Scalar Field on a higher-spin Background via Fedosov quantization

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Abstract

Conformal higher-spin gravity is the log-divergent part of the effective action of the scalar field coupled to background fields via higher-spin currents, as was defined by Segal and Tseytlin, which can be worked out over the flat space background. We revisit the problem of the scalar field in a higher-spin background and propose a manifestly covariant version thereof. The construction utilizes the Fedosov quantization of the cotangent bundle and the action is written with the help of the trace on a curved phase space that is provided by the Feigin–Felder–Shoikhet cocycle. The same construction allows one to formulate quantum mechanics on a curved space, the phase space being the cotangent bundle.

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Contents

1 Introduction

Conformal higher-spin gravity [1–3] is a rare example of a higher-spin extension of (conformal) gravity where the usual field theory concepts directly apply. For example, despite the infinite spectrum of states it is a perturbatively local field theory, a feature enforced by the Weyl symmetry; it has an action and there is some understanding of what the underlying geometry is.[1] The action can be extracted as the anomalous part of the effective action of a (massless) scalar field in a higher-spin background, which is similar to how the Yang–Mills and the Weyl gravity actions arise as conformal anomalies in the presence of the background of gauge fields and of conformal gravity, respectively.

Despite its conceptual simplicity, e.g. the action can be extracted order by order over the flat space background [3, 19, 20], realizing conformal and higher-spin symmetries in a manifestly covariant way has been an open problem. This problem was solved recently in [21]. Essentially, the notion of a higher-spin covariant derivative is encoded in the Fedosov approach to deformation quantization of the cotangent bundle (of spacetime). The notion of a higher-spin invariant measure, which is needed to write down an action, is encoded in the trace operation. Surprisingly, defining a trace (over the deformed algebra of functions) *explicitly* within the Fedosov approach took some time and was done by Feigin, Felder and Shoikhet in [22], who relied on Shoikhet's proof [23] of an extension of Kontsevich formality conjectured by Tsygan [24], known as Shoikhet–Tsygan–Kontsevich formality.

In this paper we address the problem of how to couple a (massless) scalar field to the background of (off-shell) conformal higher-spin fields. This problem was bypassed in [21] thanks to Segal's approach to conformal higher-spin gravity that allows one to fix the action directly by the gauge invariance and, hence, the actual problem *boiled down* to covariantizing the construction.

In more detail, the route between the free scalar field and conformal higher-spin fields is as follows. Being a free theory, the massless scalar on flat space possesses an infinite tower of on-shell conserved currents for all integer spin $s \geq 0$, which come from the invariance of the d'Alembert equation under the action of conformal Killing tensors [25]. The existence of these conserved currents opens the possibility of introducing interactions between the scalar field ϕ, and gauge fields of arbitrary spin, starting with the Noether coupling and completing it to all orders. For instance, the currents of spin 1 and 2,

$$J_\mu := \tfrac{i}{2}\left(\phi^*\,\partial_\mu\phi - \phi\,\partial_\mu\phi^*\right), \qquad T_{\mu\nu} := \phi^*\,\partial_\mu\partial_\nu\phi - \tfrac{2n}{n-1}\,\partial_{(\mu}\phi^*\partial_{\nu)}\phi + \phi\,\partial_\mu\partial_\nu\phi^* - (\text{traces})\,, \tag{1.1}$$

can be used to introduce gauge fields, say A_μ and $h_{\mu\nu}$ respectively, to the free scalar action, via

$$S[\phi, A] = \int_{\mathbb{R}^n} \mathrm{d}^n x\, \tfrac{1}{2}\,\phi^*\Box\phi + e\,A_\mu\,J^\mu\,, \qquad \text{and} \qquad S[\phi, h] = \int_{\mathbb{R}^n} \mathrm{d}^n x\, \tfrac{1}{2}\,\phi^*\Box\phi + \kappa\,T^{\mu\nu}\,h_{\mu\nu}\,, \tag{1.2}$$

where e and κ are coupling constants. Since both currents are divergenceless on-shell (meaning modulo the scalar field equation of motion $\Box\phi \approx 0$), and the spin 2 one is also traceless, the gauge transformations

$$\delta_\varepsilon A_\mu = \partial_\mu\varepsilon\,, \qquad \delta_\xi h_{\mu\nu} = \partial_{(\mu}\xi_{\nu)} + \eta_{\mu\nu}\,\sigma\,, \tag{1.3}$$

[1] Numerous $3d$ higher-spin gravities [4–11] lack local degrees of freedom except for [12]. The action for chiral higher-spin gravity [13–15] is available in the light-cone gauge, while covariant equations of motion are known [16–18], and there is no clear understanding of the geometry behind.

together with the transformations of the scalar field

$$\delta_\varepsilon \phi = -ie\,\varepsilon\,\phi\,, \qquad \text{and} \qquad \delta_\xi \phi = \xi^\mu \partial_\mu \phi\,, \tag{1.4}$$

leave the respective actions invariant up to second order in the coupling constants. The spin 1 case can be completed to a gauge-invariant action to all orders by adding a quadratic term in the gauge field A_μ, which amounts to re-constructing scalar electrodynamics,

$$S[\phi, A] = \tfrac{1}{2} \int_{\mathbb{R}^n} \mathrm{d}^n x\, \phi^* \Box_A \phi\,, \qquad \text{where} \qquad \Box_A = (\partial_\mu + ie\,A_\mu)(\partial^\mu + ie\,A^\mu)\,. \tag{1.5}$$

The spin 2 case is technically more involved, though similar in spirit. It requires infinitely many correction terms, which can be re-summed into the action for the conformally-coupled scalar field,

$$S[\phi, g] = \tfrac{1}{2\kappa} \int_M \mathrm{d}^n x \sqrt{-g}\ \phi^* \left(\nabla^2 - \tfrac{n-2}{4(n-1)}\,R\right)\phi\,, \qquad g_{\mu\nu} := \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad \nabla^2 := g^{\mu\nu}\,\nabla_\mu \nabla_\nu\,, \tag{1.6}$$

expanded around flat spacetime. In both of these low spin cases, the Noether coupling is completed by higher order terms in the gauge fields and suitable deformations of their gauge symmetries. The output of this procedure is an action, quadratic in the scalar field, and non-linear in the gauge fields. The all order coupling of the former to the latter is encoded in a covariant differential operator—the square of the covariant derivative in the spin 1 case and the conformal Laplacian in the spin 2 case. From this point of view, these gauge fields are *background fields* for the scalar field ϕ.

One can then integrate out the scalar field to derive an action for the background fields. To be more precise, the effective action for the scalar field ϕ can be interpreted as an action for the background fields, a point of view already advocated by Sakharov [26] in his approach to gravity as an 'induced theory'. Indeed, focusing on the spin 2 case above, the effective action boils down to the computation of the determinant of the conformal Laplacian (since the scalar field action is quadratic). In practice, one needs to resort to a regularization scheme, say for instance the use of a UV cut-off. Expanding the effective action in powers of the cut-off, several coefficients consist of local functionals of the metric g and its derivatives, which are diffeomorphism-invariant. The latter can be considered as potential actions for the metric, whose diffeomorphism invariance stems from a successful deformation of the linear gauge symmetries generated by ξ in (1.3). Weyl transformations, that is rescalings of the metric and the scalar field ϕ for the form

$$g \longmapsto \Omega^2\, g \qquad \text{and} \qquad \phi \longmapsto \Omega^{-\frac{n-2}{2}}\,\phi\,, \tag{1.7}$$

for an arbitrary (but nowhere vanishing) parameter Ω also leave invariant the action (1.6) and define an all order completion of the linear gauge transformations generated by σ. In even dimensions, only one term in the effective action is also invariant under Weyl transformations, namely the coefficient of the logarithmically divergent piece. For $n = 4$, this term is essentially the integral of the Weyl tensor squared (up to a total derivative and a topological term), which is the action for Weyl gravity.

The lessons of these low spin examples is that one can leverage the existence of conserved currents to derive an action for gauge fields introduced as sources of the aforementioned currents. For a *massless* free scalar, the latter are also traceless which leads to an additional, 'Weyl-type', symmetry. Insisting on preserving this linear gauge symmetry at the non-linear level, and after having integrated out the scalar field, leads to a unique action for the gauge fields under consideration.

This procedure generalizes to the higher-spin currents, thereby producing a coupling of the original complex scalar to a background of higher-spin gauge fields, via a differential operator, covariant under the associated higher-spin symmetries which define a non-linear completion of the linear gauge transformations

$$\delta_{\xi,\sigma} h_{\mu_1\dots\mu_s} = \partial_{(\mu_1}\xi_{\mu_2\dots\mu_s)} + \eta_{(\mu_1\mu_2}\,\sigma_{\mu_3\dots\mu_s)}\,, \tag{1.8}$$

for all integers $s \geq 1$. These were identified as the linear symmetries of *conformal higher-spin gravity* (CHSGra), a higher-spin generalization of conformal (super)gravity proposed by Fradkin and Tseytlin [27] at the free level, and studied further at the cubic level [28]. Accordingly, Tseytlin proposed to define conformal higher-spin gravity as the coefficient of the logarithmically divergent piece of the effective action of a scalar field in a higher-spin background [1]. However, as the spin 2 case already illustrates, working out *perturbatively* the exact expression of the relevant differential operator encoding this coupling for all spins $s > 2$ seems unrealistic. This is not to say that with a perturbative approach to this problem it is impossible to get/recover manifestly (higher-spin) covariant objects. It allows one to compute the conformal higher-spin gravity action at the lowest orders, and confirm that the quadratic piece is the expected one [27], as argued in [1] and worked out in details in [3] (see also [29] for a similar approach with $\mathcal{N} = 4$ super Yang–Mills theory, and [20] from a worldline perspective).

A. Segal proposed an elegant solution to the problem of coupling a (complex) scalar field to a background of higher-spin fields and computing its effective action, by resorting to symbol calculus, and more generally, to deformation quantization [2]. Without delving into technical details—that we shall review in the bulk of the paper—the idea is to translate action and its gauge symmetries which formally read

$$S[\phi, h_s] = \tfrac{1}{2}\,\langle\phi|\,\widehat{H}[h_s]\,|\phi\rangle\,, \qquad \delta_\varepsilon \widehat{H} = \widehat{\varepsilon}^\dagger \circ \widehat{H} + \widehat{H} \circ \widehat{\varepsilon}\,, \qquad \delta_\varepsilon\,|\phi\rangle = -\widehat{\varepsilon}\,|\phi\rangle\,, \tag{1.9}$$

where $\widehat{H}$ and $\widehat{\varepsilon}$ are differential operators respectively encoding the coupling to background fields h_s and gauge parameters (which appear as coefficients of these operators), into the language of symbols, i.e. functions on the cotangent bundle $T^*\mathbb{R}^n$. This approach has some computational advantages, and in particular, the cubic part of the action for CHSGra was derived [2] in this framework.

One of the drawback of both approaches outlined above, however, is that they are defined around flat spacetime. Working out the expression of conserved currents for a free scalar field on a more general background can be rather challenging, although the case of Weyl-flat space (and $\mathcal{N} = 1$ supersymmetrization thereof) has been successfully worked out [30]. More generally, formulating CHSGra around an arbitrary background or in a manifestly covariant manner, has been the subject of several works [30–34] (see also [35–39] for supersymmetric extensions, and [40] for an approach to conformal gravity using 'unfolding').

In this paper, we shall expand on the framework developed in [31] and [21], wherein Segal's ideas were combined with techniques from Fedosov quantization in order to obtain a background-independent formulation of CHSGRa, and work out how to express the coupling of a scalar field to a conformal higher-spin fields within this setting. Having both the CHSGra action, and the action for a matter scalar field coupled to it, in the same formalism would allow one to probe various aspects of this coupling.

This paper is organized as follows: in Section 2 we review the main constructions of Fedosov quantization for the cotangent bundle of any manifold (our spacetime), before introducing in Section 3 an analogue of the

Wigner function which we use to build an action for a scalar field coupled to a background of higher-spin fields in this framework. In Section 4, we illustrate our proposal by detailing the case of the conformally-coupled scalar, and show how Weyl symmetry is embedded in our formulation. We also explain how higher-spin couplings arise, together with the gauge symmetries. We conclude the paper in Section 5 with a discussion of possible further directions to be explored, and complement it with a short review of Weyl calculus in Appendix A, some computational details about Weyl transformations in the formalism presented here in Appendix B, a quick review the definition of the FFS cocycle in Appendix C, and finally a curvature expansion of the Fedosov connection is presented in Appendix D.

2 Elements of Fedosov quantization

Before spelling out our action for a complex scalar coupled to an arbitrary higher-spin background, we shall briefly review some constructions proposed by Fedosov in his seminal paper [41] on the deformation quantization of symplectic manifolds (see also his textbook [42] for more details). Readers familiar with these ideas may safely skip this section, while unfamiliar readers interested in complementary references may consult [31, App. A] which we closely follow, as well as [21] where these techniques have been used in the context of conformal higher-spin gravity.

Building the Fedosov connection. The ingredient we need is a flat connection on the Weyl bundle,

$$\mathcal{W}_{\mathcal{X}} := S(T\mathcal{X}) \otimes \hat{S}(T^*\mathcal{X}) \twoheadrightarrow \mathcal{X}\,, \tag{2.1}$$

where $\mathcal{X}$ denotes our n-dimensional spacetime manifold, and $\hat{S}(\dots)$ the completion of the symmetric algebra. To be concrete, a typical section of this bundle locally takes the form

$$\Gamma(\mathcal{W}_{\mathcal{X}}) \ \ni\ \mathsf{a}(x;y,p) = \sum_{k,l} \mathsf{a}_{a_1\dots a_k}^{b_1\dots b_l}(x)\, y^{a_1}\dots y^{a_k}\, p_{b_1}\dots p_{b_l}\,, \tag{2.2}$$

where $\{y^a\}$ and $\{p_b\}$, for $a,b = 1,\dots,n := \dim\mathcal{X}$, respectively define a basis of its cotangent and tangent space over the point $x \in \mathcal{X}$. The above section is *polynomial* in p, but is allowed to be a *formal power series* in y, in accordance with the fact that the Weyl bundle is the tensor product of the symmetric algebra of $T\mathcal{X}$, and the *completion* of the symmetric algebra of $T^*\mathcal{X}$.

The fiber at each point is isomorphic, upon extending it over $\mathbb{R}[\![\hbar]\!]$, to that of the Weyl algebra $\mathcal{A}_{2n}$ generated by the $2n$ variables y and p, whose associative (but non-commutative) product $*$ is given by

$$\big(f * g\big)(y,p) = f(y,p)\, \exp\Big(\tfrac{\hbar}{2}\,\big[\tfrac{\overleftarrow{\partial}}{\partial y}\cdot\tfrac{\overrightarrow{\partial}}{\partial p} - \tfrac{\overleftarrow{\partial}}{\partial p}\cdot\tfrac{\overrightarrow{\partial}}{\partial y}\big]\Big)\, g(y,p)\,, \tag{2.3}$$

where we denoted the contraction of Latin indices by a dot, i.e. $y\cdot p = y^a\, p_a$. This product is called the Moyal–Weyl product, see Appendix A for a review of its derivation from the perspective of symbol calculus. Note that the operation[2]

$$\hbar^\dagger = -\hbar\,, \qquad (y^a)^\dagger = y^a\,, \qquad (p_a)^\dagger = p_a\,, \tag{2.4}$$

[2]One can think of it as essentially complex conjugation, upon considering $\hbar$ as a *purely imaginary* formal parameter. When deriving the Moyal–Weyl product from the point of view of symbol calculus, as recalled in Appendix A, the $\hbar$ factor in its definition appears multiplied by the imaginary unit, which we chose to absorb in $\hbar$ itself here to simplify computations.

which also acts by complex conjugation on coefficients, defines an anti-involution of the Weyl algebra, that is

$$(f * g)^\dagger = g^\dagger * f^\dagger\,, \tag{2.5}$$

for any pair of elements f and g. The sections of the Weyl bundle can therefore be multiplied, using the Moyal–Weyl product fiberwise, and thereby making $\mathcal{W}_\mathcal{X}$ into a bundle of associative algebras. The Weyl algebra can be endowed with a grading, namely

$$\deg(y^a) = 1 = \deg(\hbar)\,, \qquad \deg(p_a) = 0\,, \tag{2.6}$$

with respect to which the Moyal–Weyl product is of degree 0.

