac{\lambda s}{m}\right) \Phi^{(s)} \right]_{a(s)} = 0 \qquad (*) \qquad , \qquad \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \xrightarrow[\text{diag.}]{} \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}$$

where $\left[D(\eta)\, \Phi^{(s)} \right]_{\nu(s)} = D(\eta)_{\nu_1}{}^{\rho}\, \Phi^{(s)}_{\rho \nu_2 \dots \nu_s}$, $D(\eta)_\nu{}^\rho := \lambda\, \delta_\nu^\rho + \frac{\eta}{\sqrt{-g}}\, \varepsilon_\nu{}^{\mu\rho} \nabla_\mu$

- $D(\eta)\, \Phi^{(s)}$ is totally symmetric, traceless & divergence-free if $\Phi^{(s)}$ satisfies these constraints

=> Wave equation $(*)$ has solutions that are linear combinations of $\Phi^{(s)}_\pm$

s.t. $D\left(\frac{\lambda s}{m}\right) \Phi^{(s)}_\pm = 0$, $D\left(-\frac{\lambda s}{m}\right) \Phi^{(s)}_\pm = 0$

where $\Phi^{(s)}_\pm := \phi^1{}_{a(s)} \pm \phi^2{}_{a(s)}$

Generalized massive HS

$\mu = \mathrm{diag}(m_1, \dots, m_N)$ where $m_i \neq m_j$ if $i \neq j$.

$\hookrightarrow$ N equations $\mathcal{D}\left(\frac{\lambda s}{m_i}\right) \Phi^{(s)}_i = 0$, $i = 1, \dots, N$

Since all m_i's are different, the solution to these equations is $\Phi^{(s)}$ s.t.

$$\mathcal{D}\left(\frac{\lambda s}{m_1}\right) \mathcal{D}\left(\frac{\lambda s}{m_2}\right) \cdots \mathcal{D}\left(\frac{\lambda s}{m_N}\right) \Phi^{(s)} = 0$$

The case $N = 2$, $m_1 = m = -m_2$ was presented above

- Finally, when μ cannot be diagonalized, direct sum of $r \times r$ Jordan blocks

$$\mu_{(r)} = \begin{pmatrix} m & \lambda & & 0 \\ & m & \lambda & \\ & & \ddots & \ddots \\ 0 & & & \lambda \\ & & & m \end{pmatrix} \quad \text{with} \quad \begin{cases} \mathcal{D}\left(\frac{\lambda m}{s}\right) \phi^{i}_{(s)} = -\frac{s}{\lambda} \phi^{i+1}_{(s)} , & i = 1, \dots, r-1 \\ \mathcal{D}\left(\frac{\lambda m}{s}\right) \phi^{r}_{(s)} = 0 \end{cases}$$

Eliminating $\phi^{i+1}_{(s)}$ in terms of $\phi^{i}_{(s)}$, $\phi^1_s =: \Phi_s$:

$$\mathcal{D}\left(\frac{\lambda s}{m}\right)^r \Phi_{(s)} = 0$$: HS generalization of 3D critical massive gravity.

• $D\left(\frac{\lambda s}{m}\right)^{r} \Phi_{(s)} = 0 \longrightarrow$ on top of spin-s UIR $D\left(\frac{m}{\lambda s}+1, s\right)$

it possesses p-fold logarithmic solutions for $p = 1, \dots, r-1$

Gauge potentials : Introduce them for the μ-deformed systems

Different choices can be made,
for which some of the framed equ$^{\text{ns}}$ above become identities or constraints.

$\hookrightarrow$ does not matter for free theory, while we expect it does for interactions!

① $\Phi_{a(s)} \rightsquigarrow$ Traceless part of Fronsdal tensor for $\varphi_{\mu(s)}$ s.t. $\varphi'' = 0$

$\Rightarrow$ e.o.m. (*) order 3 for $\varphi_{\mu(s)}$

② $\Phi_{a(s)} \rightsquigarrow$ dual de Witt – Freedman curvature for $h_{\mu(s)}$ unconstrained

$\Rightarrow$ e.o.m. (*) order $s+1$

③ $\Phi_{a(s)} \rightsquigarrow$ 3D spin-s Cotton tensor for potential $\varphi_{\mu(s)}$

$\Rightarrow$ e.o.m (*) order $2s$ for $\varphi_{\mu(s)}$.

