

# NORMALIZED VECTOR SOLUTIONS OF NONLINEAR SCHRÖDINGER SYSTEMS

XIAOMENG HUANG, ANGELA PISTOIA, CHRISTOPHE TROESTLER, AND CHUNHUA WANG

ABSTRACT. Given  $\mu > 0$  we look for solutions  $\lambda \in \mathbb{R}$  and  $v_1, \dots, v_k \in H^1(\mathbb{R}^N)$  of the system

$$\begin{cases} -\Delta v_i + \lambda v_i + V_i(x)v_i = \sum_{j=1}^k \beta_{ij} v_i v_j^2 & \text{in } \mathbb{R}^N, \quad i = 1, \dots, k, \\ \int_{\mathbb{R}^N} (v_1^2 + \dots + v_k^2) dx = \mu, \end{cases}$$

where  $N = 1, 2, 3$ ,  $V_i : \mathbb{R}^N \rightarrow \mathbb{R}$  and  $\beta_{ij} \in \mathbb{R}$  satisfy  $\beta_{ij} = \beta_{ji}$  and  $\beta_{ii} > 0$ . Under suitable assumptions on the  $\beta_{ij}$ 's, given a non-degenerate critical point  $\xi_0$  of a suitable linear combination of the potentials  $V_i$ , we build solutions whose components concentrate at  $\xi_0$  as the prescribed global mass  $\mu$  is either large (when  $N = 1$ ) or small (when  $N = 3$ ) or it approaches some critical threshold (when  $N = 2$ ).

## 1. INTRODUCTION

A problem widely studied in the last decades concerns the existence of solutions  $(\lambda_i, v_i) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ ,  $i = 1, \dots, k$  of the systems

$$-\Delta v_i + \lambda_i v_i + V_i(x)v_i = \sum_{j=1}^k \beta_{ij} v_i v_j^2 \quad \text{in } \mathbb{R}^N, \quad i = 1, \dots, k, \quad (1.1)$$

with prescribed masses, namely

$$\int_{\mathbb{R}^N} v_i^2 = \mu_i, \quad i = 1, \dots, k. \quad (1.2)$$

Here  $N = 1, 2, 3$ ,  $V_i \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  and  $\beta_{ij} \in \mathbb{R}$  satisfy  $\beta_{ij} = \beta_{ji}$ . We will consider the *focusing* case, i.e.  $\beta_{ii} > 0$ .

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Solutions to (1.1)–(1.2) naturally arise in the study of solitary waves to time-dependent nonlinear Schrödinger equations as

$$i\partial_t \Phi_i + \Delta \Phi_i - V_i(x)\Phi_i + \sum_{j=1}^k \beta_{ij} \Phi_i |\Phi_j|^2 = 0, \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}, \quad (1.3)$$

which has application in nonlinear optics and in the study of Bose-Einstein condensates [10, 15]. Solitary wave solutions to (1.3) are obtained imposing the ansatz  $\Phi_i(x, t) = e^{i\lambda_i t} v_i(x)$ , where the real constant  $\lambda_i$  and the real valued function  $v_i$  satisfy Equation (1.1). Despite the problem having some relevance in physical problems, only a few existence (or non-existence) results seem to be known.

The natural approach to produce solutions to (1.1)–(1.2) consists in finding critical points of the energy

$$J(v_1, \dots, v_k) := \frac{1}{2} \sum_{i=1}^k \int_{\mathbb{R}^N} (|\nabla v_i|^2 + V_i(x)v_i^2) \, dx - \frac{1}{4} \sum_{i,j=1}^k \int_{\mathbb{R}^N} \beta_{ij} v_i^2 v_j^2 \, dx \quad (1.4)$$

constrained on the product of spheres

$$S := S_{\mu_1} \times \dots \times S_{\mu_k}, \quad \text{with } S_{\mu_i} := \left\{ v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |v|^2 \, dx = \mu_i \right\}. \quad (1.5)$$

The Langrange multipliers are nothing but the unknown real numbers  $\lambda_1, \dots, \lambda_k$ .

The study of existence of solutions to (1.1)–(1.2) strongly depends on the dimension  $N$ . Indeed, when  $N = 2$ , the scaling  $u(x) = v(x/t)$  leaves both the ratio  $\int |\nabla v|^2 \, dx / \int |v|^4 \, dx$  and the mass invariant, which is why the power  $p = 3$  when  $N = 2$  is called  *$L^2$ -critical*. In the following, we agree that  $N = 2$  is the *critical* regime and we say that  $N = 1$  is the *subcritical* regime and  $N = 3$  is the *supercritical* regime.

The situation in the case of a single equation (i.e.  $k = 1$ ) is quite well understood. A complete review of the available results in this context goes beyond the aim of this paper. We only quote the pioneering paper by Jeanjean [11] where the author studies the autonomous case (i.e. the potential is a constant) using a variational argument, which have been widely employed in the successive literature. We also quote the recent paper by Pellacci, Pistoia, Vaira and Verzini in [20], where the authors tackle the problem using a different point of view. They use the well-known Lyapunov-Schmidt method keeping the mass as the natural parameter in the reduction process and build solutions with large mass in the subcritical regime, with small mass in the supercritical regime and with mass close to a certain threshold value in the critical regime. We also refer the interested reader to the references therein.

In striking contrast, very few papers concern with the existence of normalized solutions to the system. Moreover, most of the known results only consider the case of 2 equations in the autonomous case. To describe them it is useful to introduce the coupling parameter  $\beta := \beta_{12} = \beta_{21}$ . The first result is due to Nguyen and Wang in [18] in dimension  $N = 1$  in an *attractive* regime, i.e.  $\beta > 0$ . In 1D the growth of the nonlinearity is subcritical so that the functional  $J$  in (1.4) is bounded from below on the constraint  $S$

in (1.5) and normalized solutions can be obtained by minimization. We observe that in higher dimensions the functional is unbounded from below on the constraint when  $\beta$  is positive, and hence their approach cannot be used. The supercritical regime (i.e.  $N = 3$ ) has been firstly studied by Bartsch, Jeanjean and Soave in [3] who developed an accurate minimax argument to find a solution for suitable choices of the parameters in the attractive case (i.e.  $\beta > 0$ ). In particular, a solution to the system (1.1)–(1.2) exists for every sufficiently small or sufficiently large  $\beta$ . We also quote some further generalizations obtained by Bartsch, Zhong and Zou [7], Bartsch and Jeanjean [2] and Li and Zou [12]. The existence of a solution in the repulsive case (i.e.  $\beta < 0$ ) has been established by Bartsch and Soave [4, 5, 6] who devise a different variational approach, based upon the introduction of a further natural constraint. In the critical regime (i.e.  $N = 2$ ) the existence of normalized solutions is a very subtle issue, heavily depending on the prescribed masses as can already be seen in the scalar case and it seems largely open. Very recently, Mederski and Szulkin [16] consider the case of  $k \geq 2$  equations and show the existence of multiple solutions provided that all the parameters  $\beta_{ij}$ 's are positive and satisfy a suitable condition. In particular, they prove that if  $\beta := \beta_{ij}$  for any  $i \neq j$  then the system has a solution when  $\beta$  is large enough. Finally, as far as we know, there is only one paper concerning the non-autonomous case. Noris, Tavares and Verzini in [19] consider the system (1.1)–(1.2) with only two equations in the presence of positive continuous trapping potentials (i.e.  $V_i \rightarrow +\infty$  as  $|x| \rightarrow \infty$ ) and prove via a variational approach the existence of positive solutions with small masses.