Having recalled the definition of the Weyl bundle, we can come back to our initial goal which is to construct a flat connection on it. As it turns out, this is relatively simple, as one can show that any 1-form connection of the form $A_0 = \mathrm{d}x^\mu\, e_\mu^a\, p_a + \dots$, where e_μ^a are the components of an invertible frame field on $\mathcal{X}$ and the dots denote higher order terms in y and p, can be extended into a flat connection on $\mathcal{W}_\mathcal{X}$,

$$\mathrm{d}A + \tfrac{1}{2\hbar}\,[A, A]_* = 0\,, \qquad \text{with} \qquad A = A_0 + (\text{corrections})\,. \tag{2.7}$$

A simple way of constructing such a flat connection is to start from

$$A_0 = \mathrm{d}x^\mu \left(e_\mu^a\, p_a + \omega_\mu^{a,b}\, p_a\, y_b\right), \tag{2.8}$$

where $\omega^{a,b} := \mathrm{d}x^\mu\, \omega_\mu^{a,b}$ are the components of the torsionless spin-connection with respect to the vielbein e_μ^a, which preserves the fiber metric η^{ab}, used to raise and lower the fiber (i.e. Latin) indices. Let us introduce

$$\delta := -\tfrac{1}{\hbar}\,[\mathrm{d}x^\mu\, e_\mu^a\, p_a, -]_*\,, \qquad \nabla := \mathrm{d} + \tfrac{1}{\hbar}\,[\mathrm{d}x^\mu\, \omega_\mu^{a,b}\, p_a\, y_b, -]_*\,, \qquad R^\nabla := \left(\mathrm{d}\omega^{a,b} + \omega^{a,}{}_c\, \omega^{c,b}\right) p_a\, y_b\,, \tag{2.9}$$

so that the curvature of ∇ is simply given by

$$\nabla^2 = \tfrac{1}{\hbar}\,[R^\nabla, -]_*\,, \tag{2.10}$$

and one can easily check that

$$\delta\nabla + \nabla\delta = 0\,, \tag{2.11}$$

as a consequence of the torsionlessness of ∇. Note that δ and ∇ are respectively of degree -1 and 0 with respect to the previously introduced grading (2.6). One can then show that there exists a unique 1-form $\boldsymbol{\gamma} \in \Omega^1(\mathcal{X}, \mathcal{W}_\mathcal{X})$ of degree ≥ 2 such that

$$A = A_0 + \boldsymbol{\gamma}\,, \tag{2.12}$$

defines a *flat* connection on the Weyl bundle, with $\boldsymbol{\gamma}$ linear in p and obeying $h\boldsymbol{\gamma} = 0$, and where

$$h := \tfrac{1}{N}\, y^a\, e_a^\mu\, \tfrac{\partial}{\partial(\mathrm{d}x^\mu)}\,, \qquad N := y^a\, \tfrac{\partial}{\partial y^a} + \mathrm{d}x^\mu\, \tfrac{\partial}{\partial(\mathrm{d}x^\mu)}\,, \tag{2.13}$$

with N the number operator returning the sum of the form degree and y-degree of its argument. Equivalently, the associated covariant derivative

$$\mathfrak{D} := \mathrm{d} + \tfrac{1}{\hbar}\,[A,-]_* \equiv -\delta + \nabla + \tfrac{1}{\hbar}[\boldsymbol{\gamma},-]_*\,, \tag{2.14}$$

defines a *differential*, i.e. squares to zero, on the Weyl bundle. The 1-form $\boldsymbol{\gamma}$ can be computed order by order in y via the recursive formulae

$$\boldsymbol{\gamma}_{(2)} = h(R^\nabla) \qquad \text{and} \qquad \boldsymbol{\gamma}_{(k+1)} = h\Big(\nabla\boldsymbol{\gamma}_{(k)} + \tfrac{1}{2\hbar}\sum_{l=2}^{k-1}[\boldsymbol{\gamma}_{(l)},\boldsymbol{\gamma}_{(k+1-l)}]_*\Big) \quad \text{for} \quad k \geq 2\,, \tag{2.15}$$

which yield

$$\begin{aligned}\boldsymbol{\gamma} = -\tfrac{1}{3}\,\mathrm{d}x^\mu\, R_{\mu a}{}^c{}_b\, y^a y^b p_c - \tfrac{1}{12}\,\mathrm{d}x^\mu\,\nabla_a R_{\mu b}{}^d{}_c\, y^a y^b y^c p_d \\ - \mathrm{d}x^\mu\,\big[\tfrac{1}{60}\,\nabla_a\nabla_b R_{\mu c}{}^e{}_d + \tfrac{2}{45}\,R_{\times a}{}^e{}_b\, R_{\mu c}{}^\times{}_d\big]\, y^a y^b y^c y^d p_e + (\cdots)\end{aligned} \tag{2.16}$$

where the dots denote terms of higher order in y.[3] Introducing the notation

$$\mathcal{R} := hR^\nabla\,, \qquad \text{and} \qquad \partial_\triangledown := h\nabla\,, \tag{2.17}$$

we can re-sum the defining relations of $\boldsymbol{\gamma}$ as

$$\boldsymbol{\gamma} = \mathcal{R} + \partial_\triangledown\boldsymbol{\gamma} + \tfrac{1}{2\hbar}\,h[\boldsymbol{\gamma},\boldsymbol{\gamma}]_*\,, \tag{2.18}$$

so that the first few orders of $\boldsymbol{\gamma}$ in y can be re-written as

$$\boldsymbol{\gamma}_{(2)} = \mathcal{R}\,, \quad \boldsymbol{\gamma}_{(3)} = \partial_\triangledown\mathcal{R}\,, \quad \boldsymbol{\gamma}_{(4)} = \partial_\triangledown^2\mathcal{R} + \tfrac{1}{2\hbar}\,h\big[\mathcal{R},\mathcal{R}\big]_*\,. \tag{2.19}$$

As mentioned above, any 1-form connection valued in the Weyl algebra whose component along p_a is an invertible vielbein can be extended to a flat connection by the same mechanism as above: the vielbein piece gives rise to the differential δ, and the components of the 1-form valued in the Weyl bundle needed to flatten the original connection can be computed recursively using its contracting homotopy h. In particular, one may start from a connection containing higher-spin components which appear as terms of higher order in p (and y) in the initial data, e.g.

$$A_0 = e^a\, p_a + \omega^{a,b}\, p_a\, y_b + e^{ab}\, p_a p_b + \omega^{ab,c}\, p_a p_b\, y_c + \dots\,, \tag{2.20}$$

and find $\boldsymbol{\gamma}$ so that $A = A_0 + \boldsymbol{\gamma}$ is flat, though the 1-form $\boldsymbol{\gamma}$ will also involve the curvature of these higher-spin components.[4] The higher components of (2.20) correspond to vielbeins and spin-connections of conformal higher-spin fields within the frame-like formulation, which was developed in [43, 44].

[3] Remark that the grading (2.6) with respect to which the defining recursion relation for $\boldsymbol{\gamma}$ is given, reduces to the degree of homogeneity in y. This is a consequence of the fact that the first correction $\boldsymbol{\gamma}_{(2)}$ is linear in p so that, not only all higher order correction stay linear in p, but also the star-commutator in (2.15) reduces to the Poisson bracket piece, i.e. to its piece of order $\hbar^1$. Consequently, no $\hbar$ correction appear in $\boldsymbol{\gamma}$ in the case of interest here.

[4] Note that in the case where the initial data A_0 contain higher-spin components (higher orders in p), one should use a slightly different degree, namely one should assign degree 1 to both y and p and degree 2 to $\hbar$, so that the Moyal–Weyl product remains of degree 0 with respect to this new grading. This is actually the gradation used originally by Fedosov [41, 42], for more details see also, e.g., [21, App. E].

Lift of symbols and invariant trace. Once the Fedosov connection $\mathfrak{D}$ is constructed, we can define the lift of the symbol of a differential operator on $\mathcal{X}$, that is a function on the cotangent bundle $T^*\mathcal{X}$, say $f(x,p) \in \mathcal{C}^\infty_{pol.}(T^*\mathcal{X}) \cong \Gamma(ST\mathcal{X})$, as the (unique) section $F(x;y,p) \in \Gamma(\mathcal{X}, \mathcal{W}_\mathcal{X})$ verifying

$$\mathfrak{D}F = 0\,, \qquad F|_{y=0} = f\,, \tag{2.21}$$

i.e. the (unique) covariantly constant section of the Weyl bundle whose order 0 in y is f. In other words, starting from a function only of x^μ and p_a, one reconstruct a flat section of the Weyl bundle, which is a function of x^μ, p_a and y^a, whose dependency on y is completely determined by the covariant constancy condition, and the coefficients of these terms proportional to y are obtained from the original function of x and p. To do so, one simply needs to solve the covariant constancy condition, which can be done iteratively via

$$F_{(0)} = f \qquad \text{and} \qquad F_{(k+1)} = h\Big(\nabla F_{(k)} + \tfrac{1}{\hbar}\sum_{l=2}^{k+1}[\boldsymbol{\gamma}_{(l)}, F_{(k+1-l)}]_*\Big) \quad \text{for} \quad k \geq 0\,, \tag{2.22}$$

where $F_{(n)}$ denotes the component of the lift F homogeneous of degree n with respect to grading (2.6), i.e. it corresponds to the homogeneity both in y and $\hbar$. This leads to

$$F(x;y,p) = f + y^a\,\nabla_a f + \tfrac{1}{2}\,y^a y^b\,\big(\nabla_a\nabla_b + \tfrac{1}{3}\,R_{da}{}^c{}_b\,p_c\,\tfrac{\partial}{\partial p_d}\big)\,f + (\dots)\,, \tag{2.23}$$

at the first few orders. This lift of (fiberwise polynomial) functions establishes a bijection between the latter and covariantly constant sections of the Weyl bundle,

$$\begin{aligned} \tau : \mathcal{C}^\infty(T^*\mathcal{X}) &\stackrel{\sim}{\longrightarrow} \operatorname{Ker}(\mathfrak{D}) \subset \Gamma(\mathcal{W}_\mathcal{X}) \\ f(x,p) &\longmapsto F(x;y,p) \equiv \tau(f)(x;y,p)\,, \end{aligned} \tag{2.24}$$

and allows us to define a *star-product*, i.e. an associative but non-commutative deformation of the pointwise product, via the simple formula[5]

$$f \star g = (F * G)\big|_{y=0}\,, \qquad f, g \in \mathcal{C}^\infty_{pol}(T^*\mathcal{X})\,, \tag{2.25}$$

where $F, G \in \Gamma(\mathcal{W}_\mathcal{X})$ are the lifts of f and g respectively. Associativity simply follows from the fact that the Moyal–Weyl product in the fiber is itself associative. To summarize, we are able to define a star-product on the cotangent bundle of our spacetime $T^*\mathcal{X}$ thanks to the fact that any function can be lifted to a flat section of the Weyl bundle, wherein we can use the Moyal–Weyl star-product to multiply the flat sections corresponding to two functions on $\mathcal{X}$, and evaluate the result at $y = 0$ thereby producing another function on $T^*\mathcal{X}$.

There exists a trace (essentially unique) on the space of covariantly constant sections of the Weyl bundle, which takes the form [21]

$$\operatorname{Tr}_A(F) = \int_{x\in\mathcal{X}} \int_{T^*_x\mathcal{X}} \mathrm{d}^n p\;\mu(F|\underbrace{A, \cdots, A}_{n\text{ times}})\,, \tag{2.26}$$

[5] Remark that, by construction, the evaluation of a covariantly constant section at $y = 0$ yields the function on the cotangent bundle that it is the lift of. In other words, this simple operation is the inverse of the lift τ, i.e. $\tau^{-1}(-) = (-)|_{y=0}$. The star-product on $T^*\mathcal{X}$ can therefore be written as $f \star g = \tau^{-1}\big(\tau(f) * \tau(f)\big)$ which makes it clear that the lift τ is a morphism of algebras between $\big(\operatorname{Ker}(\mathfrak{D}), *\big)$ and $\big(\mathcal{C}^\infty(T^*\mathcal{X}), \star\big)$, the star-product on the latter being 'pulled-back' from the Moyal–Weyl one defined fiberwise.

where $\mu : \mathcal{A}_{2n}^{\wedge n} \otimes \mathcal{A}_{2n} \longrightarrow \mathbb{R}[p_a]$ is a multilinear map valued in polynomials in p, obtained from the Feigin–Felder–Shoikhet cocycle [22]. The fact that μ is obtained from a Hochschild cocycle for the Weyl algebra ensures two important properties of this trace: it is invariant under the gauge transformations

$$\delta_\xi A = \mathrm{d}\xi + \tfrac{1}{\hbar}\,[A,\xi]_*\,, \qquad \xi \in \Gamma(\mathcal{W}_{\mathcal{X}})\,, \tag{2.27}$$

of the flat connection A up to boundary terms, i.e.

$$\delta_\xi\,\mathrm{Tr}_A(F) = \int_{\mathcal{X}}\int \mathrm{d}^n p\,\left[\mathrm{d}(\dots) + \tfrac{\partial}{\partial p_a}(\dots)_a\right], \tag{2.28}$$

and it is cyclic, also up to boundary terms,

$$\mathrm{Tr}_A([F,G]_*) = \int_{\mathcal{X}}\int \mathrm{d}^n p\,\left[\mathrm{d}(\dots) + \tfrac{\partial}{\partial p_a}(\dots)_a\right], \tag{2.29}$$

for any covariantly constant sections F and G.

The detailed expression for μ is given in Appendix C, for the moment it is enough for our purpose to know that, for a flat connection A which is linear in p as the example reviewed previously, the associated trace of any lifted symbol F boils down to

$$\mathrm{Tr}_A(F) = \int_{\mathcal{X}} \mathrm{d}^n x\,|e| \int_{T_x^*\mathcal{X}} \mathrm{d}^n p\, \sum_{k\geq 0} \mu^{\nabla}_{a_1\dots a_k}(x)\,\frac{\partial^k}{\partial p_{a_1}\dots\partial p_{a_k}} F\big|_{y=0}\,, \tag{2.30}$$

where $\mu^{\nabla}_{a_1\dots a_k}(x)$ are polynomials in the curvature of ∇ and covariant derivatives thereof.

What has been reviewed above is just the Fedosov approach to deformation quantization for the particular case of the symplectic manifold being the cotangent bundle (of the spacetime), which was also studied by Fedosov himself [45]. A development since [45] is the construction of the invariant trace by Feigin, Felder and Shoikhet [22]. Let us briefly explain now, see [21] for more details, how this is related to conformal higher-spin fields. To begin with, an off-shell description of conformal higher-spin fields requires the Fedosov connection A and a covariantly constant section of the Weyl bundle F. Different types of scalar matter, i.e. whether we start out with $\mathcal{L} \sim \phi\Box\phi$ or $\mathcal{L} \sim \phi\Box^k\phi$, $k > 1$, lead to different spectra of (higher-spin) currents and, hence, to different spectra of sources/background (higher-spin) fields. An immediate consequence is that it is necessary to fix the background value for F to land on a specific theory. We will consider F of the form

$$F = p^2 + \sum_{s>2} h^{a_1\dots a_s}(x)\,p_{a_1}\dots p_{a_s} + \dots\,, \tag{2.31}$$

where the presence of p^2 here implies that we are coupling the usual free scalar field $\mathcal{L} \sim \phi\Box\phi$ to (higher-spin) background fields $h^{a_1\dots a_s}$. The formalism is flexible enough to allow one to realize conformal higher-spin fields both in the frame-like, cf. (2.20), and in the metric-like ways, as below. In this paper, we prefer to keep A purely gravitational, i.e. it is completely expressed in terms of a vielbein e^a. With the help of the ξ gauge-symmetry one can move between the frame-like and metric-like formulations.

Fock space bundle. Having constructed a bundle of Weyl algebra, let us now proceed with the definition of a vector bundle associated with the Fock representation. As a vector space, the latter can be identified with the subspace of $\mathcal{A}_{2n} \cong \mathbb{R}[y^a, p_b]$ consisting of polynomials (or even formal power series) in y, that we shall denote by $\mathfrak{F}_n \equiv \mathbb{R}[y^a]$. The representation is given by the *quantization map*,

$$\big(\rho(f)\varphi\big)(y) = f(y,p)\,\exp\Big(-\hbar\,\tfrac{\overleftarrow{\partial}}{\partial p}\cdot\big[\tfrac{1}{2}\,\tfrac{\overleftarrow{\partial}}{\partial y}+\tfrac{\overrightarrow{\partial}}{\partial y}\big]\Big)\,\varphi(y)\big|_{p=0}\,, \tag{2.32}$$

for any element $f(y,p) \in \mathcal{A}_{2n}$ of the Weyl algebra and $\varphi(y) \in \mathfrak{F}_n$ of the Fock space. That it defines a representation of the Weyl algebra means that it verifies

$$\rho(f)\circ\rho(g) = \rho(f*g)\,, \qquad f,g \in \mathcal{A}_{2n}\,. \tag{2.33}$$

The name 'quantization map' comes from the fact that it allows one to associate, to any (polynomial) function of $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$, which are nothing but elements of the Weyl algebra, a differential operator acting on the space of 'wave functions', i.e. smooth functions on $\mathbb{R}^n$, which we consider as elements of the Fock space (via for instance their Taylor series). Put differently, the pair $(\mathcal{W}_{2n}, \mathfrak{F}_n)$ can be thought of as a *flat model* for the quantization of a cotangent bundle $T^*\mathcal{X}$ with $\dim\mathcal{X} = n$, wherein the Weyl algebra models the algebra of functions on $T^*\mathcal{X}$, while the Fock space models smooth functions on the base manifold $\mathcal{X}$, on which functions on the cotangent bundle act as differential operators.