① Constrained Fronsdal potential

- $\{\omega^{\{s\}}\} = \{ e^{a(s-1)} , \omega^{a(s-1),b} \} \cup$ Background spin-2 sector : $\{ h^a , \bar{\omega}^{ab} \}$

valued in $\boxed{s-1}$ and $\begin{array}{|l|}\hline \quad s-1 \quad \\ \hline \square \\ \hline \end{array} \simeq \boxed{s-1}$ of $so(1,2)^{\text{Lor.}}$

- Single tower of zero-forms ($N = 1$) :

  - $\nabla e_{a(s-1)} = h^b \, \omega_{a(s-1),b}$

  - $$\nabla \omega_{a(s-1),b} = h^c \left(h_b \, \Phi_{a(s-1)c} - h_a \, \Phi_{a(s-2)bc} \right) + \tau \left(h_b \, e_{a(s-1)} - h_a \, e_{a(s-2)b} + \frac{s-2}{s-1} \left[h^c \, e_{cba(s-3)} \eta_{aa} - h^c \, e_{ca(s-2)} \eta_{ab} \right] \right)$$

  - $$\nabla_b \Phi_{a(s)} = \Phi_{ba(s)} + \frac{\mu}{s+1} \varepsilon_{ba}{}^{c} \, \Phi_{a(s-1)c}$$

$\vdots$

- Integrability : $\tau = (s-1)^2$.

• First equation : zero-torsion $\longrightarrow$ inject in second equation :

$$F_{\mu_1 \cdots \mu_s} = \Phi_{\mu_1 \cdots \mu_s} \qquad (1)$$

where

$$F_{\mu_1 \cdots \mu_s} := \left(\Box - s(s-3)\right) \varphi_{\mu(s)} - s \nabla_{(\mu_1} \nabla^{\nu} \varphi_{\mu_2 \cdots \mu_s)\nu}$$
$$+ \frac{s(s-1)}{2} \left[\nabla_{(\mu_1} \nabla_{\mu_2} - 2 g_{(\mu_1 \mu_2} \right] \varphi'_{\mu_3 \cdots \mu_s)}$$

$\lambda \overset{!}{=} 1$ and

$$\varphi_{\mu_1 \cdots \mu_s} := s \, e_{(\mu_1 | \mu_2 \cdots \mu_s)} \quad \text{using } h^a_\mu \quad AdS_3 \text{ backgrnd.}$$

where φ satisfies Fronsdal constraint $\varphi'' \overset{!}{\equiv} 0$.

Eq. (1) imposes

Eq (*) imposes

$$\boxed{\begin{cases} F' = 0 \\ F_{\mu_1 \cdots \mu_s} + \frac{s}{\mu} \varepsilon_{(\mu_1}{}^{\nu e} \nabla_{|\nu} F_{e|\mu_2 \cdots \mu_s)} = 0 \end{cases}}$$

. For $s = 1$: TM photon. For $s = 2$: Linearized TMG

Rem 1) In $D \geq 4$ the Lopatin-Vasiliev set $F = 0$ and zero-torsion constraints, leaving equation

Weyl = Rieman "$\Phi^{\{s\}} = \nabla^s \varphi$", $\Phi^{\{s\}} \sim \boxed{}_{so(1,D-1)}$

Here in 3D, primary zero-form $\rightsquigarrow$ traceless part of Fronsdal 2 ∂'s.