In this paper, we study the system (1.1) when we prescribe the *global mass* of the solution  $v_1, \dots, v_k$ . More precisely, given  $\mu > 0$  we look for solutions  $\lambda \in \mathbb{R}$  and  $\mathbf{v} := (v_1, \dots, v_k)$ ,  $v_i \in H^2(\mathbb{R}^N)$  of the system

$$\begin{cases} -\Delta v_i + \lambda v_i + V_i(x)v_i = \sum_{j=1}^k \beta_{ij} v_i v_j^2 & \text{in } \mathbb{R}^N, \quad i = 1, \dots, k, \\ \int_{\mathbb{R}^N} (v_1^2 + \dots + v_k^2) \, dx = \mu. \end{cases} \quad (1.6)$$

For sake of simplicity we will assume that  $V_i, |\nabla V_i| \in L^\infty(\mathbb{R}^N)$  for every  $i = 1, \dots, k$ .

Let us introduce the necessary ingredients to state our result. Let  $U$  be the positive radial solution of

$$-\Delta U + U = U^3 \quad \text{in } \mathbb{R}^N.$$

It is well known that  $U$  and its first and second derivatives decay exponentially [13, 14]. We assume that  $\mathbf{U} := (U_1, \dots, U_k)$  is a *synchronized* radial positive solution to the limit system

$$-\Delta U_i + U_i = \sum_{j=1}^k \beta_{ij} U_i U_j^2 \quad \text{in } \mathbb{R}^N, \quad i = 1, \dots, k, \quad (1.7)$$

i.e.  $U_i = \sigma_i U$ , with  $\sigma_i > 0$  for  $i = 1, \dots, k$  solutions of the algebraic system

$$\sum_{j=1}^k \beta_{ij} \sigma_j^2 = 1, \quad i = 1, \dots, k. \quad (1.8)$$

We also assume that it is *non-degenerate*, i.e. the set of the solutions of the linear system

$$-\Delta\phi_i + \phi_i - \sum_{j=1}^k \beta_{ij}(U_j^2\phi_i + 2U_iU_j\phi_j) = 0 \text{ in } \mathbb{R}^N, \quad i = 1, \dots, k, \quad (1.9)$$

is a  $N$ -dimensional space generated by

$$\Phi_i := \left( \frac{\partial U_1}{\partial x_i}, \dots, \frac{\partial U_k}{\partial x_i} \right) = \left( \sigma_1 \frac{\partial U}{\partial x_i}, \dots, \sigma_k \frac{\partial U}{\partial x_i} \right), \quad i = 1, \dots, N. \quad (1.10)$$

Examples of this kind of solutions can be found in Examples A.2 and A.4.

Set

$$\mu_0 := \sum_{i=1}^k \int_{\mathbb{R}^N} U_i^2(x) dx = \gamma \sum_{i=1}^k \sigma_i^2 \quad \text{with } \gamma = \int_{\mathbb{R}^N} U^2 dx. \quad (1.11)$$

Next, we introduce the *global potential* (see (1.11))

$$\Gamma(x) = \gamma \sum_{i=1}^k \sigma_i^2 V_i(x). \quad (1.12)$$

We assume that  $\xi_0 \in \mathbb{R}^N$  is a non-degenerate critical point of  $\Gamma$ . Without loss of generality we can suppose that, in a neighborhood of  $\xi_0$ ,

$$\Gamma(x) = \Gamma(\xi_0) + \frac{1}{2} \sum_{i=1}^N \frac{\partial^2 \Gamma}{\partial x_i^2}(\xi_0) (x - \xi_0)_i^2 + \mathcal{O}(|x - \xi_0|^3), \quad \text{with } \frac{\partial^2 \Gamma}{\partial x_i^2}(\xi_0) \neq 0. \quad (1.13)$$

We will also assume that each single potential  $V_i$  is  $C^4$  in a neighbourhood of  $\xi_0$ .

Finally, we say that a family  $\mathbf{v} = \mathbf{v}_\mu$  of solutions of (1.6), indexed on  $\mu$ , *concentrates* at  $\xi_0 \in \mathbb{R}^N$  if

$$\mathbf{v}_\mu(x) = \frac{1}{\epsilon_\mu} \mathbf{U} \left( \frac{x - \xi_\mu}{\epsilon_\mu} \right) + \phi_\mu(x),$$

where, as  $\mu \rightarrow \mu^* \in [0, +\infty]$ , for  $\epsilon_\mu \rightarrow 0$ ,  $\xi_\mu \rightarrow \xi_0$ , and the remainder  $\phi_\mu$  is a higher order term, in some suitable sense.

Finally, we can state our main result.

**Theorem 1.1.** (1) *There exists  $\kappa = \kappa(N) > 0$  such that*

- (i) *in the subcritical regime, i.e.  $N = 1$ , for any  $\mu > \kappa$  there exist a solution  $(\lambda_\mu, \mathbf{v}_\mu)$  to (1.6) with  $\mathbf{v}_\mu$  concentrating at  $\xi_0$*
  - (ii) *in the supercritical regime, i.e.  $N = 3$ , for any  $0 < \mu < \kappa$  there exist a solution  $(\lambda_\mu, \mathbf{v}_\mu)$  to (1.6) with  $\mathbf{v}_\mu$  concentrating at  $\xi_0$*
- and in both cases*

$$\lambda_\mu \sim \left( \frac{\mu_0}{\mu} \right)^{\frac{3}{N-2}} \rightarrow +\infty \text{ as } \mu \rightarrow \infty \text{ or } \mu \rightarrow 0, \text{ respectively.}$$

- (2) *In the critical regime, i.e.  $N = 2$ , we suppose that  $V_i(\xi_0) = c$  and  $\nabla V_i(\xi_0) = 0$  for every  $i = 1, \dots, k$ . Moreover we also assume  $\Delta\Gamma(\xi_0) \neq 0$ .*

*Then there exists  $\delta > 0$  such that for any  $\mu_0 - \delta < \mu < \mu_0$  (if  $\Delta\Gamma(\xi_0) > 0$ ) or*

$\mu_0 < \mu < \mu_0 + \delta$  (if  $\Delta\Gamma(\xi_0) < 0$ ) there exists a solution  $(\lambda_\mu, \mathbf{v}_\mu)$  to (1.6) with  $\mathbf{v}_\mu$  concentrating at  $\xi_0$  and

$$\lambda_\mu \sim \left( \frac{\Delta\Gamma(\xi_0)}{\mu_0 - \mu} \right)^{\frac{1}{2}} \rightarrow +\infty \text{ as } \mu \rightarrow \mu_0.$$

Let us make some comments.