Given now an arbitrary smooth manifold $\mathcal{X}$, one can consider the 'bundle of Fock spaces' defined as

$$\mathfrak{F}_{\mathcal{X}} := S(T^*\mathcal{X}) \twoheadrightarrow \mathcal{X}\,, \tag{2.34}$$

whose sections are

$$\Gamma(\mathfrak{F}_{\mathcal{X}}) \ni \Phi(x;y) = \sum_{k\geq 0} \tfrac{1}{n!}\,\Phi_{a_1\dots a_k}(x)\,y^{a_1}\dots y^{a_k}\,, \tag{2.35}$$

that we shall extend as formal power series in $\hbar$. A Fedosov connection A defines a flat covariant derivative on this Fock bundle, whose local expression is

$$\mathfrak{D} = \mathrm{d} + \tfrac{1}{\hbar}\,\rho(A)\,, \tag{2.36}$$

with ρ is the quantization map above. A simple computation leads to

$$\rho(p_a) = -\hbar\,\tfrac{\partial}{\partial y^a}\,, \qquad \rho(y^a p_b) = -\tfrac{\hbar}{2}\,\big(y^a\,\tfrac{\partial}{\partial y^b} + \tfrac{\partial}{\partial y^b}\,y^a\big) = -\hbar\,\big(y^a\,\tfrac{\partial}{\partial y^b} + \tfrac{1}{2}\,\delta^a_b\big)\,, \tag{2.37}$$

and more generally,

$$-\tfrac{1}{\hbar}\,\rho(y^{a_1}\dots y^{a_n}p_b) = y^{a_1}\dots y^{a_n}\,\tfrac{\partial}{\partial y^b} + \tfrac{n}{2}\,\delta_b^{(a_1}y^{a_2}\dots y^{a_n)}\,, \tag{2.38}$$

so that, upon choosing ∇ to be a metric connection, and A the flat connection (2.12) built from it as explained above, one finds

$$\mathfrak{D}\Phi = \Big(-\delta + \nabla + \rho(\boldsymbol{\gamma})\Big)\Phi\,, \qquad \text{with} \qquad \delta = e^a\,\tfrac{\partial}{\partial y^a}\,, \tag{2.39}$$

which in particular, contains the *same acyclic piece* δ as in the Fedosov connection (2.12), which is an operator of degree $n-1$ in y. As a consequence, we can solve for covariantly constant sections of the Fock bundle in a similar manner as we did in the Weyl bundle: expanding the condition

$$\mathfrak{D}\Phi = 0\,, \qquad \text{with} \qquad \Phi\big|_{y=0} = \phi\,, \tag{2.40}$$

order by order in y yields

$$\delta\Phi_{(n+1)} = \nabla\Phi_{(n)} + \tfrac{1}{\hbar}\sum_{k=2}^{n+1}\rho(\boldsymbol{\gamma}_{(k)})\Phi_{(n+1-k)}\,, \tag{2.41}$$

which gives us a definition of the order $n+1$ term in the y expansion of Φ thanks to the contracting homotopy (2.13) introduced before, i.e.

$$\Phi_{(n+1)} = h\Big(\nabla\Phi_{(n)} + \tfrac{1}{\hbar}\sum_{k=2}^{n+1}\rho(\boldsymbol{\gamma}_{(k)})\Phi_{(n+1-k)}\Big)\,. \tag{2.42}$$

The whole covariant section $|\Phi\rangle$ only depends on its value at $y=0$, which is a function on $\mathcal{X}$, thereby establishing a bijection

$$\begin{aligned} \tau : \mathcal{C}^\infty(\mathcal{X}) &\xrightarrow{\ \sim\ } \mathrm{Ker}(\mathfrak{D}) \subset \Gamma(\mathfrak{F}_\mathcal{X}) \\ \phi(x) &\longmapsto \Phi(x;y) \equiv \tau(\phi)(x;y)\,, \end{aligned} \tag{2.43}$$

between $\mathcal{C}^\infty(\mathcal{X})$ and covariantly constant sections of the Fock bundle (that we denoted by the same symbol τ as the isomorphism between functions on the cotangent bundle and flat sections of Weyl bundle, in a slight abuse of notation). The first few order of the covariantly constant section associated with $\phi(x)$ read

$$\Phi(x;y) = \phi + y^a\nabla_a\phi + \tfrac{1}{2}\,y^a y^b\,\big(\nabla_a\nabla_b - \tfrac{1}{6}\,R_{ab}\big)\phi + \dots \tag{2.44}$$

where R_{ab} denotes the Ricci tensor of ∇, and the dots denote terms of order 3 or higher in y.

We can now define a quantization map in this curved setting, that is to say, a way to associate to any symbol $f \in \mathcal{C}^\infty_{pol}(T^*\mathcal{X})$, that is any fiberwise polynomial function on the cotangent bundle of $\mathcal{X}$, a differential operator $\widehat{f}$ which acts on 'wave functions', i.e. functions $\phi \in \mathcal{C}^\infty(\mathcal{X})$ on the base, defined as follows[6]

$$\big(\widehat{f}\phi\big)(x) := \rho(F)\Phi\big|_{y=0}\,, \tag{2.45}$$

where $F \in \Gamma(\mathcal{W}_\mathcal{X})$ and $\Phi \in \Gamma(\mathfrak{F}_\mathcal{X})$ are the lifts of f and ϕ respectively as a covariantly constant section of the Weyl and Fock bundles. This defines a representation of the star-product algebra $\big(\mathcal{C}^\infty_{pol}(T^*\mathcal{X}),\star\big)$ on the space of 'wave functions' $\mathcal{C}^\infty(\mathcal{X})$, i.e.

$$\widehat{f}\circ\widehat{g} = \widehat{f\star g} \qquad \forall f,g \in \mathcal{C}^\infty_{pol}(T^*\mathcal{X})\,, \tag{2.46}$$

where $\star$ is the star-product defined in (2.25). Here again, this is simply a consequence of the fact that $(\mathfrak{F}_n,\rho)$ is a representation of $(\mathcal{A}_{2n},*)$, i.e. we sort of 'pullback' the algebra and representation structure from the fiber to the base manifold $\mathcal{X}$. Note that this approach was already outlined in [31, App. A].

[6] Let us note that, as for the star-product, writing the quantization map in terms of the lift τ, namely $\widehat{f}\phi = \tau^{-1}\big[\rho\big(\tau(f)\big)\tau(\phi)\big]$, makes it apparent that the latter defines a morphism of pairs algebra-module between $\big(\mathcal{C}^\infty(T^*\mathcal{X}),\mathcal{C}^\infty(\mathcal{X})\big)$ and flat sections of the Weyl and Fock bundles. This also shows that the quantization map on $\mathcal{C}^\infty(\mathcal{X})$ is 'pulled-back' from that on flat sections of the Fock bundle, in complete parallel with the definition of the star-product on $T^*\mathcal{X}$.

3 Wigner function and quadratic actions

We have now all the ingredients needed to re-express the coupling of a scalar field to an arbitrary higher-spin background. The latter is encoded in a pair of fields,[7] namely a flat connection A and a covariantly constant section F of the Weyl bundle [31, 50],

$$\mathrm{d}A + \tfrac{1}{2\hbar}\,[A,A]_* = 0\,, \qquad \mathrm{d}F + \tfrac{1}{\hbar}\,[A,F]_* = 0\,, \tag{3.1}$$

which is invariant under the gauge transformations

$$\delta_\xi A = \mathrm{d}\xi + \tfrac{1}{\hbar}\,[A,\xi]_*\,, \qquad \delta_{\xi,w} F = \tfrac{1}{\hbar}\,[F,\xi]_* + \{F,w\}_*\,, \qquad \mathrm{d}w + \tfrac{1}{\hbar}\,[A,w]_* \stackrel{!}{=} 0\,, \tag{3.2}$$

where $\xi, w \in \Gamma(\mathcal{W}_\mathcal{X})$ are 0-form valued in the Weyl bundle, with w required to be covariantly constant, while ξ is unconstrained. The sum $\varepsilon = \frac{1}{\hbar}\xi + w$ corresponds to the symbol of an arbitrary differential operator, such as $\widehat{\varepsilon}$ appearing in (1.9), and it splits into its Hermitian and anti-Hermitian part, respectively w and ξ (though both are real, the latter is dressed with $\hbar$ that we take as imaginary in the sense that $\hbar^\dagger = -\hbar$).

As usual when dealing with gauge theories, matter fields consists of sections of vector bundles associated with representation of the gauge algebra (meaning here, the algebra in which gauge fields take values). Accordingly, we add the scalar field ϕ to the previous system in the guise of its lift as a covariantly constant section of the Fock bundle,

$$\mathrm{d}\Phi + \tfrac{1}{\hbar}\,\rho(A)\Phi = 0\,, \tag{3.3}$$

which transforms in the corresponding representation,

$$\delta_{\xi,w}\Phi = -\rho(\tfrac{1}{\hbar}\,\xi + w)\Phi\,, \tag{3.4}$$

thereby preserving the covariant constancy condition. Now all we need is an action functional implementing the coupling of ϕ to the higher-spin background in a gauge-invariant manner.

Around flat space, Segal's approach consisted in considering a quadratic action for a *complex* scalar field ϕ in flat spacetime,

$$S[\phi] = \int_{\mathbb{R}^n} \mathrm{d}^n x\ \phi^*(x)\,(\widehat{H}\phi)(x)\,, \tag{3.5}$$

for some differential operator $\widehat{H}$ which encode the coupling of ϕ to a background of gauge fields, the latter being related to the 'coefficients' of this operator. For instance, in the case of the conformally-coupled scalar, $\widehat{H}$ would be the conformal Laplacian whose expression depends on a metric g (via its inverse contracting two covariant derivatives, and via the Ricci scalar term), and which implements the coupling of ϕ to conformal gravity. The above action can formally be written as

$$S[\phi] = \langle\phi|\,\widehat{H}\,|\phi\rangle\ , \tag{3.6}$$

so that it becomes relatively simple to see that it is invariant under the following infinitesimal transformations

$$\delta_\varepsilon\,|\phi\rangle = -\widehat{\varepsilon}\,|\phi\rangle\ , \qquad \delta_\varepsilon \widehat{H} = \widehat{\varepsilon}^\dagger \circ \widehat{H} + \widehat{H}\circ\widehat{\varepsilon}, \tag{3.7}$$

[7] Such a description is obtained from an approach known as the 'parent formulation' of gauge theories, developed in [46–49] and references therein.

where ε is another, arbitrary, differential operator. Assuming that the space of operators we are working with possesses a *trace*, we can further re-write the action as

$$S[\phi] = \mathrm{Tr}\Big(\widehat{H} \circ |\phi\rangle\langle\phi|\Big)\,, \tag{3.8}$$

that is the trace of the operator $\widehat{H}$ composed with the projector $|\phi\rangle\langle\phi|$. In this form, the action can be more easily translated in terms of symbols, leading to

$$S[\phi] = \int_{T^*\mathbb{R}^n} \mathrm{d}^n p\, \mathrm{d}^n x\; \big(H \star W_\phi\big)(x,p) \tag{3.9}$$

where $H(x,p)$ and $W_\phi(x,p)$ are the symbols of the kinetic operator $\widehat{H}$ and the projector $|\phi\rangle\langle\phi|$, also known as the Wigner function, respectively. The integration over the cotangent bundle $T^*\mathbb{R}^n$ defines a trace over the space of symbols, at least those which are compactly supported or vanish at infinity sufficiently fast. Indeed, in this case one finds

$$\mathrm{Tr}(f \star g) = \int_{T^*\mathbb{R}^n} \mathrm{d}^n x\, \mathrm{d}^n p\; (f \star g)(x,p) = \int_{T^*\mathbb{R}^n} \mathrm{d}^n x\, \mathrm{d}^n p\; f(x,p)\, g(x,p) = \mathrm{Tr}(g \star f)\,, \tag{3.10}$$

for any symbols f and g, since all higher order terms in the star product are total derivatives on $T^*\mathbb{R}^n$, and hence can be ignored for the aforementioned suitable class of symbols. The transformation rule, in terms of symbols, becomes

$$\delta_\varepsilon H = \varepsilon^\dagger \star H + H \star \varepsilon\,, \qquad \text{and} \qquad \delta_\varepsilon W_\phi = -\varepsilon \star W_\phi - W_\phi \star \varepsilon^\dagger\,, \tag{3.11}$$

under which the action transform as

$$\delta_\varepsilon S[\phi] = -\,\mathrm{Tr}\left([H \star W_\phi\,, \varepsilon^\dagger]_\star\right) = 0\,, \tag{3.12}$$

i.e. the action is left invariant as a consequence of the cyclicity of the trace.

We have seen in the previous section how to define a star-product and construct the associated invariant trace via the FFS cocycle for any, possibly curved, manifold $\mathcal{X}$ so that we only need to find a suitable generalization of the Wigner function to curved settings. One can think of the Wigner function as a bilinear map

$$W : \mathfrak{F}_n \otimes \mathfrak{F}_n \longrightarrow \mathcal{A}_{2n}\,, \tag{3.13}$$

taking two elements of the Fock representation and constructing an element of the Weyl algebra out of them. For our purpose, what matters is that it possesses the following couple of properties (whose proof are recalled in Appendix A).

(i) First, it intertwines the left and right multiplication in the Weyl algebra with the Fock action

$$F * W[\Phi,\Psi] = W[\rho(F)\Phi,\Psi]\,, \qquad W[\Phi,\Psi] * F^\dagger = W[\Phi,\rho(F)\Psi]\,, \tag{3.14}$$

for any element $F(y,p) \in \mathcal{A}_{2n}$ and any pair of Fock space states $\Phi(y), \Psi(y) \in \mathfrak{F}_n$.

(*ii*) Second, integrating it over momenta yields

$$\int_{\mathbb{R}^n} \mathrm{d}^n p \, \frac{\partial^k}{\partial p_{a_1} \dots \partial p_{a_k}} W[\Phi, \Psi] = \delta_{k,0} \, \Phi(y) \, \Psi(y) \,, \tag{3.15}$$

for any Fock space elements $\Phi, \Psi \in \mathfrak{F}_n$ which are seen as embedded in the Weyl algebra on the right hand side.

A first naive guess for a curved version $\mathcal{W}_\phi$ of the Wigner function associated with a scalar field $\phi \in \mathcal{C}^\infty(\mathcal{X})$ is to simply apply the above bilinear map to two copies of its lift as covariantly constant sections of the Fock bundle, i.e.

$$\mathcal{W}_\phi(x; y, p) := W[\Phi, \Phi] = \int \mathrm{d}^n u \; e^{\frac{1}{\hbar} \, p \cdot u} \, \Phi(x; y + \tfrac{1}{2} \, u) \, \Phi^\dagger(x; y - \tfrac{1}{2} \, u) \,. \tag{3.16}$$

First of all, let us note that this is a covariantly constant section of the Weyl bundle. Indeed, upon writing it as $\mathcal{W}_\phi = W[\Phi, \Phi]$ in order to highlight the fact that it is bilinear in the covariantly constant section of the Fock bundle Φ, one finds that it verifies

$$\tfrac{1}{\hbar} \, [A, \mathcal{W}_\phi]_* = \tfrac{1}{\hbar} A * W[\Phi, \Phi] + W[\Phi, \Phi] * (\tfrac{1}{\hbar} \, A)^\dagger = W[\rho(\tfrac{1}{\hbar} \, A)\Phi, \Phi] + W[\Phi, \rho(\tfrac{1}{\hbar} \, A)\Phi] \,, \tag{3.17}$$

where we used the properties (*i*). We can then use the covariant constancy of Φ, to show that

$$\mathrm{d}\mathcal{W}_\phi + \tfrac{1}{\hbar} \, [A, \mathcal{W}_\phi]_* = 0 \,, \tag{3.18}$$

i.e. our curved version the Wigner function $\mathcal{W}_\phi$ is a covariantly constant section of the Weyl bundle. Moreover, properties (*i*) also ensure that $\mathcal{W}_\phi$ transforms as

$$\delta_{\xi,w} \mathcal{W}_\phi = \tfrac{1}{\hbar} \, [\mathcal{W}_\phi, \xi]_* - \{\mathcal{W}_\phi, w\}_* \,. \tag{3.19}$$

which implies that its star-product with the covariantly constant lift F behaves as

$$\delta_{\xi,w}(F * \mathcal{W}_\phi) = [F * \mathcal{W}_\phi, \tfrac{1}{\hbar} \, \xi - w]_* \,, \tag{3.20}$$

under the gauge transformations of the system. As a consequence, the functional[8]

$$S[\phi] = \mathrm{Tr}_A(F * \mathcal{W}_\phi) \,, \tag{3.21}$$

is well-defined, being the trace of the star-product of two covariantly constant sections of the Weyl bundle, as well as gauge invariant under all transformations listed above thanks to the cyclicity of the FFS trace, which holds *up to boundary terms*. Let us remark that, contrary to the action for CHS gravity which is expressed as the FFS trace of a symbol that dies off at infinity *both* in spacetime and in the fiber/momenta directions [21], this is not necessarily the case here: the p-dependency of the integrand may not allows us to discard boundary terms for arbitrary gauge parameters. In other words, we expect that the gauge parameters ξ and w should be restricted so as to ensure that the boundary terms appearing when checking the cyclicity/gauge invariance of the FFS trace (see [21, App. C]) can actually be neglected. Modulo this subtlety, eq. (3.21) gives a manifestly

[8]Note that the dependence on conformal higher spin fields in the action (3.21) is a little subtle to read-off: as explained in [21], they can be moved around between A and F via gauge transformations, which can therefore encode field redefinitions.

covariant and higher-spin invariant form of a coupling between the scalar field and a background of conformal higher-spin fields, which is one of the main results of the paper. On the other hand, irrespectively of the action principle, the equations of motion $\rho(F)\Phi\big|_{y=0}=0$ are well-defined and, in particular, gauge invariant. As it turns out, in the case where A is linear in p, this expression simplifies to

$$S[\phi]=\int_{\mathcal{X}}\mathrm{d}^n x\,|e|\int_{T^*_x\mathcal{X}}\mathrm{d}^n p\;W[\rho(F)\Phi,\Phi]\big|_{y=0}=\int_{\mathcal{X}}\mathrm{d}^n x\,|e|\,\phi^*(x)\,(\widehat{f}\phi)(x)\,, \tag{3.22}$$

as a consequence of the properties (i) and (ii) of the Wigner function, and the fact that the trace takes the form (2.30).

4 Conformally-coupled scalar and higher-spins

Let us give two examples to show how the formalism and the action (3.21) can reproduce what it has to, e.g. the coupling to low-spin background fields and to higher-spin background. The latter problem was studied in $d=4$ for a coupling to a spin-three field in [32].

4.1 Conformally-invariant Laplacian

As an illustration, let us show how we can recover the conformally-coupled scalar. This boils down to identifying the symbol of the conformal Laplacian,

$$\nabla^2-\tfrac{n-2}{4(n-1)}\,R\,, \tag{4.1}$$

which we can do in a couple of ways: either by working out its quantization, or by imposing that it transforms correctly under the above gauge transformations.