2) Using gauge invariance of field equations, the De Donder gauge for φ_s :

$$\begin{cases} \left[\Box - s(s-3)\right]\left(\varphi_{\mu_1 \dots \mu_s} + \frac{s}{\mu}\, \varepsilon_{(\mu_1|}{}^{\nu e}\, \nabla_\nu\, \varphi_{e|\mu_2 \dots \mu_s)}\right) = 0 \\ \varphi'_{\mu(s-2)} = 0 \qquad \nabla^\nu \varphi_{\nu\mu(s-1)} = 0 \end{cases}$$

Expanding φ in lowest-energy UIRs gives

$$(e_0, s_0) : \quad (s, s) \oplus (s, -s) \oplus \left(1 + \frac{\mu}{s}, s\right)$$

(e_0, s_0) : $(s, s) \oplus (s, -s) \oplus \left(1+\frac{\mu}{s}, s\right)$

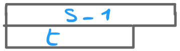
Boundary modes spin-s singleton
from gauge potential sector
$$\left(\Box - s(s-3)\lambda^2\right)\varphi = 0$$

TM spin-s mode carried by zero-form
$\Phi_{a(s)}$ sector. If $\mu = \lambda\, s(s-1)$, propagating mode drops out
but log. mode appears due to degeneracy with singleton
in gauge sector. (~critical gravity)

- In limit $\mu \to 0$, massive spin-s mode $\longrightarrow$ $D(1,s)$ non-unit. for $s>1$
spin-s analog of $(1,2)$ state in conformal CS gravity
(PM graviton depth-2) [Afshar, Cvetkovic, Ertl, Grumiller, Johansson 2012]
$e_0 = s + d - 1 - t$

② Unconstrained potentials

$$\omega^{\{s\}} = \{ \omega_{m(s-1),\, n(t)} \}, \qquad t = 0, 1, \dots, s-1$$

valued in 

of $gl(3)$.

Unfolded system :

- $$\nabla \omega_{m(s-1),\, n(t)} - h^p\, \omega_{m(s-1),\, n(t)p}$$
$$- \rho_t \left(h_n\, \omega_{m(s-1),\, n(t-1)} - \frac{s-1}{s-t}\, h_m\, \omega_{m(s-2)n,\, n(t-1)} \right) = 0 \quad , \; t < s-1$$

- $$\nabla \omega_{m(s-1),\, n(s-1)} - \rho_{s-1} \left(h_n\, \omega_{m(s-1),\, n(s-2)} - (s-1)\, h_m\, \omega_{m(s-2)n,\, n(s-2)} \right)$$
$$= h^p h^q\, \varepsilon_{pq}{}^{c_s}\, \varepsilon_{m_1 n_1}{}^{c_1} \cdots \varepsilon_{m_{s-1} n_{s-1}}{}^{c_{s-1}}\, \Phi_{c_1 \dots c_s} \quad , \; t = s-1$$

- $$\nabla_b \Phi_{a(s)} = \Phi_{ba(s)} + \frac{\mu}{s+1}\, \varepsilon_{ba}{}^{c}\, \Phi_{a(s-1)c}$$
$$\vdots$$

Cartan integrability : $\rho_t = \lambda^2\, t(s-t)$

• Gauge - invariance under

$$\delta_\epsilon \omega_{m(s-1),n(t)} = \nabla \epsilon_{m(s-1),n(t)} - h^p \epsilon_{m(s-1),n(t)p} - \rho_t \left(h_n \epsilon_{m(s-1),n(t-1)} - \frac{s-1}{s-t} h_m \epsilon_{m(s-2)n,n(t-1)} \right) = 0 \quad , \; t < s-1$$

$$\delta_\epsilon \omega_{m(s-1),n(s-1)} = \nabla \epsilon_{m(s-1),n(s-1)} - \rho_{s-1} \left(h_n \epsilon_{m(s-1),n(s-2)} - (s-1) h_m \epsilon_{m(s-2)n,n(s-2)} \right)$$

All ω's expressed as derivatives of

$$h_{\mu_1 \dots \mu_s} := s \, h^{m_2}_{(\mu_2} \dots h^{m_s}_{\mu_s} \omega_{\mu_1)| m_2 \dots m_s} \qquad \text{unconstrained}$$