**Remark 1.2.** We build the solution using a Ljapunov-Schmidt procedure taking the mass  $\mu$  as a parameter in the same spirit of [20]. The profile of the solution at the main order looks like the synchronized solution to the limit system (1.7). However, in contrast to the previous work, here the solution must also be corrected at second order by means of the solution of the linear problem (2.4) where the values of the potentials at  $\xi_0$  appear. We observe that in the case of the single equation once we fix the non-degenerate critical point  $\xi_0$  of  $V$  we can assume (without loss of generality) that  $V(\xi_0) = 0$  up to replacing  $\lambda$  with the new parameter  $\lambda - V(\xi_0)$ . This no longer holds in the case of the system, because if the single parameter  $\lambda$  is replaced by the parameters  $\lambda_i = \lambda - V_i(\xi_0)$  they are different, unless all the potentials have the same value at the point  $\xi_0$ .

**Remark 1.3.** The main term of the solution found in Theorem 2.1 in the critical regime (i.e.  $N = 2$ ) is not good enough to detect its mass. We need to improve the ansatz up to the second order and to keep the size of the error term small enough. That is why we need to assume that all the potentials have the same expansion (up to the first order) close to the point  $\xi_0$ , i.e. all the functions  $V_i$ 's take the same value at the point, which also turns out to be a common critical point. It would be extremely interesting to determine whether this extra assumption is merely a tool to simplify the computations or if it has a deeper significance.

**Remark 1.4.** Byeon in [8] considers a singularly perturbed system with only two equations similar to system (2.2). He proves the existence of solutions concentrating at the same point which is a common non-degenerate critical point of both the potentials. In Theorem 2.1 we show that the concentration phenomenon is actually governed by the critical points of the *global* potential  $\Gamma$  rather than by the critical points of the individual potentials. Moreover, our approach allows to consider systems with more than two components.

**Remark 1.5.** The existence of solutions concentrating at the point  $\xi_0$  strongly depends on the nature of the critical point. In particular in the critical regime there exists a solution with a mass smaller than  $\mu_0$  if  $\xi_0$  is a minimum point (since  $\Delta\Gamma(\xi_0) > 0$ ) or with a mass larger than  $\mu_0$  if  $\xi_0$  is a maximum point (since  $\Delta\Gamma(\xi_0) < 0$ ). It would be interesting to prove that such conditions are also necessary. More precisely, it could be challenging to prove that if  $\xi_0$  is a minimum or a maximum point then there are no solutions blowing-up at  $\xi_0$  with masses approaching  $\mu_0$  from above or below, respectively.

**Remark 1.6.** In [20] we conjectured that the constant  $\alpha_N$  defined in (4.1) is positive for any  $N$ . This is true in the 1-dimensional case as proved in [20, Remark 3.5]. In Section 4 we provide numerical evidence that this is still true for dimensions  $N \in \{2, \dots, 8\}$ . The validity of the conjecture is in our opinion an interesting open problem.

The proof paper is organized as follows. In Section 2, we find a solution to the perturbed Schrödinger equation (2.2) via the classical Ljapunov-Schmidt reduction. For

sake of completeness we repeat the main steps of the proofs taking into account that a second order expansion of the main term of the solution we are looking for is needed. In Section 3, we select the solutions with the prescribed norm. In Section 4 we discuss the numerical approach used to study the sign of  $\alpha_N$  defined in (4.1). Finally, in Appendix A we study the existence of non-degenerate synchronized solutions to System (1.7).

*Notation:* In what follows we agree that notation  $f = \mathcal{O}(g)$  or  $f \lesssim g$  stand for  $|f| \leq C|g|$  for some  $C > 0$  uniformly with respect all the variables involved, unless specified.

## 2. EXISTENCE OF SOLUTIONS TO A SINGULARLY PERTURBED SYSTEM

Set

$$\epsilon := \lambda^{-\frac{1}{2}} \quad \text{and} \quad u_i := \epsilon v_i, \quad i = 1, \dots, k. \quad (2.1)$$

Problem (1.6) turns out to be equivalent to

$$\begin{cases} -\epsilon^2 \Delta u_i + u_i + \epsilon^2 V_i(x) u_i = \sum_{j=1}^k \beta_{ij} u_i u_j^2 & \text{in } \mathbb{R}^N, \quad i = 1, \dots, k, \\ \epsilon^{-2} \int_{\mathbb{R}^N} u_1^2 + \dots + u_k^2 dx = \mu. \end{cases} \quad (2.2)$$

$$\epsilon^{-2} \int_{\mathbb{R}^N} u_1^2 + \dots + u_k^2 dx = \mu. \quad (2.3)$$

It is clear that  $(\lambda(\mu), \mathbf{v}(\mu))$ ,  $\mathbf{v}(\mu) := (v_1(\mu), \dots, v_k(\mu))$  solves (1.6) if and only if  $(\epsilon(\mu), \mathbf{u}(\mu))$ ,  $\mathbf{u}(\mu) := (u_1(\mu), \dots, u_k(\mu))$  solves (2.2)–(2.3). As a consequence, the first step is building a solution  $\mathbf{u} = \mathbf{u}(\epsilon)$  to the singularly perturbed Schrödinger system (2.2) which concentrates at a given point  $\xi_0$  as  $\epsilon \rightarrow 0$ .

In this section, we mainly prove the following result.