Let us start with the former. Considering that the quantization map yields $\widehat{p}_a=-\hbar\nabla_a$, we should consider the Ansatz $f=p^2+\alpha\,R$ for the symbol of the conformal Laplacian, where α is a numerical coefficient to be fixed. It is then enough to compute the lift of this symbol, up to order 2 in y,

$$F=\tau(p^2+\alpha\,R)=p^2+\tfrac{1}{3}\,y^a y^b\,R_a{}^c{}_b{}^d\,p_c p_d+\alpha\,\big(R+y^a\,\nabla_a R+\tfrac{1}{2}\,y^a y^b\,\nabla_a\nabla_b R\big)+\dots \tag{4.2}$$

as well as that of the scalar field ϕ at order 2 in y given previously in (2.44), and use

$$\rho(p^2)|_{y=0}=\hbar^2\,\partial_y^2\,,\qquad\qquad \rho(y^a y^b p_c p_d)|_{y=0}=\tfrac{\hbar^2}{2}\,\delta_c^{(a}\,\delta_d^{b)}\,, \tag{4.3}$$

to finds that the quantization of the Ansatz f reads

$$\widehat{f}\phi=\hbar^2\,\big(\nabla^2+[\tfrac{\alpha}{\hbar^2}-\tfrac{1}{4}]\,R\big)\phi\,, \tag{4.4}$$

which implies

$$\alpha=\frac{\hbar^2}{4(n-1)}\qquad\Longrightarrow\qquad f=p^2+\tfrac{\hbar^2}{4(n-1)}\,R\,, \tag{4.5}$$

upon imposing that it reproduces the conformal Laplacian. Note that this computation also shows that, perhaps contrary to one's intuition, the symbol of the ordinary Laplacian is not p^2, but should instead be corrected by a curvature dependent term $\frac{\hbar^2}{4}\,R$.

Let us now turn our attention to the symmetries of our action, focusing on Weyl symmetry. Having constructed the 1-form connection A from a torsionless and metric connection, its coefficients when expanded order by order in y are tensors built out of the vielbein and its derivatives only, and hence have a definite behavior under Weyl transformations[9]

$$\delta_\sigma^{\rm Weyl} e^a = \sigma\, e^a\,. \tag{4.6}$$

These Weyl transformations can be realized as gauge symmetries of A, by suitably choosing the gauge parameters $\xi_{\rm Weyl}, w_{\rm Weyl} \in \Gamma(\mathcal{W}_\mathcal{X})$. In other words, we can embed the *geometric* transformations that are Weyl rescalings, as *gauge* transformations of the system of fields A, F and Φ (which are affected by both types of parameters, ξ and w). To explicitly find the gauge parameter $\xi_{\rm Weyl}$, one needs to solve the condition

$$\mathrm{d}\xi_{\rm Weyl} + \tfrac{1}{\hbar}\,[A, \xi_{\rm Weyl}]_* \stackrel{!}{=} \delta_\sigma^{\rm Weyl} A\,, \tag{4.7}$$

for $\xi_{\rm Weyl}$ in terms of σ. This can be done as before, namely order by order in y, using the contracting h. More precisely, for $\xi_{\rm Weyl} = \sum_{k\geq 1} \xi_{(k)}$ with $\xi_{(k)}$ of order k in y and linear in p, one finds the recursion

$$\xi_{(k+1)} = h\big(\nabla\xi_{(k)} + \tfrac{1}{\hbar}\sum_{l=2}^{k}[A_{(l)}, \xi_{(k+1-l)}]_* - \delta_\sigma^{\rm Weyl} A_{(k)}\big)\,, \tag{4.8}$$

which yields

$$\xi_{\rm Weyl} = -\sigma\, y\cdot p - \nabla_a\sigma\,(y^a\, y\cdot p - \tfrac{1}{2}\, y^2\, p^a) - \tfrac{1}{3}\,\nabla_a\nabla_b\sigma\,\big(y^a y^b\, y\cdot p - \tfrac{1}{2}\, y^2\, y^a\, p^b\big) + \dots\,, \tag{4.9}$$

where as usual, the dots denote higher order terms in y. Now we can focus on the symbol of our differential operator, that we assume to be of the form $p^2 + \alpha\, R$ for some coefficient α to be fixed by requiring that, here again, Weyl transformation can be implemented as gauge symmetries. In other words, we want to impose

$$\Big(\tfrac{1}{\hbar}\,[F, \xi_{\rm Weyl}]_* + \{F, w_{\rm Weyl}\}_*\Big)\big|_{y=0} \stackrel{!}{=} \delta_\sigma^{\rm Weyl}\big(p^2 + \alpha\, R\big)\,, \tag{4.10}$$

with $F = \tau(p^2 + \alpha\, R)$ its covariantly constant lift, and where the gauge parameter $w_{\rm Weyl}$ is assumed to be proportional to the lift of the Weyl parameter σ, i.e.

$$w_{\rm Weyl} = \beta\,\tau(\sigma) \equiv \beta\sum_{k\geq 0}\tfrac{1}{k!}\, y^{a_1}\dots y^{a_k}\,\nabla_{a_1}\dots\nabla_{a_k}\sigma\,, \tag{4.11}$$

with β a coefficient to be determined as well. Note that at this point, the choice of $w_{\rm Weyl}$ is merely an educated guess: it should be covariantly constant, and related to the Weyl parameter σ, hence this is the simplest option—which turns out to be the correct one as we shall see. Using the previous formulae, one finds on the one hand,

$$\Big(\delta_{\xi_{\rm Weyl}, w_{\rm Weyl}} F\Big)\big|_{y=0} = 2\sigma\,(\beta+1)\, p^2 + \tfrac{\hbar^2}{2}\,\beta\,\Box\sigma + 2\sigma\,\alpha\beta\, R\,, \tag{4.12}$$

while on the other hand

$$\delta_\sigma^{\rm Weyl}(p^2 + \alpha\, R) = -2\alpha\,\big(\sigma\, R + (n-1)\,\Box\sigma\big)\,, \tag{4.13}$$

[9] For instance, recall that the spin-connection transforms as $\delta_\sigma^{\rm Weyl}\omega^{a,b} = 2\, e^{[a}\,\nabla^{b]}\sigma$ under a Weyl rescaling.

which implies

$$\beta = -1\,, \qquad \text{and} \qquad \alpha = \tfrac{\hbar^2}{4(n-1)}\,, \tag{4.14}$$

thereby fixing the symbol of the conformal Laplacian in accordance with the previous discussion.

As a final consistency check, one can compute the gauge transformation of the lift of the scalar ϕ generated by the parameter ξ_{Weyl} and w_{Weyl} identified previously, and recover

$$\delta_{\xi_{\text{Weyl}},w_{\text{Weyl}}}\Phi\big|_{y=0} = -\tfrac{n-2}{2}\,\sigma\,\phi\,, \tag{4.15}$$

as expected for a conformally-coupled scalar field.

4.2 Higher-spin background

Let us recall that A is kept purely gravitational and background conformal higher-spin fields are placed into F as an uplift of[10]

$$f = p^2 + \tfrac{\hbar^2}{4(n-1)}\,R + \sum_{s>2} h^{a_1\dots a_s}(x)\,p_{a_1}\dots p_{a_s}\,. \tag{4.16}$$

It is instructive to work out the gauge transformations of this symbol generated by the gauge parameters

$$\xi = \xi_{\text{Weyl}} - \tau\Big(\sum_{s>2}\xi^{a_1\dots a_{s-1}}(x)\,p_{a_1}\dots p_{a_{s-1}}\Big)\,, \tag{4.17}$$

$$w = w_{\text{Weyl}} + \tau\Big(\sum_{s>2}\sigma^{a_1\dots a_{s-2}}(x)\,p_{a_1}\dots p_{a_{s-2}}\Big)\,, \tag{4.18}$$

that is, we simply append to the gauge parameters identified previously the covariantly constant uplift of arbitrary monomials in p. Indeed, in this manner the gauge variation of A is unaffected by this new term,

$$\delta_\xi A \equiv \delta_{\xi_{\text{Weyl}}}A\,, \tag{4.19}$$

and thus boils down to a Weyl transformation of the gravitational sector. It does, however, affect the gauge transformation of f. Computing $\delta_{\xi,w}F|_{y=0}$ and extracting the piece of order $s>2$ in p, one finds

$$\delta_{\xi,\sigma}h^{a_1\dots a_s} = 2\,\nabla^{(a_1}\xi^{a_2\dots a_s)} + 2\,\eta^{(a_1a_2}\sigma^{a_3\dots a_s)} + (s-2)\,\sigma\,h^{a_1\dots a_s} + \dots \tag{4.20}$$

where the dots denote curvature corrections. The first two terms correspond to the 'naive' covariantization of the linearized gauge transformations initially proposed by Fradkin and Tseytlin for conformal higher-spin fields, i.e. the flat space ones wherein partial derivatives are replaced by covariant derivatives. The third term tells us that the Weyl weight of a conformal higher-spin field with spin s is $s-2$, which is also in accordance with expectations [27].[11] This can be seen as another sign of relevance for this framework in the problem of formulating CHS gravity in a manifestly covariant manner.

[10] If one wants to consider all integer spins, a spin-one has to be included, which is naively missing above. Alternatively, it is possible to truncate the system to even spins only.

[11] Note that the Weyl weight of a metric-like field $\phi_{\mu_1\dots\mu_s}$ is $2s-2$, e.g. it is 2 for metric $g_{\mu\nu}$. Its fiber version, to which $h^{a_1\dots a_s}$ should be compared to, is obtained by contracting it with s inverse vielbeins e^μ_a, giving Weyl weight of $s-2$.

Higher-spin currents. As a final application, we can derive the higher-spin currents for an arbitrary curved spacetime. To do so, let us split the previous symbol (4.16) into that of the conformal Laplacian and the conformal higher-spin fields,

$$f = p^2 + \tfrac{\hbar^2}{4(n-1)}\, R + f_{hs}(x,p)\,, \qquad f_{hs}(x,p) := \sum_{s>2} h^{a_1\dots a_s}\, p_{a_1}\dots p_{a_s}\,, \tag{4.21}$$

according to which the action obtained from f is the sum of the conformally-coupled scalar and a Noether coupling part,

$$S_{\text{Noether}}[h,\phi] = \tfrac{1}{2}\,\text{Tr}_A(F_{hs} * \mathcal{W}_\phi) = \tfrac{1}{2}\int_{\mathcal{X}} \mathrm{d}^n x\, |e|\, \phi^* \big[\rho(F_{hs})\Phi\big]|_{y=0}\,, \tag{4.22}$$

corresponding to the contribution of the higher-spin currents coupled to higher-spin sources/background fields $h^{a_1\dots a_s}$. In other words, we can identify the higher-spin current by putting the above functional in the form

$$S_{\text{Noether}}[h,\phi] = \tfrac{1}{2}\int_{\mathcal{X}} \mathrm{d}^n x\, |e| \sum_{s>2} h^{a_1\dots a_s}\, J_{a_1\dots a_s}(\phi)\,, \tag{4.23}$$

where the spin s current $J_{a_1\dots a_s}$ here is by definition bilinear in the scalar field ϕ.

This computation involve the action of the quantization map on the lift of f_{hs}, which is of arbitrary order in p. As a consequence, the relevant terms to compute in this lift, meaning those that will contribute to the final result after applying the quantization of F_{hs} to Φ and setting $y=0$, are those that are y-independent or contain *exactly* the same number of y's and p's. Indeed, the quantization map applied to a monomial of order l in y and m in p reads

$$\rho(y^{a_1}\dots y^{a_l}\, p_{b_1}\dots p_{b_m}) = (-\hbar)^m \sum_{k=0}^{\min(l,m)} \tfrac{1}{2^k}\, \tfrac{m!}{(m-k)!}\, \tfrac{l!}{k!(l-k)!}\, y^{(a_1}\dots y^{a_{l-k}}\, \delta^{a_{l+1-k}}_{(b_1}\dots \delta^{a_l)}_{b_k}\, \tfrac{\partial}{\partial y^{b_{k+1}}}\dots \tfrac{\partial}{\partial y^{b_m)}}\,, \tag{4.24}$$

so that when setting $y=0$, only monomials with $l \leq m$, i.e. less y's than p's, remain. This would be difficult to compute for arbitrary spin $s>2$, so we will focus on the curvature independent part of the current. The relevant part of the lift of f_{hs} is therefore given by its 'covariant Taylor series',

$$F_{hs} = \sum_{k\geq 0} \tfrac{1}{k!}\, y^{a_1}\dots y^{a_k}\, \nabla_{a_1}\dots\nabla_{a_k} f_{hs} + \dots\,, \tag{4.25}$$

where the dots denote curvature corrections. Applying the quantization map on this (partial) lift, and evaluating the result at $y=0$, one ends up with

$$\rho(F_{hs})|_{y=0} = \sum_{s>2} (-\hbar)^s \sum_{k=0}^{s} \tfrac{1}{2^k}\, \tfrac{s!}{k!(s-k)!}\, \nabla_{a_1}\dots\nabla_{a_k} h^{a_1\dots a_k\, a_{k+1}\dots a_s}\, \tfrac{\partial}{\partial y^{a_{k+1}}}\dots \tfrac{\partial}{\partial y^{a_s}} + \dots\,. \tag{4.26}$$

Under the same restrictions, the lift of the scalar field reads

$$\Phi = \sum_{k\geq 0} \tfrac{1}{k!}\, y^{a_1}\dots y^{a_k}\, \nabla_{a_1}\dots\nabla_{a_k}\phi + \dots\,, \tag{4.27}$$

so that,

$$\rho(F_{hs})\Phi|_{y=0} = \sum_{s>2}(-\hbar)^s \sum_{k=0}^{s} \tfrac{1}{2^k}\tfrac{s!}{k!(s-k)!}\,\nabla_{a_1}\dots\nabla_{a_k} h^{a_1\dots a_k\, a_{k+1}\dots a_s}\nabla_{a_{k+1}}\dots\nabla_{a_s}\phi + \dots\,, \tag{4.28}$$

again keeping only curvature independent terms. Upon integration by parts, one finds

$$J_{a_1\dots a_s} = \big(-\tfrac{\hbar}{2}\big)^s \sum_{k=0}^{s} \tfrac{(-1)^k\, s!}{k!(s-k)!}\,\nabla_{(a_1}\dots\nabla_{a_k}\phi^*\,\nabla_{a_{k+1}}\dots\nabla_{a_s)}\phi + \dots\,, \tag{4.29}$$

as one may have expected. This the covariantized version of the well-known 'dipole' generating function $\phi^*(x-y)\phi(x+y)$ that yields conserved quasi-primary (higher-spin) currents with an admixture of descendants in the flat space. The curvature corrections can systematically be worked out, see [32] for the spin-three example in the bottom-up approach. However, it is clear that the higher the spin the more non-linearities in the Riemann tensor R and its derivatives will enter. Therefore, eq. (3.21) seems to be the most compact way of writing the coupling of the free scalar field to a higher-spin background.

First order correction in curvature. If we focus on the spin-3 case, then we only need to compute the lift of $f_{s=3} = h^{abc}\,p_a p_b p_c$ to order 3. Pushing the computation of the lift of any symbol $f(x,p)$ presented in (2.23) to the next order, thanks to the recursion (2.22), yields

$$\tau(f) = \Big(1 + y^a\nabla_a + \tfrac{1}{2}\,y^a y^b\left[\nabla_a\nabla_b + \tfrac{1}{3}\,R_{da}{}^c{}_b\,p_c\,\tfrac{\partial}{\partial p_d}\right] \tag{4.30}$$

$$+ \tfrac{1}{6}\,y^a y^b y^c\left[\nabla_a\nabla_b\nabla_c + \tfrac{1}{2}\,\nabla_a R_{db}{}^e{}_c\,p_e\,\tfrac{\partial}{\partial p_d} + R_{da}{}^e{}_b\,p_e\,\tfrac{\partial}{\partial p_d}\,\nabla_c\right] \tag{4.31}$$

$$+ \tfrac{\hbar^2}{12}\,y^a\left[R_{ab}{}^d{}_c\,\tfrac{\partial^2}{\partial p_b\partial p_c}\nabla_d - \tfrac{1}{4}\,\nabla_b R_{ac}{}^e{}_d\,p_e\,\tfrac{\partial^3}{\partial p_b\partial p_c\partial p_d}\right] + \dots\Big)\,f\,, \tag{4.32}$$

and applying it to $f_{s=3}$, one finds that $F_{s=3} = \tau(f_{s=3})$ is given by

$$F_{s=3} = \Big(h^{abc} + y^i\,\nabla_i h^{abc} + \tfrac{1}{2}\,y^i y^j\left[\nabla_i\nabla_j h^{abc} + R_{di}{}^a{}_j h^{bcd}\right] \tag{4.33}$$

$$+ \tfrac{1}{6}\,y^i y^j y^k\left[\nabla_i\nabla_j\nabla_k h^{abc} + \tfrac{3}{2}\,\nabla_i R_{dj}{}^a{}_k\,h^{bcd} + 3\,R_{di}{}^a{}_j\,\nabla_k h^{bcd}\right] + \dots\Big)\,p_a p_b p_c \tag{4.34}$$

$$+ \tfrac{\hbar^2}{2}\,y^i\left[R_{ib}{}^d{}_c\,\nabla_d h^{abc} - \tfrac{1}{4}\,\nabla_b R_{ic}{}^a{}_d\,h^{bcd}\right]p_a + \dots\,, \tag{4.35}$$

where the dots denote terms of order 4 or higher, and its quantization evaluated at $y=0$ reads

$$-\tfrac{1}{\hbar^3}\,\rho(F_{s=3})|_{y=0} = h^{abc}\,\tfrac{\partial^3}{\partial y^a\partial y^b\partial y^c} + \tfrac{3}{2}\,\nabla_a h^{abc}\,\tfrac{\partial^2}{\partial y^b\partial y^c} + \tfrac{1}{4}\left[3\,\nabla_a\nabla_b h^{abc} - R_{ab}\,h^{abc}\right]\tfrac{\partial}{\partial y^c} \tag{4.36}$$

$$+ \tfrac{1}{8}\left[\nabla_a\nabla_b\nabla_c h^{abc} - \nabla_a R_{bc}\,h^{abc} - R_{ab}\,\nabla_c h^{abc}\right]. \tag{4.37}$$