• Equation for $t = s-1$: $R_{p_1 q_1 | \dots | p_s q_s} = (-\frac{1}{2})^s \, \varepsilon_{p_1 q_1}{}^{m_1} \dots \varepsilon_{p_s q_s}{}^{m_s} \, \Phi_{m_1 \dots m_s}$

for spin-s Riemann $\quad R_{\mu(s),\nu(s)} = \nabla_{\mu_1} \dots \nabla_{\mu_s} h_{\nu_1 \dots \nu_s} + \dots + \mathcal{O}(\lambda^2)$

$\sim$ 

gauge invariant

• Take s duals of R :

$$\tilde{R}_{m_1 \dots m_s} := \varepsilon_{m_1}{}^{p_1 q_1} \dots \varepsilon_{m_s}{}^{p_s q_s} R_{p_1 q_1 | \dots | p_s q_s}(h)$$

$$\Rightarrow \quad \tilde{R}_{m(s)} = \Phi_{m(s)} \quad \text{last equation}, \quad \nabla^m \tilde{R}_{m\, n(s-1)} \equiv 0$$

identically

Implies
$$\begin{cases} \tilde{R}'_{m(s-2)} = 0 & \text{and} \\ \tilde{R}_{\mu(s)} + \frac{s}{\mu} \varepsilon_{\mu_1}{}^{e\sigma} \nabla_e \tilde{R}_{\sigma \mu_2 \dots \mu_s} = 0 \end{cases}$$

• Degrees of freedom studied in $Mink_3$ for $s=2$ & $s=3$ by

[Bergshoeff, Hohm, Townsend & Kovacevic, Rosseel, Yihao Yin]

The unfolded analysis made on $\Phi_{a(s)}$ makes it direct, $\forall\, s$.

- Extension to two towers $\{\Phi^{i=1,2}\}$

$$\Phi_{a(s)} \longrightarrow \Phi^{1}_{(s)} \quad : \quad \tilde{R}_{m(s)} = \Phi^{1}_{m(s)} ,$$

Then,

$$\begin{cases} \varepsilon_{\nu}{}^{\rho\sigma} \nabla_{\rho} \Phi^{1}{}_{\sigma\nu(s-1)} + \frac{m}{s} \Phi^{2}{}_{\nu(s)} = 0 \\ \varepsilon_{\nu}{}^{\rho\sigma} \nabla_{\rho} \Phi^{2}{}_{\sigma\nu(s-1)} + \frac{m}{s} \Phi^{1}{}_{\nu(s)} = 0 \end{cases}$$

yields

$$\begin{cases} \Box \tilde{R}_{\nu(s)}(h) - \left(\frac{m^2}{s^2} - \lambda^2 (s+1) \right) \tilde{R}_{\nu(s)} = 0 \\ \tilde{R}'_{\nu(s-2)} = 0 \end{cases}$$

$\Rightarrow$ New massive spin-s expressed in terms of unconstrained $h_{\mu(s)}$ potential.

How to make trace constraint $\Phi^{a}{}_{ab(s-2)} = 0$ an identity?

$\hookrightarrow$ Yet another 1-form module.

③ Potentials for 3D conformal spin-s

[Th. Basile, R. Bonezzi, N.B.]

- The Cotton tensor is identically traceless and div-free.

  It is built from $2s-1$ ∂'s of $h_{\mu(s)}$ gauge potential

  $\Rightarrow$ needs more connection 1-forms.

- Solution is known : [Pope-Townsend, Fradkin-Linetsky '89]

$$A_\mu^{A(s-1),B(s-1)} \quad \text{valued in} \quad \begin{array}{|c|}\hline s-1 \\ \hline \\ \hline\end{array} \quad \text{of} \quad so(2,3).$$

- See [Shaynkman, Tipunin, Vasiliev 2006, Vasiliev 2010]

  for general case $so(2,d)$. Also Preitschopf-Vasiliev 1999

- One can even enlarge to $\begin{array}{|c|}\hline s-1 \\ \hline \\ \hline\end{array}$ of $gl(5)$

So as to recognize a double series of connections $\{ \overset{(i)}{\omega}{}^{m(s-1),n(t)} \}_{i=1,2}$

each of $gl(3)$ type studied above.