**Theorem 2.1.** *There exists  $\epsilon_0 > 0$  such that for any  $\epsilon \in (0, \epsilon_0)$  there exists a unique solution  $\mathbf{u}_\epsilon = (u_{1,\epsilon}, \dots, u_{k,\epsilon})$  to (2.2) such that*

$$u_{i,\epsilon}(x) = U_i \left( \frac{x - \xi_\epsilon}{\epsilon} \right) - \epsilon^2 Z_i \left( \frac{x - \xi_\epsilon}{\epsilon} \right) + \psi_{i,\epsilon}(x), \quad i = 1, \dots, k,$$

for some  $\xi_\epsilon \rightarrow \xi_0$  as  $\epsilon \rightarrow 0$ . The functions  $Z_1, \dots, Z_k \in H^1(\mathbb{R}^N)$  are the radial solutions to the linear system

$$-\Delta Z_i + Z_i - \sum_{j=1}^k \beta_{ij} (U_j^2 Z_i + 2U_i U_j Z_j) = V_i(\xi_0) U_i \text{ in } \mathbb{R}^N, \quad i = 1, \dots, k \quad (2.4)$$

and the remainder terms  $\psi_{i,\epsilon}$  satisfy

$$\left( \int_{\mathbb{R}^N} \epsilon^2 |\nabla \psi_{i,\epsilon}|^2 + \psi_{i,\epsilon}^2 dx \right)^{1/2} = \mathcal{O} \left( \epsilon^{\frac{N}{2}+3} \right).$$

Moreover, the map  $(0, \epsilon_0) \rightarrow (H^1(\mathbb{R}^N))^k : \epsilon \mapsto \mathbf{u}_\epsilon$  is continuous.

**2.1. Preliminaries.** Note that the functions  $u_i$  solve system (2.2) if and only if the scaled functions  $u_i(\epsilon \cdot + \xi)$ , which will still be denoted by  $u_i$  solve the following system

$$-\Delta u_i + u_i + \epsilon^2 V_i(\epsilon x + \xi) u_i = \sum_{j=1}^k \beta_{ij} u_i u_j^2 \text{ in } \mathbb{R}^N, \quad i = 1, \dots, k. \quad (2.5)$$

Here, we choose the concentration point

$$\xi := \epsilon \tau + \xi_0, \quad \tau \in \mathbb{R}^N.$$

Let  $H^1(\mathbb{R}^N)$  be equipped with the standard scalar product

$$\langle \phi, \psi \rangle := \int_{\mathbb{R}^N} \nabla \phi \nabla \psi + \int_{\mathbb{R}^N} \phi \psi,$$

which induces the standard norm denoted by  $\|\cdot\|$ . We also denote by  $\mathbf{i}^* : L^{4/3}(\mathbb{R}^N) \rightarrow H^1(\mathbb{R}^N)$  the adjoint operator of the embedding  $\mathbf{i} : H^1(\mathbb{R}^N) \hookrightarrow L^{4/3}(\mathbb{R}^N)$ , i.e.

$$\mathbf{i}^* f = u \iff -\Delta u + u = f \text{ in } \mathbb{R}^N.$$

We also observe that there exists  $C > 0$  such that

$$\|u\| \leq C \|f\|_{L^{4/3}(\mathbb{R}^N)}, \quad i = 1, \dots, k. \quad (2.6)$$

We set  $H := H^1(\mathbb{R}^N) \times \dots \times H^1(\mathbb{R}^N)$ , equipped with the scalar product and the induced norm (respectively)

$$\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{i=1}^k \langle u_i, v_i \rangle \quad \text{and} \quad \|\mathbf{u}\|^2 = \sum_{i=1}^k \|u_i\|^2, \quad (2.7)$$

where  $\mathbf{u} = (u_1, \dots, u_k)$ ,  $\mathbf{v} = (v_1, \dots, v_k) \in H$ . Finally, we can rewrite system (2.5) as

$$u_i = \mathbf{i}^* \left\{ \sum_{j=1}^k \beta_{ij} u_i u_j^2 - \epsilon^2 V_i(\epsilon \cdot + \xi) u_i \right\} \text{ in } \mathbb{R}^N, \quad i = 1, \dots, k. \quad (2.8)$$

We look for a solution to (2.8) as

$$\mathbf{u} = \underbrace{\mathbf{U} - \epsilon^2 \mathbf{Z}}_{=: \mathbf{W}} + \phi, \quad (2.9)$$

where  $\mathbf{U} := (U_1, \dots, U_k)$  is the non-degenerate synchronized solution of the limit system (1.7), the correction term  $\mathbf{Z} := (Z_1, \dots, Z_k)$  solves the linear system (2.4) and the remainder term  $\phi := (\phi_1, \dots, \phi_k) \in K^\perp$  where (see (1.10))

$$K := \text{span} \left\{ \Phi_i := \left( \frac{\partial U_1}{\partial x_i}, \dots, \frac{\partial U_k}{\partial x_i} \right) : i = 1, \dots, N \right\} \quad (2.10)$$

and

$$K^\perp = \{ \phi := (\phi_1, \dots, \phi_k) \in H : \langle \phi, \Phi_i \rangle = 0, \quad i = 1, \dots, N \}. \quad (2.11)$$

Note that  $\mathbf{W} \in K^\perp$  because  $U$  and the  $Z_i$  are radial. We rewrite the system (2.8) as follows

$$\mathcal{L}_{\epsilon, \tau}(\phi) - \mathcal{E}_{\epsilon, \tau} - \mathcal{N}_{\epsilon, \tau}(\phi) = 0. \quad (2.12)$$

Here the linear operator  $\mathcal{L}_{\epsilon,\tau}$  is defined by

$$\mathcal{L}_i(\phi_1, \dots, \phi_k) := \phi_i - \mathbf{i}^* \left\{ \sum_{j=1}^k \beta_{ij} (W_j^2 \phi_i + 2W_i W_j \phi_j) - \epsilon^2 V_i(\epsilon \cdot + \xi) \phi_i \right\}, \quad (2.13)$$

the nonlinear term  $\mathcal{N}_{\epsilon,\tau}(\phi)$  is defined by

$$\mathcal{N}_i(\phi_1, \dots, \phi_k) = \mathbf{i}^* \left\{ \sum_{j=1}^k \beta_{ij} (W_i \phi_j^2 + \phi_i \phi_j^2 + 2W_j \phi_i \phi_j) \right\} \quad (2.14)$$

and the error term  $\mathcal{E}_{\epsilon,\tau}$  is defined by

$$\mathcal{E}_i := \mathbf{i}^* \left\{ \sum_{j=1}^k \beta_{ij} W_i W_j^2 - \epsilon^2 V_i(\epsilon \cdot + \xi) W_i \right\} - W_i. \quad (2.15)$$

Then, problem (2.8) turns out to be equivalent to the system

$$\Pi^\perp \{ \mathcal{L}_{\epsilon,\tau}(\phi) - \mathcal{E}_{\epsilon,\tau} - \mathcal{N}_{\epsilon,\tau}(\phi) \} = 0 \quad (2.16)$$

and

$$\Pi \{ \mathcal{L}_{\epsilon,\tau}(\phi) - \mathcal{E}_{\epsilon,\tau} - \mathcal{N}_{\epsilon,\tau}(\phi) \} = 0, \quad (2.17)$$

where  $\Pi : H \rightarrow K$  and  $\Pi^\perp : H \rightarrow K^\perp$  are the orthogonal projections.