Similarly, we can use (2.42) to compute the lift of the scalar field ϕ to order 3,

$$\Phi = \phi + y^a\nabla_a\phi + \tfrac{1}{2}\,y^a y^b\left(\nabla_a\nabla_b - \tfrac{1}{6}\,R_{ab}\right)\phi + \tfrac{1}{6}\,y^a y^b y^c\left(\nabla_a\nabla_b\nabla_c - \tfrac{1}{4}\,\nabla_a R_{bc} - \tfrac{1}{2}\,R_{ab}\,\nabla_c\right)\phi + \dots\,, \tag{4.38}$$

so that one finds

$$-\tfrac{1}{\hbar^3}\,\rho(F_{s=3})\Phi|_{y=0} = h^{abc}\left(\nabla_a\nabla_b\nabla_c - \tfrac{3}{8}\,\nabla_a R_{bc} - \tfrac{3}{4}\,R_{ab}\,\nabla_c\right)\phi + \tfrac{3}{2}\,\nabla_c h^{abc}\left(\nabla_a\nabla_b - \tfrac{1}{4}\,R_{ab}\right)\phi \\ + \tfrac{3}{4}\,\nabla_a\nabla_b h^{abc}\,\nabla_c\phi + \tfrac{1}{8}\,\nabla_a\nabla_b\nabla_c h^{abc}\,\phi\,. \tag{4.39}$$

which leads to the following expression

$$J_{abc} = -\tfrac{\hbar^3}{8}\Big(\nabla_{(a}\nabla_b\nabla_{c)}\phi\,\phi^* - 3\,\nabla_{(a}\nabla_b\phi\,\nabla_{c)}\phi^* + 3\,\nabla_{(a}\phi\,\nabla_b\nabla_{c)}\phi^* - \phi\,\nabla_{(a}\nabla_b\nabla_{c)}\phi^* \tag{4.40}$$

$$- 3\,R_{(ab}\,\nabla_{c)}\phi\,\phi^* + 3\,R_{(ab}\,\phi\,\nabla_{c)}\phi^*\Big)\,, \tag{4.41}$$

for the spin-3 current. More generally, the currents up to first order in the curvature tensor (and its derivatives) are obtained from the generating function

$$\mathcal{J}(x|u) = e^{-\frac{\hbar}{2}\,u\cdot[\nabla_1-\nabla_2]}\Big(1 - \tfrac{\hbar^2}{8}\,\mathrm{sinhc}(\tfrac{\hbar}{4}\,u\cdot\nabla_3)\,R_{ab}(x_3)\,u^a u^b + \mathcal{O}(R^2)\Big)\,\phi(x_1)\,\phi^*(x_2)\big|_{x_i=x}\,, \tag{4.42}$$

where

$$\mathcal{J}(x|u) := \sum_{s\geq 0}\tfrac{(-\hbar)^s}{2^s\,s!}\,J_{a_1\dots a_s}(x)\,u^{a_1}\dots u^{a_s}\,, \qquad \mathrm{sinhc}(z) := \frac{\sinh(z)}{z}\,, \tag{4.43}$$

and ∇_i denotes the covariant derivative with respect to x_i (see Appendix D for the derivation of this formula). Let us remark here that, in order to make contact with, say the computation of [32] for the spin-three current or even the standard computation of the energy-momentum tensor, one should find the correct field redefinition bringing the components of monomials in p into the appropriate field frame (combination of conformal higher spin fields and derivatives thereof), see e.g. [2, App. F] and [3] for an instance of the same issue around a flat background.

5 Discussion

The present paper is a natural continuation of the quest to covariantize the construction of conformal higher-spin gravities started in [21]. Now, both the action for conformal higher-spin gravity $S_{CHS}[h_s]$ and the coupling of the scalar matter to the higher-spin background, $\langle\Phi|\,\widehat{H}[\phi_s]\,|\Phi\rangle$, can be written in a covariant way. The result completes the study initiated in [32], where the mixing between covariant spin-three and spin-one currents that couple to background fields have been discussed in $n=4$. In addition, one can consider the matter coupled conformal higher-spin gravity, see [19] for some amplitudes in this theory over flat background. Note, however, that while the scalar matter can be coupled to a higher-spin background for any n the conformal anomaly recipe gives $S_{CHS}[h_s]$ only for n even.

The results open up the possibility of considering more general matter fields in the relevant higher-spin background, such as the higher-derivative scalar fields (also known as higher order singletons [51]), or spinor (and its higher-derivative counterpart), see [52]. The latter would in principle require the use of the supersymmetry version of the FFS cocycle, i.e. the representative of the cohomology class of the Clifford–Weyl algebra dual to the unique Hochschild homology class of the same algebra [53].

Another possible application of the results is to conformally-invariant differential operators. Conformal geometry (in the sense of gauge symmetries realized by diffeomorphisms and Weyl transformations) is a part of the higher-spin system. As we showed, one can derive the conformal Laplacian as a particular instance of the scalar field coupled to the conformal gravity background. Generalizations such as Paneitz [54] or Fradkin–Tseytlin [55] operators and GJMS operators [56] can also be recovered by considering $F = (p^2)^k + \dots$ that would lead to operators of type $(\nabla^2)^k + \dots$, i.e. starting with the kth power of the Laplacian, and corrected by curvature terms.

It would also be interesting to apply the deformation quantization techniques to the self-dual conformal higher-spin gravity [57, 58] that is natural to formulate on twistor space. Here, the underlying space $\mathbb{CP}^3$ is already symplectic. The twistor description of low-spin fields, $s = 1, 2$, requires usual (holomorphic) connections and vector-valued one-forms, which can be understood as differential operators of zeroth and first order. An extension to higher-spin calls for differential operators of arbitrary order, i.e. to the quantization of the cotangent bundle again (see also [59] for additional discussions of the quantization of the cotangent in relation with the definition of higher-spin diffeomorphisms).

Let us also note that the present paper bridges a gap in the phase space approach to quantum mechanics. Indeed, one can attempt to extend the Fedosov construction to accommodate all the usual ingredients required in quantum mechanics. The trace is, obviously, given by the Feigin–Felder–Shoikhet cocycle; wave functions can be understood as covariantly constant elements in the Fock representation obtained via the quantization map. Wigner function takes exactly the same form as in the flat space, but in the fiber. The basic ingredients above do not rely on the phase space being a cotangent bundle and should extend to arbitrary symplectic manifolds (a polarization is needed to define the Fock space). This seems to depart from the usual approach of symbol calculus on curved background, e.g. [60–66] and references therein.

Finally, it would be interesting to construct the $3d$ matter-coupled conformal higher-spin gravity, where the 'dynamics' of conformal higher-spin fields is given by the Chern–Simons action (as there is no conformal anomaly in $3d$). Such a theory, namely the one based on fermionic matter, can be seen to exist with the help of the argument based on the parity anomaly [10] (see e.g. [67–69] for original papers on the derivation of Chern–Simons theory from the parity anomaly and [70] for the spin-three case). An alternative idea along the AdS/CFT correspondence lines was recently explored in [71, 72].

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A A brief review of Weyl calculus

Let us give a brief summary of the definition and construction of the Wigner function in flat space (following e.g. the textbook [73], or the papers [2, 3, 74, 75]).

Quantization map in flat space. The deformation quantization of $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$, amounts to defining an isomorphism

$$\mathcal{C}^\infty(T^*\mathbb{R}^n) \xrightarrow{\ \sim\ } \mathcal{D}(\mathbb{R}^n)\,, \tag{A.1}$$

where $\mathcal{D}(\mathbb{R}^n)$ stands for the space of differential operators on $\mathbb{R}^n$. This map is referred to as a 'quantization map' since, as we will recall shortly, it allows one to define a star-product on the algebra of functions $T^*\mathbb{R}^n$, and hence a quantization thereof. To do so, we can take advantage of the Fourier transform in flat space, that we denote by

$$(\mathcal{F}f)(u,v) := \int_{\mathbb{R}^{2n}} \tfrac{\mathrm{d}^n x\, \mathrm{d}^n p}{(2\pi\hbar)^n}\, f(x,p)\, e^{-\frac{i}{\hbar}\,(x\cdot u + p\cdot v)}\,, \tag{A.2}$$

for a symbol $f(x,p)$. Given a choice of quantization for the phase space coordinates $x^\mu \to \hat{x}^\mu$ and $p_\mu \to \hat{p}_\mu$, where hatted symbols denote the corresponding operator, we want to associate

Schematically, we want to write something like "$\widehat{f}(\hat{x},\hat{p}) \sim f(x,p)\,\delta(x-\hat{x})\,\delta(p-\hat{p})$", where $f(x,p)$ is the symbol of the operator $\widehat{f}$. This sketchy formula can be given a precise sense, using the Fourier representation of the Dirac distribution, leading to

$$\widehat{f}(\hat{x},\hat{p}) = \int_{\mathbb{R}^{2n}} \tfrac{\mathrm{d}^n u\, \mathrm{d}^n v}{(2\pi\hbar)^n}\, (\mathcal{F}f)(u,v)\, e^{\frac{i}{\hbar}\,(u\cdot\hat{x} + v\cdot\hat{p})}\,, \tag{A.3}$$

and which is called the *Weyl ordering* of operators. Note that the exponential operator can be re-written as

$$\exp\big(\tfrac{i}{\hbar}\,(u\cdot\hat{x} + v\cdot\hat{p})\big) = e^{\frac{i}{2\hbar}\,u\cdot v}\, \exp\big(\tfrac{i}{\hbar}\,u\cdot\hat{x}\big)\, \exp\big(\tfrac{i}{\hbar}\,v\cdot\hat{p}\big)\,, \tag{A.4}$$

since we assume $[\hat{x}^\mu, \hat{p}_\nu] = i\hbar\,\delta^\mu_\nu$. Choosing the usual coordinate representation,

$$\hat{x}^\mu = x^\mu\,, \qquad \hat{p}_\mu = -i\hbar\,\partial_\mu\,, \tag{A.5}$$

the action of this operator on a wave function $\varphi(x)$ is given by

$$\begin{aligned}
(\widehat{f}\,\varphi)(x) &= \int \tfrac{\mathrm{d}^n u\, \mathrm{d}^n v}{(2\pi\hbar)^n}\, (\mathcal{F}f)(u,v)\, e^{\frac{i}{2\hbar}\,u\cdot v}\, e^{\frac{i}{\hbar}\,u\cdot x}\, \varphi(x+v) &\text{(A.6)}\\
&= \int \tfrac{\mathrm{d}^n u\, \mathrm{d}^n v}{(2\pi\hbar)^n}\, \tfrac{\mathrm{d}^n x'\, \mathrm{d}^n p}{(2\pi\hbar)^n}\, f(x',p)\, e^{-\frac{i}{\hbar}\,p\cdot v}\, e^{\frac{i}{\hbar}\,u\cdot(x-x'+\frac{v}{2})}\, \varphi(x+v) &\text{(A.7)}\\
&= \int \tfrac{\mathrm{d}^n v\, \mathrm{d}^n p}{(2\pi\hbar)^n}\, f(\tfrac{x+v}{2},p)\, e^{\frac{i}{\hbar}\,p\cdot(x-v)}\, \varphi(v)\,, &\text{(A.8)}
\end{aligned}$$

where the first equation is obtained using (A.4) and the action of the translation operator, the second line is merely the definition of the Fourier transform, and the last one is the result of integrating over u, which gives a Dirac distribution, and then evaluating it by integrating over x'. Upon Taylor expanding f and integrating by part, one can put this formula into an operatorial form

$$(\widehat{f}\varphi)(x) = f(x,p)\, \exp\Big(-i\hbar\, \tfrac{\overleftarrow{\partial}}{\partial p}\cdot\big[\tfrac{1}{2}\tfrac{\overleftarrow{\partial}}{\partial x} + \tfrac{\overrightarrow{\partial}}{\partial x}\big]\Big)\, \varphi(x)\big|_{p=0}\,. \tag{A.9}$$

The Moyal–Weyl star-product can be recovered from the composition of the two operators associated with two symbols via the above symbol, or quantization, map. More precisely, it can be defined as the symbol of the composition of the quantization of two symbols, i.e.

$$\widehat{f} \circ \widehat{g} = \widehat{f \star g}\,. \tag{A.10}$$

To do so, let us start by recalling that the action of a symbol f given above exhibits the *kernel* of that associated operator, namely

$$(\widehat{f}\,\phi)(x) = \int_{\mathbb{R}^n} \mathrm{d}^n q\, K_f(x,q)\,\phi(q)\,, \qquad \text{with} \qquad K_f(x,q) := \int_{\mathbb{R}^n} \tfrac{\mathrm{d}^n p}{(2\pi\hbar)^n}\, f\big(\tfrac{x+q}{2},p\big)\, e^{\frac{i}{\hbar}\, p\cdot(x-q)}\,. \tag{A.11}$$

The symbol of the operator $\widehat{f}$ can be extract back from its kernel, via its inverse transform

$$f(x,p) = \int_{\mathbb{R}^n} \mathrm{d}^n q\, K_f(x+\tfrac{1}{2}\,q, x-\tfrac{1}{2}\,q)\, e^{-\frac{i}{\hbar}\,p\cdot q}\,, \tag{A.12}$$

and therefore, using this together with the fact that the integral kernel of the composition of two operators is given by

$$K_{\widehat{f}\circ\widehat{g}}(x,x') = \int_{\mathbb{R}^n} \mathrm{d}^n q\, K_{\widehat{f}}(x,q)\, K_{\widehat{g}}(q,x')\,, \tag{A.13}$$

one ends up with

$$\big(f \star g\big)(x,p) = \tfrac{1}{(\pi\hbar)^{2n}} \int \mathrm{d}^n v_1\, \mathrm{d}^n v_2\, \mathrm{d}^n w_1\, \mathrm{d}^n w_2\; e^{\frac{2i}{\hbar}\,(v_1\cdot w_2 - v_2\cdot w_1)}\, f(x+v_1, p+w_1)\, g(x+v_2, p+w_2)\,. \tag{A.14}$$

Upon Taylor expanding the two functions f and g around (x,p), and integrating by part, one

$$\big(f \star g\big)(x,p) = f(x,p)\, \exp\Big(\tfrac{i\hbar}{2}\,\big[\tfrac{\overleftarrow{\partial}}{\partial x}\cdot\tfrac{\overrightarrow{\partial}}{\partial p} - \tfrac{\overleftarrow{\partial}}{\partial p}\cdot\tfrac{\overrightarrow{\partial}}{\partial x}\big]\Big)\, g(x,p)\,, \tag{A.15}$$

Note that the Moyal–Weyl star-product is *Hermitian*, meaning that it satisfies

$$(f \star g)^* = g^* \star f^*\,, \tag{A.16}$$

where $(-)^*$ denotes the complex conjugation, i.e. the latter is an anti-involution of the Weyl algebra.

One can think of the quantization map as providing a *representation* of the Weyl algebra: identifying the latter as the subalgebra of *polynomial* functions on $T^*\mathbb{R}^n$, wave functions which are nothing but functions on $\mathbb{R}^n$, the base of the cotangent bundle $T^*\mathbb{R}^n$, are acted upon by the former via the quantization map. This subspace can be thought of as a Fock space, which carries a representation of the Weyl algebra as can be seen from the defining relation (A.10).

The integration over the cotangent bundle $T^*\mathbb{R}^n$ defines a trace over the space of symbols, at least those which are compactly supported or vanish at infinity sufficiently fast. Indeed, in this case one finds

$$\mathrm{Tr}(f\star g) = \int_{T^*\mathbb{R}^n} \mathrm{d}^n x\, \mathrm{d}^n p\; (f\star g)(x,p) = \int_{T^*\mathbb{R}^n} \mathrm{d}^n x\, \mathrm{d}^n p\; f(x,p)\, g(x,p) = \mathrm{Tr}(g \star f)\,, \tag{A.17}$$

for any symbols f and g, since all higher order terms in the star product are total derivatives on $T^*\mathbb{R}^n$, and hence can be ignored for the aforementioned suitable class of symbols.

Wigner function in flat space. Having worked out how to translate the action and composition of differential operators in terms of their symbol, as well as their trace, we can now turn our attention to the computation of matrix elements for these operators, expressing the transition probability from one state to another. Since the latter can be expressed as

$$\langle\psi|\,\widehat{H}\,|\phi\rangle = \mathrm{Tr}\Big(\widehat{H}\circ|\phi\rangle\langle\psi|\Big)\,, \tag{A.18}$$

we have everything we need to derive such quantities using symbols, provided that we know that of the projector $|\phi\rangle\langle\psi|$. In light of the relation between the symbol of an operator and its integral kernel, we may first focus on that of the projector. This integral kernel is easily computed,

$$\big(\,|\phi\rangle\langle\psi|\varphi\rangle\,\big)(x) \stackrel{!}{=} \phi(x)\int_{\mathbb{R}^n}\mathrm{d}^n q\,\psi^*(q)\,\varphi(q) \qquad\Longrightarrow\qquad K_{|\phi\rangle\langle\psi|}(x,q)\equiv\phi(x)\,\psi^*(q)\,, \tag{A.19}$$

which leads to

$$W[\phi,\psi](x,p) = \int_{\mathbb{R}^n}\mathrm{d}^n q\;\phi(x+\tfrac{q}{2})\,\psi^*(x-\tfrac{q}{2})\,e^{-\frac{i}{\hbar}\,p\cdot q}\,, \tag{A.20}$$

for its symbol. It obey the following useful properties

$$\xi\star W[\phi,\psi] = W[\widehat{\xi}\,\phi,\psi]\,, \qquad W[\phi,\psi]\star\xi^\dagger = W[\phi,\widehat{\xi}\,\psi]\,, \tag{A.21}$$

in accordance with the fact that it is the symbol of the projector $|\phi\rangle\langle\psi|$, and

$$\int_{\mathbb{R}^n}\mathrm{d}^n p\;W[\phi,\psi](x,p) = \phi(x)\,\psi^*(x)\,. \tag{A.22}$$

Now we can replace the right hand side of (A.18) with its symbol counterpart, leading to

$$\mathrm{Tr}(H\star W[\phi,\psi]) = \mathrm{Tr}\Big(W[\widehat{H}\phi,\psi]\Big) = \int_{T^*\mathbb{R}^n}\mathrm{d}^n x\,\mathrm{d}^n p\;W[\widehat{H}\phi,\psi] = \int_{\mathbb{R}^n}\mathrm{d}^n x\;\psi^*(x)\,(\widehat{H}\phi)(x)\,, \tag{A.23}$$

upon using the previously listed properties of $W[\phi,\psi]$, thereby reproducing the expected result for the quantity $\langle\psi|\,\widehat{H}\,|\phi\rangle$ from a quantum mechanical point of view. The *Wigner function* W_ϕ associated with a wave function ϕ is the symbol of the projector $|\phi\rangle\langle\phi|$, i.e.

$$W_\phi(x,p) := W[\phi,\phi](x,p) \equiv \int_{\mathbb{R}^n}\mathrm{d}^n q\;e^{-\frac{i}{\hbar}\,q\cdot p}\,\phi(x+\tfrac{q}{2})\,\phi(x-\tfrac{q}{2})\,, \tag{A.24}$$

whose integral over p is nothing but the probability density defined by ϕ.