- $A := A^{M(s-1),\,N(s-1)}\; z_{M_1} \cdots z_{M_{s-1}}\; w_{N_1} \cdots w_{N_{s-1}}$

- $M = (m, +, -)$ with light-cone directions $z_\pm := z_3 \pm z_{0'}$

  $\partial^\pm := \frac{1}{2}(\partial^3 \pm \partial^{0'})$

  $m = 0, 1, 2$ $so(1,2)$.

  $\eta_{+-} = 2 = (\eta^{+-})^{-1}$

- $(z_M \partial_z^M - s + 1)A = 0 = (w_N \partial_w^N - s + 1)A$, $z_M \partial_w^M A = 0$.

- $so(2,3)$ generators $J_{MN} := 2(z_{[M} \partial^z_{N]} + w_{[M} \partial^w_{N]})$

- $P_m := J_{m+}$ translation

- $D_0 := d + h^m P_m = d + h^m [2 z_m \partial_z^- + 2 w_m \partial_w^- - z_+ \partial^z_m - w_+ \partial^w_m]$

- Weight $\Delta := z_+ \partial_z^+ + w_+ \partial_w^+ - z_- \partial_z^- - w_- \partial_w^-$.

- $e^{m(s-1)} := A^{m(s-1),\,+\cdots+}$, $F^{m(s-1)} := A^{m(s-1),\,-\cdots-}$ glued to $\Phi_{a(s)}$.

$$\begin{cases} D_0 A^{M(s-1),N(s-1)} = 0 \quad , \quad \Delta > -(s-1) \\ dF^{m(s-1)} = h^r h^s \varepsilon_{rsn} \Phi^{m(s-1)n} \quad , \quad \Delta = -(s-1) . \end{cases}$$

In $gl(5)$ covariant way : $D_0 = d + \frac{1}{2} \Omega^M{}_N J^N{}_M$

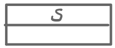
where only $\Omega_{m-} = 2 h_m$ non-zero (flat space here)

- As usual in HS, introduce V^M s.t.

$$V^M V_M = 0 \quad \text{nul} \quad , \quad \text{fix} \quad V_M = \delta_{M,+} \quad , \quad \text{and}$$

- $H_0^M := D_0 V^M = \Omega^M{}_N V^N = (h^m, 0, 0)$

$$\Rightarrow \boxed{\begin{aligned} & D_0 A^{M(s-1),N(s-1)} = H_{0M} \wedge H_{0N} \Phi^{M(s),N(s)} , \\ \text{and} \quad & \Phi^{M(s),N(s)} V_N \overset{!}{=} 0 \overset{!}{=} \Phi^{M(s-1),PQ\,N(s-2)} \eta_{PQ} \end{aligned}}$$

$\Phi \sim$ 

$\Delta(\Phi) = -(s-1)$

manifestly $gl(5)$-cov.

$\hookrightarrow$ non-zero components $\Phi^{m(s),n-\cdots-} \sim$ 

$\sim$ 

$so(1,2) \longrightarrow \Phi^{m(s)}$.

- Decompose

$$A^{M(s-1),N(s-1)} = X^{M(s-1),N(s-1)} + Z^{\{M(s-1),N(s-3)}\eta^{NN\}}$$

$\hookrightarrow$ $so(2,3)$ valued

and $\quad Tr := \dfrac{\partial^2}{\partial w^M \partial w_M}$. It obeys $[\,Tr, D_0\,] = 0$.

$$\Rightarrow \begin{cases} D_0 X^{M(s-1),N(s-1)} = H_{0M} \wedge H_{0N} \; \Phi^{M(s),N(s)} \\ D_0 Z^{M(N-1),N(s-3)} = 0 \end{cases}$$

as expected : Z is decoupled $\rightarrow$ trivial cycle .

- Useful to keep Z for technical reasons :

branching of $gl(5)$ w.r.t. $gl(3)$ subgroup .