## 2.2. Solving (2.16).

**Proposition 2.2.** *For any compact set  $T \subset \mathbb{R}^N$  there exists  $\epsilon_0 > 0$  and  $C > 0$  such that for any  $\epsilon \in [0, \epsilon_0]$  and for any  $\tau \in T$  there exists a unique  $\phi = \phi_{\epsilon,\tau} \in K^\perp$  in a neighborhood of 0 which solves equation (2.16) and*

$$\|\phi_{\epsilon,\tau}\| \leq C\epsilon^3. \quad (2.18)$$

Moreover, the map  $\epsilon \mapsto \phi_{\epsilon,\tau}$  is continuous and the map  $\tau \mapsto \phi_{\epsilon,\tau}$  is  $C^1$  and satisfies

$$\|\partial_\tau \phi_{\epsilon,\tau}\| \leq C\epsilon^3. \quad (2.19)$$

*Proof.* Let us sketch the main steps of the proof.

(i) First of all, we prove that the linear operator  $\mathcal{L}_{\epsilon,\tau}$  is uniformly invertible in  $K^\perp$ , namely there exists  $\epsilon_0 > 0$  and  $C > 0$  such that

$$\|\Pi^\perp \mathcal{L}_{\epsilon,\tau}(\varphi)\| \geq C\|\varphi\| \quad \text{for any } \epsilon \in [0, \epsilon_0], \tau \in T \text{ and } \varphi \in K^\perp.$$

Observe that

$$\Pi^\perp \mathcal{L}_{\epsilon,\tau}(\varphi) = \Pi^\perp \mathcal{L}_0(\varphi) + \epsilon^2 \tilde{\mathcal{L}}_{\epsilon,\tau}(\varphi),$$

where the linear operator  $\tilde{\mathcal{L}}_{\epsilon,\tau}$  is uniformly bounded and the linear operator  $\mathcal{L}_0$  defined by

$$(\mathcal{L}_0)_i(\varphi_1, \dots, \varphi_k) = \varphi_i - \mathbf{i}^* \left\{ \sum_{j=1}^k \beta_{ij} (U_j^2 \varphi_i + 2U_i U_j \varphi_j) \right\}. \quad (2.20)$$

The non-degeneracy assumption (1.9) means that  $K = \ker \mathcal{L}_0$ . Given that  $\mathcal{L}_0$  is self-adjoint,  $\Pi^\perp \mathcal{L}_0 = \mathcal{L}_0$  and, because it is a compact perturbation of the identity, it is invertible.



(ii) Next, we compute the size of the error  $\mathcal{E}_{\epsilon,\tau}$  in terms of  $\epsilon$ . We observe that by (1.7)

$$U_i = \mathbf{i}^* \left\{ \sum_{j=1}^k \beta_{ij} U_i U_j^2 \right\}$$

and by (2.4)

$$Z_i = \mathbf{i}^* \left\{ \sum_{j=1}^k \beta_{ij} (U_j^2 Z_i + 2U_i U_j Z_j) + V_i(\xi_0) U_i \right\}.$$

Moreover, by the mean value theorem (recalling that  $|\nabla V_i| \in L^\infty$ ),

$$V_i(\epsilon x + \epsilon \tau + \xi_0) = V_i(\xi_0) + \mathcal{O}(\epsilon(1 + |x|)).$$

Combining the above facts we have

$$\begin{aligned} \mathcal{E}_i = \mathbf{i}^* \left\{ \epsilon^4 \sum_{j=1}^k \beta_{ij} (U_i Z_j^2 + 2U_j Z_i Z_j) - \epsilon^6 \sum_{j=1}^k \beta_{ij} Z_i Z_j^2 \right\} \\ + \mathbf{i}^* \left\{ \epsilon^4 V_i(\epsilon \cdot + \xi) Z_i - \epsilon^2 (V_i(\epsilon \cdot + \xi) - V_i(\xi_0)) U_i \right\} \end{aligned} \quad (2.21)$$

and

$$\|\mathcal{E}_i\| \leq C\epsilon^3, \quad i = 1, \dots, k.$$

(iii) The existence part follows by a standard contraction mapping argument. The contraction relies on the inequality  $\|\mathcal{N}_{\epsilon,\tau}(\phi_1) - \mathcal{N}_{\epsilon,\tau}(\phi_2)\| \leq c\|\phi_1 - \phi_2\|$  where  $c = \mathcal{O}(\|\phi_1\| + \|\phi_2\|)$  which can be deduced combining the mean value theorem and (2.14). The fact that a small ball around  $\phi = 0$  is mapped into itself comes from point (ii) and the following estimate  $\|\mathcal{N}_{\epsilon,\tau}(\phi)\| \leq C\|\phi\|^2$  (which follows by (2.14)) valid in a neighbourhood of  $\phi = 0$ . Point (ii) and this last inequality also imply the bounds on  $\|\phi_{\epsilon,\tau}\|$ . Finally, the continuity of the fix point  $\phi_{\epsilon,\tau}$  follows from the same continuity of the contracting map (we choose  $\epsilon_0$  small enough so that, for all  $\epsilon \in [0, \epsilon_0]$ ,  $\epsilon T$  lies in the neighborhood of  $\xi_0$  where all  $V_i$ 's are of class  $C^4$ ). See e.g. [17, Proposition 3.5] for more details.

(iv) We show that the map  $\tau \mapsto \phi_{\epsilon,\tau}$  is a  $C^1$ . Our arguments are inspired by those developed in [17, Proposition 3.5] (see also [9, Proposition 5.2]). We apply the Implicit Function Theorem to the  $C^1$ -function  $\mathbf{G} : \mathbb{R}^N \times K^\perp \rightarrow K^\perp$  defined by

$$\mathbf{G}(\tau, \varphi) := \varphi - \Pi^\perp \{ \mathbf{F}(\mathbf{W} + \varphi) - \mathbf{W} \}$$

where  $\mathbf{W}$  is defined in (2.9) and the function  $\mathbf{F} : H \rightarrow H$  is defined as (see (2.8))

$$F_i(\mathbf{u}) := \mathbf{i}^* \left\{ \sum_{j=1}^k \beta_{ij} u_i u_j^2 - \epsilon^2 V_i(\epsilon \cdot + \xi) u_i \right\}, \quad i = 1, \dots, k.$$