To conclude this appendix, let us prove the identity (A.22) and a small variation on it (the intertwining property (A.21) can be proved by direct computation using the integral formulae for the star-product and the quantization map), by expressing the Wigner function in terms of star-product. To achieve this, recall that the star-product of a phase factor $e^{\frac{i}{\hbar}\,q\cdot p}$, where q is a fixed parameter, with any symbol $f(x,p)$ yields

$$e^{\frac{i}{\hbar}\,q\cdot p}\star f(x,p) = e^{\frac{i}{\hbar}\,q\cdot p}f(x+\tfrac{q}{2},p)\,, \qquad f(x,p)\star e^{\frac{i}{\hbar}\,q\cdot p} = e^{\frac{i}{\hbar}\,q\cdot p}f(x-\tfrac{q}{2},p)\,, \tag{A.25}$$

i.e. it implements translations in x up to a phase.[12] Integrating these formulae over q yields

$$\big(f\star\delta_p\big)(x,p)=\int_{\mathbb{R}^n}\mathrm{d}^nq\,e^{-\frac{i}{\hbar}\,q\cdot p}\,f(x+\tfrac{q}{2},p)\,,\qquad \big(\delta_p\star f\big)(x,p)=\int_{\mathbb{R}^n}\mathrm{d}^nq\,e^{-\frac{i}{\hbar}\,q\cdot p}\,f(x-\tfrac{q}{2},p)\,, \tag{A.26}$$

where δ_p is the Dirac distribution in the space of momenta p_a. With these simple identities at hand, one finds

$$\phi\star\delta_p\star\psi^*=\int_{\mathbb{R}^n}\mathrm{d}^nq\,\phi(x)\star\big[e^{-\frac{i}{\hbar}\,q\cdot p}\psi^*(x-\tfrac{1}{2}\,q)\big] \tag{A.27}$$

$$=\int_{\mathbb{R}^n}\mathrm{d}^nq\,\big[\phi(x)\star e^{-\frac{i}{\hbar}\,q\cdot p}\big]\,\psi^*(x-\tfrac{1}{2}\,q) \tag{A.28}$$

$$=\int_{\mathbb{R}^n}\mathrm{d}^nq\;\phi(x+\tfrac{q}{2})\,\psi^*(x-\tfrac{q}{2})\,e^{-\frac{i}{\hbar}\,p\cdot q}=W[\phi,\psi]\,, \tag{A.29}$$

where to pass from the first to the second line, one should notice that since ϕ only depends on x, its star-product with any other Weyl algebra element will produce only derivatives with respect to p on the latter.

Now this expression makes it relatively easy to evaluate the integral over momenta of the Wigner function and its derivatives with respect to p. Indeed, since the only term of this star-product that depends on momenta is the Dirac distribution, the result is of the form

$$\phi\star\delta_p\star\psi^*\sim\sum_{k,l\geq 0}\partial_x^k\phi\times\partial_p^{k+l}\delta(p)\times\partial_x^l\psi^*\,, \tag{A.30}$$

so that the integral over p schematically reads

$$\int\mathrm{d}^np\;W[\phi,\psi]\sim\sum_{k,l\geq 0}\int\mathrm{d}^np\;\partial_p^{k+l}\delta(p)\times(\partial_x^k\phi\,\partial_x^l\psi^*)\,, \tag{A.31}$$

which identically vanishes for $k+l>0$ since both ϕ and ψ do not depend on p, and yields (A.22) for $k=0=l$. On top of that, since taking partial derivative with respect to x or p commutes with the star-product, the derivatives of the Wigner function with respect to p are of the form $\partial_p^kW[\phi,\phi]\sim\phi\star\partial_p^k\delta_p\star\psi^*$, and hence the same argument shows that the integral over the momenta identically vanishes,

$$\int_{\mathbb{R}^n}\mathrm{d}^np\;\tfrac{\partial^k}{\partial p_{a_1}\dots\partial p_{a_k}}\,W[\phi,\psi]=0\,,\qquad\forall k>0\,. \tag{A.32}$$

B More on Weyl transformations

In this appendix, we provide more details concerning the computation of the gauge variation of the symbol $p^2+\alpha\,R$. For convenience, let us introduce the tensor

$$\mathcal{P}_{ab}{}^{cd}:=\delta^c_{(a}\delta^d_{b)}-\tfrac{1}{2}\,\eta_{ab}\,\eta^{cd}\,, \tag{B.1}$$

[12]To be more precise, the action of translation on elements depending on x *only* is generated by

$$e^{\frac{i}{\hbar}\,q\cdot p}\star\phi(x)\star e^{-\frac{i}{\hbar}\,q\cdot p}=\phi(x+q)\,,$$

which can be recovered from the formulae (A.25).

with which the gauge parameter $\xi_{\rm Weyl}$, identified in (4.9) as the one generating Weyl transformations for the components of the 1-form connection A, is given by

$$\xi_{\rm Weyl} = -\sigma\, y\cdot p - \mathcal{P}_{bc}{}^{ad}\, y^b y^c p_d\, \nabla_a\sigma - \tfrac{1}{3}\, y^{(a}\, \mathcal{P}_{cd}{}^{b)e} y^c y^d p_e\, \nabla_a\nabla_b\sigma + \dots\,, \tag{B.2}$$

plus terms of order 3 and higher in y, but all linear in p. In order to compute the gauge transformation of $F = \tau(p^2 + \alpha\, R)$ generated by $\xi_{\rm Weyl}$, and $w_{\rm Weyl}$ given by

$$w_{\rm Weyl} = \beta\, \tau(\sigma) = \beta\,\big(1 + y^a\, \nabla_a + \tfrac{1}{2}\, \nabla_a\nabla_b + \dots\big)\, \sigma\,, \tag{B.3}$$

one needs to compute the star-product between elements of the Weyl algebra which are at most quadratic in p. For our purpose, it will be enough to compute neglecting terms with less, or as many, y's than p's. We therefore only need the lift of p^2 and R up to order 2 in y,

$$\tau(p^2) = p^2 + \tfrac{1}{3}\, y^a y^b\, R_a{}^c{}_b{}^d\, p_c p_d + \dots\,, \qquad \tau(R) = R + y^a\, \nabla_a R + \tfrac{1}{2}\, y^a y^b\, \nabla_a\nabla_b R + \dots\,, \tag{B.4}$$

which yields

$$\begin{aligned}
\tfrac{1}{\hbar}\, \big[p^2, \xi_{\rm Weyl}\big]_* &= 2\sigma\, p^2 + 4\, \mathcal{P}_{bc}{}^{da}\, y^b\, p^c p_d\, \nabla_a\sigma + \tfrac{2}{3}\, y^c y^d\, \mathcal{P}_{cd}{}^{a(\bullet}\, p^{b)}\, p_a\, \nabla_b\nabla_\bullet\sigma & \text{(B.5)}\\
&\qquad + \tfrac{4}{3}\, y^{(b}\, \mathcal{P}_{de}{}^{c)a}\, y^e\, p^d p_a\, \nabla_b\nabla_c\sigma + \dots & \text{(B.6)}\\
&= 2\, p^2\, \big(\sigma + y^a\, \nabla_a\sigma + \tfrac{1}{3}\, y^a y^b\, \nabla_a\nabla_b\sigma\big) + \tfrac{2}{3}\, (y\cdot p\, y^a - \tfrac{1}{2}\, y^2\, p^a)\, p^b\, \nabla_a\nabla_b\sigma + \dots\,, & \text{(B.7)}
\end{aligned}$$

while the commutator of $\xi_{\rm Weyl}$ with other terms in the lift of p^2 or R do not contribute terms with less y's than p's, and

$$\begin{aligned}
\big\{\tau(p^2), w_{\rm Weyl}\big\}_* &= \Big(2\, \big[p^2 + \tfrac{1}{3}\, R_a{}^c{}_b{}^d\, y^a y^b\, p_c p_d\big] + \tfrac{\hbar^2}{2}\, \big[\eta^{ab} + \tfrac{1}{3}\, R_a{}^c{}_b{}^d\, y^a y^b\big]\, \tfrac{\partial^2}{\partial y^a \partial y^b}\Big) w_{\rm Weyl} + \dots & \text{(B.8)}\\
&= 2\beta\, p^2\, \big(\sigma + y^a\, \nabla_a\sigma + \tfrac{1}{2}\, y^a y^b\, \nabla_a\nabla_b\sigma\big) + \tfrac{2\beta}{3}\, \sigma\, R_a{}^c{}_b{}^d\, y^a y^b\, p_c p_d + \beta\, \tfrac{\hbar^2}{2}\, \Box\sigma + \dots & \text{(B.9)}\\
\big\{\tau(R), w_{\rm Weyl}\big\}_* &= 2\, \tau(R)\, w_{\rm Weyl} = 2\beta\, \sigma\, R + \dots & \text{(B.10)}
\end{aligned}$$

where again the dots denote terms of order 3 or higher in y. Putting everything together, we end up with

$$\begin{aligned}
\delta_{\xi_{\rm Weyl}, w_{\rm Weyl}} F &= 2\sigma\, \big[(\beta+1)\, p^2 + \alpha\beta\, R\big] + \beta\, \tfrac{\hbar^2}{2}\, \Box\sigma + 2(\beta+1)\, p^2\, y^a\, \nabla_a\sigma & \text{(B.11)}\\
&\qquad + y^a y^b\, p_c p_d\, \big([\beta + \tfrac{2}{3}]\, \eta^{cd}\, \delta_a^\times \delta_b^\bullet + \tfrac{2}{3}\, \eta^{\times c}\, \delta_a^d \delta_b^\bullet - \tfrac{1}{3}\, \eta_{ab}\, \eta^{\times c} \eta^{\bullet d}\big)\, \nabla_\times\nabla_\bullet\sigma + \dots & \text{(B.12)}
\end{aligned}$$

whose value at $y=0$, which we gave earlier in (4.15), can be compared to the Weyl variation of $p^2 + \alpha\, R$ and imposing that the two agree implies

$$\alpha = \tfrac{\hbar^2}{4(n-1)}\,, \qquad \beta = -1\,. \tag{B.13}$$

From now on, we will fix these values, and will denote the gauge transformations generated by $\xi_{\rm Weyl}$ and $w_{\rm Weyl}$ with the same symbol as for a Weyl transformation generated by σ,

$$\delta_{\xi_{\rm Weyl}, w_{\rm Weyl}} \equiv \delta_\sigma\,, \tag{B.14}$$

since the two agree with the aforementioned values of α and β. As a final cross-check, let us compute the Weyl transformation of the equation of motion

$$\widehat{f}\phi = \left(\Box - \tfrac{n-2}{4(n-1)}\,R\right)\phi\,, \tag{B.15}$$

which, in our formalism, is obtained by evaluating

$$\delta_\sigma\big(\rho(F)\,\Phi\big) = \rho(\delta_\sigma F)\,\Phi + \rho(F)\,\delta_\sigma\Phi\,, \tag{B.16}$$

at $y=0$, where recall that $\Phi = \tau(\phi)$ is the lift of ϕ as a flat section of the Fock bundle, whose first order in y are given in (2.44). To compute the first term, we only need to use the simple quantization formula (4.3), to find

$$\rho(\delta_\sigma F)\,\Phi|_{y=0} = \hbar^2\,\tfrac{n-4}{6}\,\left(\Box\sigma + \tfrac{1}{n-1}\,R\,\sigma\right)\phi\,. \tag{B.17}$$

To compute the second term, we should also use

$$\rho(F)|_{y=0} = \hbar^2\,\eta^{ab}\,\tfrac{\partial^2}{\partial y^a\partial y^b} - \tfrac{\hbar^2}{12}\,\tfrac{n-4}{n-1}\,R\,, \tag{B.18}$$

as well as

$$\tfrac{1}{\hbar}\,\rho(\xi_{\text{Weyl}}) = -\big(\sigma + y^a\,\nabla_a\sigma\big)\,\big(y\cdot\tfrac{\partial}{\partial y} + \tfrac{n}{2}\big) + \tfrac12\,y^2\,\nabla^a\,\tfrac{\partial}{\partial y^a} - \tfrac{n+1}{3}\,y^ay^b\,\nabla_a\nabla_b\sigma + \tfrac16\,y^2\,\Box\sigma + \dots\,, \tag{B.19}$$

and

$$\rho(w_{\text{Weyl}}) = -\big(1 + y^a\,\nabla_a + \tfrac12\,[\nabla_a\nabla_b - \tfrac16\,R_{ab}] + \dots\big)\sigma\,. \tag{B.20}$$

Acting with the last two operators on the lift of ϕ, one finds

$$\delta_\sigma\Phi = -\rho(\tfrac{1}{\hbar}\,\xi_{\text{Weyl}} + w_{\text{Weyl}})\,\Phi \tag{B.21}$$
$$= -\tfrac{n-2}{2}\,\sigma\,\phi - \tfrac{n}{2}\,y^a\nabla_a(\sigma\,\phi) - \tfrac{n+2}{4}\,y^ay^b\,\sigma\,\big(\nabla_a\nabla_b - \tfrac16\,R\big)\,\phi \tag{B.22}$$
$$- \tfrac{n}{2}\,y^ay^b\,\nabla_a\sigma\nabla_b\phi - \tfrac{n-2}{6}\,y^ay^b\,\phi\,\nabla_a\nabla_b\sigma + \tfrac12\,y^2\,\big(\nabla\sigma\cdot\nabla\phi + \tfrac16\,\phi\,\Box\sigma\big) + \dots\,, \tag{B.23}$$

which leads to

$$\rho(F)\,\delta_\sigma\Phi|_{y=0} = -\hbar^2\,\tfrac{n+2}{2}\,\sigma\,\Box\phi - \hbar^2\,\tfrac{n-4}{6}\,\phi\,\Box\sigma + \hbar^2\,\tfrac{n(3n-4)+4}{24(n-1)}\,\sigma\,R\,\phi\,. \tag{B.24}$$

Collecting the two terms (B.17) and (B.24), we finally obtain the action of a Weyl transformation on the equation of motion,

$$\delta_\sigma\big(\widehat{f}\phi\big) = -\tfrac{n+2}{2}\,\sigma\,\big(\Box - \tfrac{n-4}{4(n-1)}\,R\big)\,\phi\,, \tag{B.25}$$

as expected: we recover the fact that the conformal Laplacian sends functions of Weyl weight $-\frac{n-2}{2}$ to functions of Weyl weight $-\frac{n+2}{2}$.

C Feigin–Felder–Shoikhet invariant trace

The Hochschild cohomology of the Weyl algebra $\mathcal{A}_{2n}$ with values in its linear dual $\mathcal{A}^*_{2n}$ is known to be concentrated in degree $2n$ and to be one-dimensional [76]. A representative for this cohomology class, that we will denote by Φ hereafter, was given explicitly by Feigin, Felder and Shoikhet [22], and reads as follows:

$$\Phi(a_0|a_1,\dots,a_{2n}) = \int_{u\in\Delta_{2n}} \exp\Big[\hbar \sum_{0\le i<j\le 2n} \big(\tfrac12 + u_i - u_j\big)\,\pi_{ij}\Big]\ \det\big|\tfrac{\partial}{\partial p^I_a},\tfrac{\partial}{\partial y^a_I}\big|_{I=1,\dots,2n} \qquad \text{(C.1)}$$

$$\times\ a_0(y_0,p_0)\,a_1(y_1,p_1)\dots a_{2n}(y_{2n},p_{2n})|_{y_k=0}\,,$$

where Δ_{2n} is the standard $2n$-simplex which can be defined as

$$\Delta_{2n} = \big\{(u_1,\dots,u_{2n})\in\mathbb{R}^{2n}\mid u_0\equiv 0\le u_1\le u_2\le\dots\le u_{2n}\le 1\big\}\,, \qquad \text{(C.2)}$$

and

$$\pi_{ij} := \frac{\partial}{\partial y^a_i}\frac{\partial}{\partial p^j_a} - \frac{\partial}{\partial p^i_a}\frac{\partial}{\partial y^a_j}\,, \qquad \text{(C.3)}$$

and the determinant is taken over the $2n\times 2n$ matrix whose entries are the operators $\frac{\partial}{\partial p^I_a}$ and $\frac{\partial}{\partial y^a_I}$ where the index I runs over 1 to $2n$, so that the argument a_0 remains unaffected by this determinant operator.