- Once set $\{ \overset{(i)}{\omega}{}^{m(s-1),n(t)} \}_{i=1,2}$ is found , easy to decouple $Z^{M(s-1),N(s-3)}$.

so as to present the spectrum of Pope-Townsend in vector components .

Unfolded system, in 3D notation : 3D conf. spin-s.

Spectrum :

$\Delta = s-1$ $\qquad e^{a(s-1)} \equiv \omega^{a(s-1)}$ $\qquad$ traceless.

$\vdots$

$\Delta = s-k-1$ $\qquad \overset{(1)}{\omega}{}^{m(s-1),n(k)}$ $\qquad$ s.t. $\qquad \overset{(1)}{\omega}{}^{p(k)\,m(s-k-1),q(k)}(\varepsilon_{pqr})^k =: \tilde{\omega}_{r(k);}{}^{m(s-k-1)}$

$\begin{array}{|c|}\hline s-1 \\ \hline k \\ \hline \end{array}\ gl(3)$ $\qquad$ obeys $\qquad \eta_{mm}\,\tilde{\omega}_{r(k);}{}^{m(s-k-1)} \overset{!}{=} 0 \equiv \tilde{\omega}_{r(k-1)p;}{}^{pm\dots}$

$\vdots$

$k \in \{1,\dots,s-2\}$

$\Delta = 0$: $\left\{ X^{m(s-1),n(s-1)}\,,\ B^{m(s-2),n(s-2)} \right\} \rightsquigarrow \begin{array}{|c|}\hline s-1 \\ \hline \\ \hline \end{array}$ & $\begin{array}{|c|}\hline s-2 \\ \hline \\ \hline \end{array}$ of $gl(3)$.

$\Delta = -(s-k-1)$: $\qquad \overset{(2)}{\omega}{}^{m(s-1),n(k)}$ $\qquad$ obeying $\qquad \overset{(2)}{\tilde{\omega}}{}_{r(k);}{}^{mnp(s-k-3)}\,\eta_{mn} \overset{!}{=} 0$

$\vdots$

$k \in \{s-2,\dots,1\}$

$\Delta = -(s-1)$: $\qquad F^{m(s-1)}$ $\qquad$ obeying $\qquad F^{pq\,m(s-3)}\,\eta_{pq} \overset{!}{=} 0$.

1602.01682

For a detailed analysis in $s=3$ case, see also Linander & Nilsson

Full system of 1st order diff. equations, integrable :

$$
\begin{cases}
\hat{\omega}_{(1)}^{m(s-1),n(k-1)} := \omega^{m(s-1),n(k-1)p}{}_{p} \,, \\
\hat{\omega}_{(2)}^{m(s-2),n(k)} := (2k-s)\, \omega^{m(s-2)p,m(k)}{}_{p} + k(s-2)\, \omega^{m(s-3)np,mn(k-1)}{}_{p}
\end{cases}
$$

- $$d\omega^{m(s-1),n(k)} + h_p\, \omega^{m(s-1),n(k)p} + c_k\, h^m\, \hat{\omega}_{(2)}^{m(s-2),n(k)} + d_k \left[h^n\, \hat{\omega}_{(1)}^{m(s-1),n(k-1)} - \frac{s-1}{s-k}\, h^m\, \hat{\omega}_{(1)}^{m(s-2)n,n(k-1)} \right] = 0$$

  $\hookrightarrow k \in \{0, \dots, s-3\}$

$$\boxed{c_k = -\frac{s-1}{(s-k-2)(s-k)} \,, \quad d_k = \frac{k}{2(s-k-1)}}$$

- $$d\omega^{m(s-1),n(s-2)} + h_p\, X^{m(s-1),n(s-2)p} + h^m B^{m(s-2),n(s-2)} + \frac{s-2}{2} \left[h^n\, \hat{X}^{m(s-1),n(s-3)} - \frac{s-1}{2}\, h^m\, \hat{X}^{m(s-2)n,n(s-3)} \right] = 0$$