Now, it is clear that  $\mathbf{G}(\tau, \phi_{\epsilon,\tau}) = 0$ . Moreover the linearized operator  $D_\varphi \mathbf{G}(\tau, \phi_{\epsilon,\tau}) : K^\perp \rightarrow K^\perp$  is defined by

$$D_\varphi \mathbf{G}(\tau, \phi_{\epsilon,\tau})[\varphi] = \varphi - \Pi^\perp \{ D_{\mathbf{u}} \mathbf{F}(\mathbf{W} + \phi_{\epsilon,\tau})[\varphi] \},$$

where the linear operator  $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}) : H \rightarrow H$  is defined by

$$(D_{\mathbf{u}}\mathbf{F}(\mathbf{u})[\mathbf{v}])_i = \mathbf{i}^* \left\{ \sum_{j=1}^k \beta_{ij} (u_j^2 v_i + 2u_i u_j v_j) - \epsilon^2 V_i(\epsilon \cdot + \xi) v_i \right\}, \quad i = 1, \dots, k.$$

We claim that the operator  $D_{\varphi}\mathbf{G}(\tau, \phi_{\epsilon, \tau})$  is invertible. Indeed, using (2.6), one shows that  $D_{\mathbf{u}}\mathbf{F}(\mathbf{u}) \rightarrow D_{\mathbf{u}}\mathbf{F}(\mathbf{U})$  in  $\mathcal{L}(H; H)$  as  $\mathbf{u} \rightarrow \mathbf{U}$ . Thus, thanks to (2.18),  $D_{\varphi}\mathbf{G}(\tau, \phi_{\epsilon, \tau}) \rightarrow D_{\varphi}\mathbf{G}(\tau, \mathbf{U}) = \Pi^\perp \mathcal{L}_0 = \mathcal{L}_0$  in  $\mathcal{L}(K^\perp; K^\perp)$ , uniformly w.r.t.  $\tau \in T$ , as  $\epsilon \rightarrow 0$ , where  $\mathcal{L}_0$  is defined by (2.20). Taking if necessary  $\epsilon_0$  smaller, the claim is proved.

(v) Finally, we prove the estimate (2.19). We know that

$$\mathbf{G}(\tau, \phi_{\epsilon, \tau}) = 0.$$

Then, differentiating at  $\tau_0$  in the direction  $\tau$  yields

$$D_\tau \mathbf{G}(\tau_0, \phi_{\epsilon, \tau_0})[\tau] + D_{\mathbf{u}}\mathbf{G}(\tau_0, \phi_{\epsilon, \tau_0})[D_\tau \phi_{\epsilon, \tau_0}[\tau]] = 0$$

and so we get

$$\|D_\tau \phi_{\epsilon, \tau_0}[\tau]\| \leq C \|D_\tau \mathbf{G}(\tau_0, \phi_{\epsilon, \tau_0})[\tau]\| \leq C \epsilon^3 |\tau|,$$

because (setting  $\phi_{\epsilon, \tau_0} = (\phi_1, \dots, \phi_k)$ )

$$D_\tau \mathbf{G}(\tau_0, \phi_{\epsilon, \tau_0})[\tau] = \epsilon^3 \Pi^\perp (\mathbf{i}^* \{ \nabla V_1(\epsilon \cdot + \epsilon \tau_0 + \xi_0) \tau (W_1 + \phi_1) \}, \dots, \mathbf{i}^* \{ \nabla V_k(\epsilon \cdot + \epsilon \tau_0 + \xi_0) \tau (W_k + \phi_k) \})$$

and

$$\|D_\tau \mathbf{G}(\tau_0, \phi_{\epsilon, \tau_0})[\tau]\| \leq C \epsilon^3 |\tau| \|\mathbf{W} + \phi_{\epsilon, \tau_0}\| \leq C \epsilon^3 |\tau|. \quad \square$$

### 2.3. Solving (2.17).

**Proposition 2.3.** *There exists  $\epsilon_0 > 0$  such that for any  $\epsilon \in [0, \epsilon_0]$  there exists a unique  $\tau_\epsilon \in \mathbb{R}^N$  such that equation (2.17) is satisfied with  $\phi = \phi_{\epsilon, \tau}$ , where  $\phi_{\epsilon, \tau}$  is given by Proposition 2.2. Moreover, the map  $\epsilon \mapsto \tau_\epsilon$  is continuous and goes to 0 as  $\epsilon \rightarrow 0$ .*

*Proof.* As  $\phi_{\epsilon, \tau}$  solves (2.16), there exist real numbers  $c_{\epsilon, \tau}^i$ ,  $i = 1, \dots, N$  such that (see (1.10))

$$\mathcal{L}_{\epsilon, \tau}(\phi_{\epsilon, \tau}) - \mathcal{N}_{\epsilon, \tau}(\phi_{\epsilon, \tau}) - \mathcal{E}_{\epsilon, \tau} = \sum_{i=1}^N c_{\epsilon, \tau}^i \Phi_i. \quad (2.22)$$

We aim to find a unique point  $\tau = \tau_\epsilon$  such that all the  $c_{\epsilon, \tau}^i$ 's are zero. We multiply (2.22) by  $\Phi_j$ . We get

$$\langle \mathcal{L}_{\epsilon, \tau}(\phi_{\epsilon, \tau}) - \mathcal{N}_{\epsilon, \tau}(\phi_{\epsilon, \tau}) - \mathcal{E}_{\epsilon, \tau}, \Phi_j \rangle = \sum_{i=1}^N c_{\epsilon, \tau}^i \langle \Phi_i, \Phi_j \rangle = c_{\epsilon, \tau}^j A, \quad (2.23)$$

because, by (1.10) and the oddness of  $\partial_i U$  along the axis  $x_i$ ,

$$\begin{aligned} \langle \Phi_i, \Phi_j \rangle &= \sum_{\ell=1}^k \langle \partial_i U_\ell, \partial_j U_\ell \rangle = \sum_{\ell=1}^k \sigma_\ell^2 \langle \partial_i U, \partial_j U \rangle = \sum_{\ell=1}^k \sigma_\ell^2 \int_{\mathbb{R}^N} g'(U) \partial_i U \partial_j U \\ &= A \delta_{ij}, \quad \text{where } A := \sum_{\ell=1}^k \sigma_\ell^2 \int_{\mathbb{R}^N} g'(U) (\partial_1 U)^2 \neq 0 \text{ and } g(t) = t^3. \end{aligned}$$