In practice, we need only the Chevalley–Eilenberg cocycle obtained from Φ by skew-symmetrisation of its arguments,[13] which we will denote by,

$$[\Phi](a_0|a_1,\dots,a_{2n}) = \sum_{\sigma\in\mathcal{S}_{2n}} (-1)^\sigma\,\Phi(a_0|a_{\sigma_1},\dots,a_{\sigma_{2n}})\,, \qquad \text{(C.4)}$$

where $(-1)^\sigma$ denotes the signature of the permutation σ. The n-cochain defined by

$$\mu(a_0|a_1,\dots,a_n) := \tfrac{1}{n!}\,\epsilon_{b_1\dots b_n}\,[\Phi](a_0|y^{b_1},\dots,y^{b_n},a_1,\dots,a_n)\,, \qquad \text{(C.5)}$$

is *almost* a Chevalley–Eilenberg cocycle, in the sense that it satisfies

$$\sum_{i=0}^{n}(-1)^i\,\mu\big([a_{-1},a_i]_*|a_0,\dots,a_n\big) + \sum_{0\le i<j\le n}(-1)^{i+j}\,\mu\big(a_{-1}|[a_i,a_j]_*,a_0,\dots,a_n\big) = \tfrac{\partial}{\partial p_a}\,\varphi_a(a_{-1}|a_0,\dots,a_n)\,, \qquad \text{(C.6)}$$

where

$$\varphi_a(a_{-1}|a_0,\dots,a_n) = \tfrac{1}{(n-1)!}\,\epsilon_{ab_1\dots b_{n-1}}\,[\Phi](a_{-1}|y^{b_1},\dots,y^{b_{n-1}},a_0,\dots,a_n)\,, \qquad \text{(C.7)}$$

i.e. it verifies the cocycle condition modulo a total derivative in p. As a first step towards simplifying the expression of μ, let us note that

$$\det\big|\tfrac{\partial}{\partial y^a_I},\tfrac{\partial}{\partial p^I_a}\big| = \sum_{\sigma\in\mathcal{S}_{n|n}}(-1)^\sigma\,\epsilon^{a_1\dots a_n}\,\frac{\partial}{\partial y^{a_1}_{\sigma_1}}\cdots\frac{\partial}{\partial y^{a_n}_{\sigma_n}}\,\epsilon_{b_1\dots b_n}\,\frac{\partial}{\partial p^{\sigma_{n+1}}_{b_1}}\cdots\frac{\partial}{\partial p^{\sigma_{2n}}_{b_n}} \qquad \text{(C.8)}$$

where $\mathcal{S}_{n|n}$ denotes the set of permutations of $2n$ elements which preserve the order of the first n and the last

[13] Recall that the skew-symmetrisation map is a morphism of complexes between the Hochschild complex of an associative algebra, and the Chevalley–Eilenberg of its commutator Lie algebra.

n elements separately, i.e. $\sigma_1 < \sigma_2 < \cdots < \sigma_n$ and $\sigma_{n+1} < \sigma_{n+2} < \cdots < \sigma_{2n}$, and

$$\begin{aligned}\tfrac{1}{n!}\,\epsilon_{a_1\ldots a_n} &\sum_{\sigma\in\mathcal{S}_{2n}} (-1)^\sigma\, y^{a_1}_{\sigma_1}\ldots y^{a_n}_{\sigma_n}\, a_1(y_{\sigma_{n+1}},p_{\sigma_{n+1}})\ldots a_n(y_{\sigma_{2n}},p_{\sigma_{2n}}) \\ &= \epsilon_{a_1\ldots a_n} \sum_{\sigma\in\mathcal{S}_{n|n}} (-1)^\sigma\, y^{a_1}_{\sigma_1}\ldots y^{a_n}_{\sigma_n} \sum_{\tau\in\mathcal{S}_n} (-1)^\tau\, a_{\tau_1}(y_{\sigma_{n+1}},p_{\sigma_{n+1}})\ldots a_{\tau_n}(y_{\sigma_{2n}},p_{\sigma_{2n}})\,,\end{aligned} \tag{C.9}$$

so that, put together, these two formulae yield

$$\det\big|\tfrac{\partial}{\partial y^a_I},\tfrac{\partial}{\partial p^I_a}\big|\Big(\tfrac{1}{n!}\,\epsilon_{a_1\ldots a_n}\sum_{\sigma\in\mathcal{S}_{2n}}(-1)^\sigma\, y^{a_1}_{\sigma_1}\ldots y^{a_n}_{\sigma_n} a_1(y_{\sigma_{n+1}},p_{\sigma_{n+1}})\ldots a_n(y_{\sigma_{2n}},p_{\sigma_{2n}})\Big) \tag{C.10}$$

$$= (2n)! \sum_{\substack{\{i_1<\cdots<i_n\}\\ \subset\{1,\ldots,2n\}}} \sum_{\sigma\in\mathcal{S}_n} (-1)^\sigma\, \epsilon_{a_1\ldots a_n} \frac{\partial a_{\sigma_1}}{\partial p_{a_1}}(y_{i_1},p_{i_1})\ldots\frac{\partial a_{\sigma_n}}{\partial p_{a_n}}(y_{i_n},p_{i_n})\,, \tag{C.11}$$

where the first sum is taken over all *ordered* subsets of n integers in the set $\{1,\ldots,2n\}$. We are now in position of writing down the cochain μ: for any $a_0,a_1,\ldots,a_n\in\mathcal{A}_{2n}$, it is given explicitly by

$$\mu(a_0|a_1,\ldots,a_n) = (2n)! \sum_{\sigma\in\mathcal{S}_n}(-1)^\sigma \int_{u\in\Delta_{2n}} \mathcal{D}(u;a_0,a_{\sigma_1},\ldots,a_{\sigma_n})|_{y=0}\,, \tag{C.12}$$

where

$$\mathcal{D}(u;-) = \sum_{f\in\Delta([n],[2n])} \exp\Big[\hbar\sum_{0\le i<j\le n}\big(\tfrac12+u_{f(i)}-u_{f(j)}\big)\,\pi_{ij}\Big]\epsilon_{a_1\ldots a_n}\big(1\otimes\tfrac{\partial}{\partial p_{a_1}}\otimes\cdots\otimes\tfrac{\partial}{\partial p_{a_n}}\big)\,, \tag{C.13}$$

and

$$\Delta([k],[l]) := \big\{f:\{1,2,\ldots,k\}\longrightarrow\{1,2,\ldots,l\}\mid f(i)<f(j)\,,\ 1\le i<j\le k\big\} \tag{C.14}$$

denotes the set of order-preserving maps from the set $[k]$ of the first k integers, to the set $[l]$ of the first l integers. Note that by convention, we put $f(0)=0$ and $u_0=0$.

Trace on the deformed algebra of functions. Suppose that $a_1,\ldots,a_n$ are linear in p, and write $\frac{\partial a}{\partial p_b}=a^b(y)$ for their derivative with respect to p. Then the above operator collapses to

$$\mathcal{D}(u;a_0,a_1,\cdots,a_n) = \sum_{f\in\Delta([n],[2n])} \exp\Big[\hbar\sum_{i=1}^n\big(u_{f(i)}-\tfrac12\big)\,\tfrac{\partial}{\partial p_a}\,\tfrac{\partial}{\partial y^a_i}\Big]\,a_0(y,p)\times\epsilon_{b_1\ldots b_n}\,a_1^{b_1}(y_1)\ldots a_n^{b_n}(y_n)\,, \tag{C.15}$$

thereby exhibiting a clear distinction between the arguments: the zeroth one will only receive derivative with respect to p, while the remaining n arguments will only receive derivatives with respect to y. Now consider the case where $a_0=F(y,p)$, and all other arguments are equal to the Fedosov connection, $a_1=\cdots=a_n=A$. Since A is linear in p we can write it as

$$A(y,p) = \mathrm{d}x^\mu\, e^a_\mu\, A_a{}^b(y)\,p_b\,, \tag{C.16}$$

and introducing the notation

$$\mathbb{A}(y_1,\dots,y_n) := \epsilon^{a_1\dots a_n}\,\epsilon_{b_1\dots b_n}\,A_{a_1}{}^{b_1}(y_1)\cdots A_{a_n}{}^{b_n}(y_n)\,, \tag{C.17}$$

we end up with

$$\mathcal{D}(u;F,A,\dots,A) = \mathrm{d}^n x\,|e| \sum_{f\in\Delta([n],[2n])} \exp\Big[\hbar\sum_{i=1}^{n}\big(u_{f(i)}-\tfrac{1}{2}\big)\,\tfrac{\partial}{\partial p_a}\,\tfrac{\partial}{\partial y_i^a}\Big]\,F(y,p)\times\mathbb{A}(y_1,\dots,y_n)\,, \tag{C.18}$$

where $|e|$ is the determinant of the vielbein. This formula exhibits a couple of properties:

- First, as we noticed earlier, the argument $F(y,p)$ is the only one to receive derivatives with respect to p. This means that in order to compute $\mu(F|A,\dots,A)$, one only needs to know $F|_{y=0}$, the y-independent part of the symbol F.
- Second, the integral over the simplex will produce some combinatorial coefficients

$$\sum_{f\in\Delta([n],[2n])}\int_{\Delta_{2n}}\big(u_{f(\ell_1)}-\tfrac{1}{2}\big)\cdots\big(u_{f(\ell_k)}-\tfrac{1}{2}\big)\,, \tag{C.19}$$

which depends on a k-tuple of integers $(\ell_1,\dots,\ell_k)$ comprised between 1 and n. In fact, one can refine this dependency a little bit by remarking that if two k-tuple are related by a permutation $\tau\in\mathcal{S}_k$, the associated coefficients are equal, so that these coefficients may as well be labeled by partitions of k.

Putting this together, one ends up with

$$\mu(F|A,\dots,A) = \mathrm{d}^n x\,|e|\sum_{k\geq 0}\mu^{\nabla}_{a_1\dots a_k}(x)\frac{\partial^k}{\partial p_{a_1}\cdots\partial p_{a_k}}F(y,p)\big|_{y=0}\,, \tag{C.20}$$

where $\mu^{\nabla}_{a_1\dots a_k}(x)$ are polynomials in the (covariant derivatives of the) curvature of ∇ which is obtained by computing the term of order $\hbar^k$ in

$$\sum_{f\in\Delta([n],[2n])}\int_{\Delta_{2n}}\exp\Big[\hbar\sum_{i=1}^{n}\big(u_{f(i)}-\tfrac{1}{2}\big)\,\tfrac{\partial}{\partial y_i^a}\otimes\tfrac{\partial}{\partial p_a}\Big]\,\mathbb{A}(y_1,\dots,y_n)\otimes F(y,p)\big|_{y_i=0}\,. \tag{C.21}$$

D Curvature expansion

Since the components of $\boldsymbol{\gamma}$ are constructed from the curvature tensor of ∇, its covariant derivatives and contractions thereof, we can re-arrange its expansion in power of the curvature, which appears through

$$\mathcal{R} \equiv -\tfrac{1}{3}\,\mathrm{d}x^\mu\,R_{\mu a}{}^c{}_b\,y^a y^b\,p_c\,, \tag{D.1}$$

namely we write $\boldsymbol{\gamma}=\sum_{k\geq 1}\boldsymbol{\gamma}^{(k)}$ where $\boldsymbol{\gamma}^{(k)}$ is of order k in $\mathcal{R}$ and its derivatives. Let us now evaluate its defining equation (2.18) at order n in $\mathcal{R}$,

$$\boldsymbol{\gamma}^{(k)} = \mathcal{R}\,\delta_{k,1} + \partial_\nabla\boldsymbol{\gamma}^{(k)} + \sum_{l=1}^{k-1}\tfrac{1}{2\hbar}\,h\big[\boldsymbol{\gamma}^{(l)},\boldsymbol{\gamma}^{(k-l)}\big]_*\,, \tag{D.2}$$

which we can re-write, for $k > 1$, as

$$\boldsymbol{\gamma}^{(k)} = \tfrac{1}{2\hbar} \sum_{l=1}^{k-1} j_\nabla h \left[\boldsymbol{\gamma}^{(l)}, \boldsymbol{\gamma}^{(k-l)}\right]_* , \qquad \text{with} \qquad j_\nabla := \frac{1}{1-\partial_\nabla} = \sum_{m=0}^{\infty} \partial_\nabla^m . \tag{D.3}$$

The first few orders in this curvature expansion

$$\boldsymbol{\gamma}^{(1)} = j_\nabla \mathcal{R} , \qquad \boldsymbol{\gamma}^{(2)} = \tfrac{1}{2\hbar}\, j_\nabla \{\mathcal{R}, \mathcal{R}\} , \qquad \boldsymbol{\gamma}^{(3)} = \tfrac{1}{2\hbar^2}\, j_\nabla \{\mathcal{R}, \{\mathcal{R}, \mathcal{R}\}\} , \tag{D.4}$$

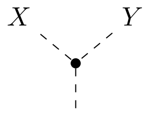
$$\boldsymbol{\gamma}^{(4)} = \tfrac{1}{2\hbar^3}\, j_\nabla \{\mathcal{R}, \{\mathcal{R}, \{\mathcal{R}, \mathcal{R}\}\}\} + \tfrac{1}{8\hbar^3}\, j_\nabla \{\{\mathcal{R}, \mathcal{R}\}, \{\mathcal{R}, \mathcal{R}\}\} , \tag{D.5}$$

where we introduce the bracket

$$\{-,-\} := h \left[j_\nabla(-), j_\nabla(-) \right]_* , \tag{D.6}$$

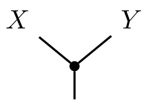
as a shorthand notation. This approach has a couple of advantages: first, it allows us to access in one go whole pieces of $\boldsymbol{\gamma}$ at arbitrary order in y, and second, the recursion in order of curvature exhibits an interesting structure, namely it appears that it is controlled by the grafting (non-planar) binary trees. Indeed, denoting the operator j_∇ by an edge, and the composition of the star-commutator with the contracting homotopy by a vertex, i.e.

$j_\nabla(X) =$ 

$\tfrac{1}{2\hbar}\, h[X, Y]_* =$ 

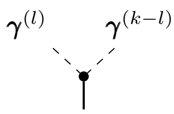
(D.7)

where the diagrams should be read from top to bottom, so that for instance

$j_\nabla \{X, Y\} \;=$ 

(D.8)

one can re-write the recursion relation (D.3) as

$\boldsymbol{\gamma}^{(k)} = \sum_{l=1}^{k-1}$ 

(D.9)

for all $k > 1$. Now it becomes relatively easy to see that the result of this recursion relation is to express $\boldsymbol{\gamma}^{(k)}$ as a sum over all *rooted planar binary trees with k leaves.* Indeed, the latter can be obtained by successively *grafting*—meaning summing over all trees resulting from attaching the root of a tree to the leaves of another one—the rooted binary tree with a single vertex to itself, k times, which is exactly what the above relation produces. Taking into account the fact that $\{-,-\}$ is antisymmetric amounts to identifying any two rooted planar binary trees which can be related by permuting the two leaves at each node, that is, one should sum over rooted *non-planar* binary trees and take into account the number of planar ones that it is equivalent to as a multiplicity.

Having worked out the recursion formula for $\boldsymbol{\gamma}$ in order of the curvature, we can do the same for the lift of symbols. Indeed, resumming the defining relation (2.22) for the lift $F(x;y,p)$ of a symbol $f(x,p)$ yields,

$$F = \partial_\triangledown F + \tfrac{1}{\hbar}\, h\, [\boldsymbol{\gamma}, F]_* \tag{D.10}$$

from which we can extract the order k piece via

$$F^{(0)} = j_\triangledown f\,, \qquad F^{(k>0)} = \tfrac{1}{\hbar} \sum_{l=0}^{k-1} j_\triangledown h\, [\boldsymbol{\gamma}^{(k-l)}, F^{(l)}]_*\,. \tag{D.11}$$

The first few orders are given by

$$F^{(1)} = \tfrac{1}{\hbar}\, j_\triangledown \{\mathcal{R}, f\}\,, \qquad F^{(2)} = \tfrac{1}{2\hbar^2}\, j_\triangledown \Big(\{\{\mathcal{R},\mathcal{R}\}, f\} + 2\{\mathcal{R}, \{\mathcal{R}, f\}\} \Big)\,, \tag{D.12}$$

and

$$F^{(3)} = \tfrac{1}{2\hbar^3}\, j_\triangledown \Big(\{\{\mathcal{R},\{\mathcal{R},\mathcal{R}\}\}, f\} + \{\{\mathcal{R},\mathcal{R}\},\{\mathcal{R}, f\}\} + \{\mathcal{R},\{\{\mathcal{R},\mathcal{R}\}, f\}\} + 2\{\mathcal{R},\{\mathcal{R},\{\mathcal{R}, f\}\}\} \Big)\,. \tag{D.13}$$

Finally, the same can be done for the lift of a function $\phi(x)$ to a covariantly constant section $\Phi(x;y)$ of the Fock bundle: the re-summed for of the recursion relation (2.42) reads

$$\Phi = \partial_\triangledown \Phi + \tfrac{1}{\hbar}\, h\, \rho(\boldsymbol{\gamma})\Phi\,. \tag{D.14}$$

which, when evaluated at order n in $\mathcal{R}$ gives us

$$\Phi^{(0)} = j_\triangledown\, \phi\,, \qquad \Phi^{(k>0)} = \tfrac{1}{\hbar} \sum_{l=0}^{k-1} j_\triangledown h\, \rho(\boldsymbol{\gamma}^{(k-l)})\Phi^{(l)}\,. \tag{D.15}$$

The first few orders read

$$\Phi^{(1)} = \tfrac{1}{\hbar}\, j_\triangledown \big(\mathcal{R} \triangleright \phi\big)\,, \qquad \Phi^{(2)} = \tfrac{1}{2\hbar^2}\, j_\triangledown \Big(\{\mathcal{R},\mathcal{R}\} \triangleright \phi + 2\,\mathcal{R} \triangleright \big(\mathcal{R} \triangleright \phi\big) \Big)\,, \tag{D.16}$$

and

$$\Phi^{(3)} = \tfrac{1}{2\hbar^3}\, j_\triangledown \Big(\{\mathcal{R},\{\mathcal{R},\mathcal{R}\}\} \triangleright \phi + \{\mathcal{R},\mathcal{R}\} \triangleright (\mathcal{R} \triangleright \phi) + \mathcal{R} \triangleright \big(\{\mathcal{R},\mathcal{R}\} \triangleright \phi\big) + 2\,\mathcal{R} \triangleright \big(\mathcal{R} \triangleright (\mathcal{R} \triangleright \phi)\big) \Big)\,, \tag{D.17}$$

where we introduced the shorthand notation

$$\bullet \triangleright (-) := h\, \rho\big(j_\triangledown(\bullet)\big)\, j_\triangledown(-)\,, \tag{D.18}$$

for the sakes of conciseness. The term of order k will be almost identical to that of F, except for the replacement of f with ϕ, and any bracket of the form $\{-, f\}$, i.e. whose second argument is f, with $(-)\triangleright\phi$. This is of course not surprising since the only difference between the two case is the representation of the Weyl algebra in which the covariantly constant section that we are solving for sits in: the adjoint for symbols like f and the Fock one for functions like ϕ.