  $\hookrightarrow k = s-2$

- $$dX^{m(s-1),n(s-1)} + h^{\{n} f^{m(s-1),n(s-2)\}} = 0 = dB^{m(s-2),n(s-2)} - \tfrac{1}{2} h_p\, f^{pm(s-2),n(s-2)} + \dots$$

  $\hookrightarrow k = s-1$.

  where $f^{m(s-1),n(k)} := \overset{(2)}{\omega}{}^{m(s-1),n(k)}$

- $$d f^{m(s-1),n(2s-2-k)} + h^{\{n} f^{m(s-1),n(2s-3-k)\}} = 0 \qquad k = s, s+1, \dots, 2s-3$$

- $$dF^{a(s-1)} = h^p h^q \epsilon_{pqr}\, \Phi^{a(s-1)r} \,.$$

- Schouten tensor

$$\tilde{f}_{\mu|\kappa(s-2);}{}^{m} := \underbrace{\varepsilon_{\kappa pq}\cdots\varepsilon_{\kappa pq}}_{s-2}\, \omega^{(2)}_{\mu|}{}^{p(s-2)m,\,q(s-2)}$$

Prove that $P_{\mu(s)} := \tilde{f}_{\mu|\mu(s-2);\mu}$

is the Schouten tensor, spin-s :

$\hookrightarrow \delta P_{\mu(s)} = \partial_\mu \partial_\mu \tilde{\sigma}_{\mu(s-2)}$ where $\tilde{\sigma}_{\mu(s-2)} := \varepsilon_\mu{}^{\alpha\beta}\cdots\varepsilon_\mu{}^{\alpha\beta}\,\partial_\alpha\cdots\partial_\alpha\sigma_{\beta(s-2)}$ + trace

$$\boxed{\delta_\sigma \varphi_{\mu(s)} = \frac{s(s-1)}{2}\,\eta_{\mu\mu}\,\sigma_{\mu(s-2)}}$$

s.t. & $\begin{cases} (s-3)\,\partial_\mu \tilde{\sigma}'_{\mu(s-4)} + 3\,(\partial\cdot\tilde{\sigma})_{\mu(s-3)} = 0 \\ \partial^\nu P_{\nu\mu(s-1)} - (s-1)\,\partial_\mu P'_{\mu(s-2)} \equiv 0 . \end{cases}$

- $\Phi^{e(s)} = (\varepsilon^{\mu\nu e}\,\partial_\mu)^{s-1}\, P_{\nu(s-1)}{}^{e}$ Cotton .

as in [Henneaux - Hörtner - Leonard 1511.07389]

OFF-shell conformal geometry

↳ Automatic via Cartan formulation : $W := \Omega + A$

spin-2 backgnd + spin-s field

Ω t.q. $d\Omega + \Omega \wedge \Omega = 0$.

Unfolded system

$\Leftrightarrow \quad F_{[2]} := dW + W \wedge W = \Phi_{[2]} := H_0^R \wedge H_0^S \, \Phi^{M(s-1)R \;\; N(s-1)}{}_{R,\;\;\; S} \; z_{M_1} \cdots z_{M_{s-1}} \, w_{N_1} \cdots w_{N_{s-1}}$

$F_{[2]} \equiv d\Omega + \Omega^2 + D_0 A = D_0 A$

- If Cotton $= 0$ then W flat, hence g s.t. $W = g^{-1} dg$.

Decompose $g = g_0 \tilde{g}$ where $\Omega = g_0^{-1} d g_0 \qquad \tilde{g} = 1 + \Sigma$

$\Rightarrow \; W = \Omega + D_0 \Sigma_1 \quad \Leftrightarrow \quad A = D_0 \Sigma$ where $\Sigma_1^{M(s-1), N(s-1)} z_M^{s-1} w_N^{s-1} = \Sigma_1$

In particular, $P_{\mu(s)} = \frac{s(s-1)}{2} \partial_\mu \partial_\mu \sigma_{\mu(s-2)}$.

- Conversely : If $A = D_0 \Sigma_1 \Rightarrow W$ flat $\Rightarrow$ Cotton $= 0$.

THANK YOU