The claim will follow at once if we prove that

$$\langle \mathcal{L}_{\epsilon, \tau}(\phi_{\epsilon, \tau}) - \mathcal{N}_{\epsilon, \tau}(\phi_{\epsilon, \tau}) - \mathcal{E}_{\epsilon, \tau}, \Phi_j \rangle = -\frac{1}{2} \epsilon^4 \left( \tau_j \frac{\partial^2 \Gamma}{\partial x_j^2}(\xi_0) + \mathfrak{o}(1) \right), \quad j = 1, \dots, N, \quad (2.24)$$

where the  $\mathfrak{o}$ 's are  $C^1$ -uniform with respect to  $\tau \in T$  as  $\epsilon \rightarrow 0$ , where  $T$  is a given compact set. Indeed, using (2.24), (2.23) may be rewritten as

$$-\frac{1}{2} \epsilon^4 \left( \tau_j \frac{\partial^2 \Gamma}{\partial x_j^2}(\xi_0) + \mathfrak{o}(1) \right) = A c_{\epsilon, \tau}^j \quad \text{for any } j = 1, \dots, N, \quad (2.25)$$

where  $A \neq 0$  is a constant and the  $\mathfrak{o}(1)$  is  $C^1$ -uniform in  $\tau \in T$  as  $\epsilon \rightarrow 0$ . Since all the  $\partial^2 \Gamma / \partial x_j^2(\xi_0)$ 's are different from zero (because  $\xi_0$  is a non-degenerate critical point of  $\Gamma$ ), a contraction mapping argument shows that

$$\tau_j = - \left( \frac{\partial^2 \Gamma}{\partial x_j^2}(\xi_0) \right)^{-1} \mathfrak{o}(1), \quad j = 1, \dots, N,$$

has a unique solution  $\tau_\epsilon = (\tau_1, \dots, \tau_N)$  for  $\epsilon$  small enough. Therefore the left hand side in (2.25) vanishes and  $c_{\epsilon, \tau_\epsilon}^j = 0$  for all  $j = 1, \dots, N$ . The map  $\epsilon \mapsto \tau_\epsilon$  is continuous because the functions in  $\mathfrak{o}(1)$  are continuous with respect to  $\epsilon$ . Moreover it is clear that  $\tau_j \rightarrow 0$ ,  $j = 1, \dots, N$ , as  $\epsilon \rightarrow 0$  and  $(\epsilon, \tau) \in \mathcal{T}$ .

Let us prove (2.24). First of all, we estimate the leading term in (2.23). By (1.10), (2.7), (2.15) (taking into account that  $U$  and  $Z_i$  are radial functions and the derivatives  $\partial U / \partial x_j$  are odd functions), and the Taylor expansions of  $V_\ell$  and of

$$\frac{\partial V_\ell}{\partial x_j}(\epsilon x + \epsilon \tau + \xi_0) = \frac{\partial V_\ell}{\partial x_j}(\xi_0) + \epsilon \sum_{i=1}^N \frac{\partial^2 V_\ell}{\partial x_j \partial x_i}(\xi_0) (x_i + \tau_i) + \mathcal{O}(\epsilon^2(1 + |x|^2)),$$

we get:

$$\begin{aligned} & -\langle \mathcal{E}_{\epsilon, \tau}, \Phi_j \rangle \\ &= -\sum_{\ell=1}^k \sigma_\ell \left\langle \mathcal{E}_\ell, \frac{\partial U}{\partial x_j} \right\rangle \\ &= \sum_{\ell=1}^k \sigma_\ell \int_{\mathbb{R}^N} V_\ell(\epsilon x + \epsilon \tau + \xi_0) (\epsilon^2 \sigma_\ell U - \epsilon^4 Z_\ell) \frac{\partial U}{\partial x_j} dx \end{aligned}$$

$$\begin{aligned}
 &= \epsilon^2 \sum_{\ell=1}^k \sigma_\ell^2 \int_{\mathbb{R}^N} V_\ell(\epsilon x + \epsilon \tau + \xi_0) U \frac{\partial U}{\partial x_j} dx + \mathfrak{o}(\epsilon^4) \\
 &= -\frac{1}{2} \epsilon^3 \sum_{\ell=1}^k \sigma_\ell^2 \int_{\mathbb{R}^N} \frac{\partial V_\ell}{\partial x_j}(\epsilon x + \epsilon \tau + \xi_0) U^2(x) dx + \mathfrak{o}(\epsilon^4) \\
 &= -\frac{1}{2} \epsilon^3 \sum_{\ell=1}^k \sigma_\ell^2 \int_{\mathbb{R}^N} \frac{\partial V_\ell}{\partial x_j}(\xi_0) U^2(x) dx - \frac{1}{2} \epsilon^4 \sum_{\ell=1}^k \sigma_\ell^2 \sum_{i=1}^N \frac{\partial^2 V_\ell}{\partial x_j \partial x_i}(\xi_0) \tau_i \int_{\mathbb{R}^N} U^2 dx + \mathfrak{o}(\epsilon^4) \\
 &\quad \underbrace{\hspace{10em}}_{=\frac{\partial \Gamma}{\partial x_j}(\xi_0)=0 \text{ (see (1.12))}} \\
 &= -\frac{1}{2} \epsilon^4 \sum_{i=1}^N \tau_i \underbrace{\left( \gamma \sum_{\ell=1}^k \sigma_\ell^2 \frac{\partial^2 V_\ell}{\partial x_j \partial x_i}(\xi_0) \right)}_{=\frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\xi_0)} + \mathfrak{o}(\epsilon^4) = -\frac{1}{2} \epsilon^4 \tau_j \frac{\partial^2 \Gamma}{\partial x_j^2}(\xi_0) + \mathfrak{o}(\epsilon^4). \tag{2.26}
 \end{aligned}$$

Note that since  $V$  is  $C^4$  in a neighbourhood of  $\xi_0$ , all  $\mathfrak{o}(\epsilon^4)$  hold in the  $C^1$ -topology with respect to  $\tau \in T$ .

Finally, it remains to prove that

$$\langle \mathcal{L}_{\epsilon, \tau}(\phi_{\epsilon, \tau}), \Phi_j \rangle = \mathfrak{o}(\epsilon^4) \quad \text{and} \quad \langle \mathcal{N}_{\epsilon, \tau}(\phi_{\epsilon, \tau}), \Phi_j \rangle = \mathfrak{o}(\epsilon^4), \tag{2.27}$$

where all  $\mathfrak{o}(\epsilon^4)$  are in the  $C^1(T)$ -topology.