Simplifying the elementary operations. Let us try to find a concise expression for the operator j_∇. To do so, first notice that

$$\partial_\nabla \alpha = h\nabla\alpha = \tfrac{1}{N}\left(y^a\nabla_a\alpha - \nabla\, N\, h\alpha\right), \tag{D.19}$$

so that, on forms valued in the Weyl algebra which are *annihilated* by the homotopy operator h, one finds

$$h\alpha = 0 \qquad \Longrightarrow \qquad \partial_\nabla\alpha = \tfrac{1}{N}\, y\cdot\nabla\alpha = y\cdot\nabla\,\tfrac{1}{N+1}\,\alpha\,, \tag{D.20}$$

where we used the fact that $y\cdot\nabla$ is obviously of degree 1 in y, and hence increases the eigenvalue of the number operator N, a fact we should take into account when moving the latter to the right of the former. Repeating this operation, we end up with

$$j_\nabla = \sum_{k\geq 0}(y\cdot\nabla)^k\,\tfrac{1}{(N+1)_k} = \sum_{k\geq 0}\tfrac{1}{k!}\,(y\cdot\nabla)^k\,\tfrac{(1)_k}{(N+1)_k} \equiv {}_1F_1\big[1;N+1;y\cdot\nabla\big]\,, \tag{D.21}$$

where $(a)_k = a(a+1)\dots(a+k-1)$ is the raising Pochhammer symbol, and where we used the fact that $(1)_k = k!$ to recognize the *confluent hypergeometric function.* Let us stress that the above expression is valid only for j_∇ acting on elements in $\mathrm{Ker}(h)$, which is enough for us since we are interested in applying it to either $\boldsymbol{\gamma}$ or a 0-form, both annihilated by h, by definition.

We can now use the integral representation of the confluent hypergeometric function,

$${}_1F_1\big[a;b;x\big] := \sum_{k\geq 0}\tfrac{x^k}{k!}\,\tfrac{(a)_k}{(b)_k} = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)}\int_0^1 \mathrm{d}t\, e^{t\,x}\,t^{a-1}\,(1-t)^{b-a-1}\,, \tag{D.22}$$

which holds for $\mathrm{Re}(b) > \mathrm{Re}(a) > 0$. In our case, both parameters are positive integer, and verify the inequality except if $N=0$, i.e. if we act on a y-independent 0-form. This case is particularly simple to treat since the hypergeometric series collapses to an ordinary exponential, i.e.

$$j_\nabla|_{\mathcal{C}^\infty(T^*\mathcal{X})} = e^{y\cdot\nabla}\,. \tag{D.23}$$

We can therefore exclude this case, that is consider $N>0$, and use the above integral representation to re-write j_∇ as

$$j_\nabla = N\int_0^1 \mathrm{d}t\, e^{t\,y\cdot\nabla}\,(1-t)^{N-1} = \int_0^1 \mathrm{d}t\, e^{(1-t)\,y\cdot\nabla}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\,t^N\,, \tag{D.24}$$

where we used the change of variable $t\to 1-t$ before recognizing the derivative. We therefore find

$$j_\nabla(\alpha) = \int_0^1 \mathrm{d}t\, e^{(1-t)\,y\cdot\nabla}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha(x,t\,\mathrm{d}x,t\,y,p) \tag{D.25}$$

$$= \alpha + y\cdot\nabla\int_0^1 \mathrm{d}t\, e^{(1-t)\,y\cdot\nabla}\,\alpha(x,t\,\mathrm{d}x,t\,y,p)\,, \tag{D.26}$$

for any Weyl algebra-valued form α such that $h\alpha=0$. Remark that this last form obtained by integrating by part, namely

$$j_\nabla = 1 - \int_0^1 \mathrm{d}t\,\tfrac{\mathrm{d}}{\mathrm{d}t}\big(e^{(1-t)\,y\cdot\nabla}\big)\,t^N\,, \tag{D.27}$$

also makes sense for $N=0$, since it reproduces (D.23).

Lifts at first order in curvature. Let us now use this operator to compute the lift of symbols and wave functions at first order in curvature. Starting with the latter, we need to first compute

$$\begin{aligned}\rho(j_\nabla\mathcal{R})\,j_\nabla\phi &= \exp\Big(-\hbar\,\partial_p\cdot\big[\tfrac12\,\partial_{y_1}+\partial_{y_2}\big]\Big)\int_0^1 \mathrm{d}u\,u^2\,e^{(1-u)\,y_1\cdot\nabla}\,\big(-\mathrm{d}x^\mu\,R_{\mu a}{}^c{}_b\,y_1^a y_1^b\,p_c\big)\,\times\,e^{y_2\cdot\nabla}\phi\big|_{p=0,y_1=y=y_2}\\ &= -\tfrac{\hbar}{2}\int_0^1 \mathrm{d}u\,u^2\,e^{(1-u)\,y\cdot\nabla}\,\big(\mathrm{d}x^\mu\,R_{\mu a}\,y^a\big)\times e^{y\cdot\nabla}\phi+(\cdots)\,,\end{aligned} \tag{D.28}$$

where the dots denote terms annihilated by the contracting homotopy h. Applying the latter composed with j_∇, we end up with

$$\begin{aligned}\Phi^{(1)}\equiv\tfrac{1}{\hbar}\,j_\nabla\,h\,\rho(j_\nabla\mathcal{R})\,j_\nabla\phi = -\tfrac12\int_{[0,1]^3}&\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}u\;e^{(1-t)\,y\cdot[\nabla_1+\nabla_2]}\\ &\times\;\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(s\,t^2\,u^2\,e^{t\,s\,y\cdot[\nabla_1+(1-u)\,\nabla_2]}\phi(x_1)\,R_{ab}(x_2)\,y^a y^b\Big)|_{x_1=x=x_2}\,,\end{aligned} \tag{D.29}$$

where ∇_i denotes the covariant derivative with respect to x_i. Evaluating this formula up to third order in y yields

$$\Phi=\phi+y^a\,\nabla_a\phi+\tfrac12\,y^a y^b\,\big(\nabla_a\nabla_b-\tfrac16\,R_{ab}\big)\phi+\tfrac16\,y^a y^b y^c\,\big(\nabla_a\nabla_b\nabla_c-\tfrac12\,R_{ab}\,\nabla_c-\tfrac14\,\nabla_a R_{bc}\big)\phi+(\dots)\,, \tag{D.30}$$

as previously derived from the defining recursion relation for the lift of the wave function ϕ in the Fock bundle. We can re-write this lift up to first order in curvature as

$$\Phi(x;y)=\Big(\tau^{(0)}(y\cdot\nabla_1)+\tau^{(1)}(y\cdot\nabla_1,y\cdot\nabla_2)\,\tfrac12\,y^a y^b\,R_{ab}(x_2)+\dots\Big)\phi(x_1)|_{x_i=x}\,, \tag{D.31}$$

with $\tau^{(0)}(z_1)=e^{z_1}$ and

$$\tau^{(1)}(z_1,z_2)=-\int_{[0,1]^3}\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}u\;e^{(1-t)\,[z_1+z_2]}\;\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(s\,t^2\,u^2\,e^{t\,s\,[z_1+(1-u)\,z_2]}\Big)=-\tfrac16\,e^{z_1}\,{}_1F_1\big[2;4;z_2\big] \tag{D.32}$$

which, upon using the integral representation (D.22), can be expressed as

$$\tau^{(1)}(z_1,z_2)=e^{z_1}\int_0^1\mathrm{d}t\;(t-1)\,t\,e^{t\,z_2}\,. \tag{D.33}$$

and hence

$$\Phi(x;y)=\Big(1+\tfrac12\,y^a y^b\int_0^1\mathrm{d}t\;t\,(t-1)\,e^{t\,y\cdot\nabla}R_{ab}+\dots\Big)\,e^{y\cdot\nabla}\phi\,. \tag{D.34}$$

Now let us turn our attention to the piece of first order in curvature of the lift of f, for which we need to compute

$$[j_\nabla \mathcal{R}, j_\nabla f]_* = \sum_{\sigma=\pm} \sigma \, \exp\Big(\tfrac{\sigma\hbar}{2}\big[\partial_{y_1}\cdot\partial_{p_2} - \partial_{p_1}\cdot\partial_{y_2}\big]\Big) \tag{D.35}$$

$$\times \int_0^1 \mathrm{d}u \, e^{(1-u)\, y_1\cdot\nabla}\,\big(-u^2\,\mathrm{d}x^\mu\, R_{\mu\, a}{}^c{}_b y_1^a y_1^b\, p_{1\, c}\big) \times e^{y_2\cdot\nabla} f(p_2)\big|_{\substack{y_1=y=y_2\\ p_1=p=p_2}} \tag{D.36}$$

$$= \sum_{\sigma=\pm} \sigma \, \exp\Big(\tfrac{\sigma\hbar}{2}\,\partial_{y_1}\cdot\partial_{p_2}\Big)\int_0^1 \mathrm{d}u\, e^{(1-u)\, y_1\cdot\nabla}\big(u^2\,\mathrm{d}x^\mu\, R_{\mu\, a}{}^c{}_b\, y_1^a y_1^b\big) \tag{D.37}$$

$$\times\,\big(-p_{1\,c} + \tfrac{\sigma\hbar}{2}\,\nabla_c\big)\, e^{y_2\cdot\nabla}\, f(p_2)\big|_{\substack{y_1=y=y_2\\ p_1=p=p_2}} \tag{D.38}$$

$$= \tfrac{\hbar}{2}\sum_{\sigma=\pm}\int_0^1 \mathrm{d}u\; e^{(1-u)[\,y+\frac{\sigma\hbar}{2}\,\partial_p]\cdot\nabla}\Big(u^2\,\mathrm{d}x^\mu\, R_{\mu\, a}{}^c{}_b\,\big[y^b\,\tfrac{\partial}{\partial p_a} + \tfrac{\sigma\hbar}{2}\,\tfrac{\partial^2}{\partial p_a\partial p_b}\big]\Big) \tag{D.39}$$

$$\times\,\big(-p'_c + \tfrac{\sigma\hbar}{2}\,\nabla_c\big)\, e^{y\cdot\nabla}\, f(p)|_{p'=p} + (\dots)\,, \tag{D.40}$$

where the dots denote terms that are annihilated by h. Applying it followed by j_∇, one finds

$$F^{(1)} \equiv \tfrac{1}{\hbar}\, j_\nabla h\,[j_\nabla\mathcal{R}, j_\nabla f]_* = \tfrac12 \sum_{\sigma=\pm}\int_{[0,1]^3} \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\; e^{(1-t)\, y\cdot[\nabla_1+\nabla_2]} \tag{D.41}$$

$$\times\,\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(t\, u^2\, e^{s\, t\, y\cdot[(1-u)\,\nabla_1+\nabla_2]+\frac{\sigma\hbar}{2}\,(1-u)\,\partial_p\cdot\nabla_1}\, y^d\, R_{da}{}^c{}_b(x_1) \tag{D.42}$$

$$\times\,\big[t\, s\, y^b\,\tfrac{\partial}{\partial p_a} + \tfrac{\sigma\hbar}{2}\,\tfrac{\partial^2}{\partial p_a\partial p_b}\big]\,\big(-p'_c + \tfrac{\sigma\hbar}{2}\,\nabla_c\big)\, f(x_2,p)\Big)\Big|_{\substack{x_1=x=x_2\\ p'=p}}\,. \tag{D.43}$$

After explicitly performing the integrals as for the lift of ϕ, one finds

$$F^{(1)}(x,p;y) = \tfrac12\, e^{y\cdot\nabla_1}\sum_{\substack{k,l\geq 0\\ \sigma=\pm}}\big(\tfrac{\sigma\hbar}{2}\big)^k\,\tfrac{(\partial_p\cdot\nabla_2)^k}{k!}\,\tfrac{(y\cdot\nabla_2)^l}{l!}\, y^a\, R_{da}{}^c{}_b(x_2)\,\tfrac{1}{(k+1)(l+k+2)(l+k+3)} \tag{D.44}$$

$$\times\,\Big[y^b\,\tfrac{\partial}{\partial p_d} + \tfrac{\sigma\hbar}{2}\,\tfrac{l+2k+4}{(k+2)}\,\tfrac{\partial^2}{\partial p_b\partial p_d}\Big]\,\big(p'_c - \tfrac{\sigma\hbar}{2}\,\nabla_c\big)\, f(x_1,p)\big|_{\substack{x_i=x,\\ p'=p}} \tag{D.45}$$

$$= \tfrac12\sum_{\sigma=\pm}\int_{[0,1]^2}\mathrm{d}s\,\mathrm{d}t\; t\,(1-t)\, e^{y\cdot\nabla_1 + t\,[y+s\,\frac{\sigma\hbar}{2}\,\partial_p]\cdot\nabla_2}\,\big[p' - \tfrac{\sigma\hbar}{2}\,\nabla_1\big]_c \tag{D.46}$$

$$\times\, y^a\,\Big(y^b + \sigma\hbar\,\big[1 + \tfrac{t}{2}\,(1-s)\, y\cdot\nabla_2\big]\,\tfrac{\partial}{\partial p_b}\Big)\, R_{da}{}^c{}_b(x_2)\,\tfrac{\partial}{\partial p_d} f(x_1,p)\big|_{\substack{x_i=x,\\ p'=p}}\,, \tag{D.47}$$

for the piece of first order in curvature of the lift $\tau(f)$ of any symbol $f(x,p)$.

Generating function. Combining the first order lift $\Phi^{(1)}$ of the scalar field given in (D.34) with the fact that

$$\rho(j_\nabla f)|_{y=0} = \exp\Big(-\hbar\partial_p\cdot\big[\tfrac12\,\nabla + \partial_y\big]\Big) f\,\big|_{y=0=p} \tag{D.48}$$

for a symbol $f = f(x,p)$, i.e. y-independent, one ends up with

$$\rho(F^{(0)})\Phi^{(1)}|_{y=0} = \tfrac{\hbar^2}{2}\int_0^1 \mathrm{d}t\;(t-1)t\,\times\, e^{-\hbar\,\partial_p\cdot[\nabla_1 + t\,\nabla_2 + \frac12\,\nabla_3]}\,\phi(x_1)\, R_{ab}(x_2)\,\tfrac{\partial^2}{\partial p_a\partial p_b} f(x_3,p)\big|_{x_i=x,p=0}\,, \tag{D.49}$$

upon using the BCH formula. Note that the commutator appearing in these manipulations can be discarded as a consequence of the fact that we are working at first order in curvature. Now applying the quantization map to the first order lift $F^{(1)}$ of a symbol given in (D.46), we end up with,

$$\rho(F^{(1)})\Phi^{(0)}|_{y=0} = \tfrac{\hbar^2}{8} \sum_{\sigma=\pm} \int_{[0,1]^2} \mathrm{d}s\, \mathrm{d}t\ t(t-1)\, e^{-\frac{\hbar}{2}\,\partial_p\cdot[2\nabla_1 + t(1-\sigma\, s)\,\nabla_2 + \nabla_3]} \\ \times \Big[(1-2\sigma) + \tfrac{\sigma\hbar}{2}\, t(1-s)\,\partial_p\cdot\nabla_2\Big]\, \phi(x_1)\, R_{ab}(x_2)\, \tfrac{\partial^2}{\partial p_a \partial p_b} f(x_3,p)\big|_{\substack{x_i=x,\\ p=0}}\,, \tag{D.50}$$

after using the BCH formula a few times as before. Performing the integrals, multiplying the result by ϕ^*, and integrating by part so that all derivatives on f are re-distributed on ϕ, ϕ^* and the curvature, one ends up with

$$\mathcal{J}(x|u) = e^{-\frac{\hbar}{2}\,u\cdot[\nabla_1-\nabla_2]}\Big(1 - \tfrac{\hbar^2}{8}\,\mathrm{sinhc}(\tfrac{\hbar}{4}\,u\cdot\nabla_3)\,R_{ab}(x_3)\,u^a u^b + \mathcal{O}(R^2)\Big)\,\phi(x_1)\,\phi^\dagger(x_2)\big|_{x_i=x}\,, \tag{D.51}$$

where

$$\mathcal{J}(x|u) := \sum_{s\geq 0} \tfrac{(-\hbar)^s}{2^s\, s!}\, J_{a_1\ldots a_s}(x)\, u^{a_1}\ldots u^{a_s}\,, \tag{D.52}$$

is the generating function for the higher spin currents, and

$$\mathrm{sinhc}(z) := \frac{\sinh(z)}{z}\,. \tag{D.53}$$

is the hyperbolic version of the sinc function.

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