Writing as usual  $\phi_{\epsilon, \tau} = (\phi_1, \dots, \phi_k)$ , recalling the definition (2.13), and taking into account that  $\Phi_j$  solves (1.9) yields

$$\begin{aligned}
 \langle \mathcal{L}_{\epsilon, \tau}(\phi_{\epsilon, \tau}), \Phi_j \rangle &= \sum_{\ell=1}^k \sigma_\ell \int_{\mathbb{R}^N} \sum_{\kappa=1}^k \beta_{\ell\kappa} (U_\kappa^2 \phi_\ell + 2U_\ell U_\kappa \phi_\kappa) \partial_j U \\
 &\quad - \sum_{\ell=1}^k \sigma_\ell \int_{\mathbb{R}^N} \sum_{\kappa=1}^k \beta_{\ell\kappa} (W_\kappa^2 \phi_\ell + 2W_\ell W_\kappa \phi_\kappa) \partial_j U \\
 &\quad + \epsilon^2 \sum_{\ell=1}^k \sigma_\ell \int_{\mathbb{R}^N} V_\ell(\epsilon x + \epsilon \tau + \xi_0) \phi_\ell \partial_j U \\
 &= \sum_{\ell=1}^k \sigma_\ell \int_{\mathbb{R}^N} \sum_{\kappa=1}^k \beta_{\ell\kappa} (2\epsilon^2 U_\kappa Z_\kappa - \epsilon^4 Z_\kappa^2) \phi_\ell \partial_j U \\
 &\quad + \sum_{\ell=1}^k \sigma_\ell \int_{\mathbb{R}^N} \sum_{\kappa=1}^k 2\beta_{\ell\kappa} (\epsilon^2 (U_\ell Z_\kappa + U_\kappa Z_\ell) - \epsilon^4 Z_\ell Z_\kappa) \phi_\kappa \partial_j U \\
 &\quad + \epsilon^2 \sum_{\ell=1}^k \sigma_\ell \int_{\mathbb{R}^N} V_\ell(\epsilon x + \epsilon \tau + \xi_0) \phi_\ell \partial_j U
 \end{aligned}$$

$= \mathfrak{o}(\epsilon^4)$  in  $C^0(T)$  because of (2.18).

The derivative with respect to  $\tau_j$  of the previous quantity enjoys the following estimate:

$$\begin{aligned} \partial_{\tau_i} \langle \mathcal{L}_{\epsilon, \tau}(\phi_{\epsilon, \tau}), \Phi_j \rangle &= \sum_{\ell=1}^k \sigma_\ell \int_{\mathbb{R}^N} \sum_{\kappa=1}^k \beta_{\ell\kappa} (2\epsilon^2 U_\kappa Z_\kappa - \epsilon^4 Z_\kappa^2) \partial_{\tau_i} \phi_\ell \partial_j U \\ &\quad + \sum_{\ell=1}^k \sigma_\ell \int_{\mathbb{R}^N} \sum_{\kappa=1}^k 2\beta_{\ell\kappa} (\epsilon^2 (U_\ell Z_\kappa + U_\kappa Z_\ell) - \epsilon^4 Z_\ell Z_\kappa) \partial_{\tau_i} \phi_\kappa \partial_j U \\ &\quad + \epsilon^2 \sum_{\ell=1}^k \sigma_\ell \int_{\mathbb{R}^N} V_\ell (\epsilon x + \epsilon \tau + \xi_0) \partial_{\tau_i} \phi_\ell \partial_j U \\ &\quad + \epsilon^3 \sum_{\ell=1}^k \sigma_\ell \int_{\mathbb{R}^N} \partial_i V_\ell (\epsilon x + \epsilon \tau + \xi_0) \phi_\ell \partial_j U \\ &= \mathfrak{o}(\epsilon^4) \text{ in } C^0(T) \text{ because of (2.19) and (2.18).} \end{aligned}$$

By (2.14),

$$\begin{aligned} \langle \mathcal{N}_{\epsilon, \tau}(\phi_{\epsilon, \tau}), \Phi_j \rangle &= \sum_{\ell=1}^k \sigma_\ell \int_{\mathbb{R}^N} \sum_{\kappa=1}^k \beta_{\ell\kappa} (W_\ell \phi_\kappa^2 + \phi_\ell \phi_\kappa^2 + 2W_\kappa \phi_\ell \phi_\kappa) \partial_j U \\ &= \mathfrak{o}(\epsilon^4) \text{ in } C^0(T) \text{ because of (2.18)} \end{aligned}$$

and, differentiating with respect to  $\tau_i$ , we easily get

$$\partial_{\tau_i} \langle \mathcal{N}_{\epsilon, \tau}(\phi_{\epsilon, \tau}), \Phi_j \rangle = \mathfrak{o}(\epsilon^4) \text{ in } C^0(T) \text{ because of (2.18) and (2.19).} \quad \square$$

*Proof of Theorem 2.1, completed.* The existence of the solution  $\mathbf{u}_\epsilon$  to problem (2.2) follows combining all the previous arguments. It suffices to define

$$\mathbf{u}_\epsilon(x) := \mathbf{W}\left(\frac{\mathbf{x} - \xi_\epsilon}{\epsilon}\right) + (\psi_{1,\epsilon}, \dots, \psi_{\mathbf{k},\epsilon})(\mathbf{x}), \quad \text{where } (\psi_{1,\epsilon}, \dots, \psi_{\mathbf{k},\epsilon})(\mathbf{x}) := \phi_{\epsilon, \tau_\epsilon}\left(\frac{\mathbf{x} - \xi_\epsilon}{\epsilon}\right)$$

with  $\xi_\epsilon := \epsilon \tau_\epsilon + \xi_0$ , where  $\tau_\epsilon$  is given by Proposition 2.3. The continuity in the  $H^1$ -topology results from the fact that, for  $\epsilon > 0$ , the norm  $u \mapsto \epsilon^{-N/2} \left( \int \epsilon^2 |\nabla u|^2 + u^2 \right)^{1/2}$  is equivalent to the usual  $H^1$ -norm. Finally the estimate on  $\psi_{i,\epsilon}$  results from (2.18).  $\square$

### 3. THE MASS OF $u_\epsilon$

In this section we find the solutions to (2.8) by selecting the solutions to (2.2) for a suitable ranges of prescribed masses  $\mu$ 's.

#### 3.1. The non-critical case.

**Theorem 3.1.** (i) *If  $N = 1$ , then there exists  $R > 0$  such that for any  $\mu > R$  problem (2.8) has a solution  $(\epsilon_\mu, \mathbf{u}_\mu)$ , where  $\mathbf{u}_\mu$  is concentrating at the point  $\xi_0$  as  $\mu \rightarrow \infty$ .*

(ii) If  $N = 3$ , then there exists  $r > 0$  such that for any  $\mu < r$  problem (2.8) has a solution  $(\epsilon_\mu, \mathbf{u}_\mu)$ , where  $u_\mu$  is concentrating at the point  $\xi_0$  as  $\mu \rightarrow 0$ . In both cases  $\epsilon_\mu^{2-N} \mu \rightarrow \mu_0$  (see (1.11)), as  $\mu \rightarrow \infty$  or  $\mu \rightarrow 0$ , respectively.

*Proof.* By Theorem 2.1 there exists a solution

$$\mathbf{u}_\epsilon = \mathbf{U} \left( \frac{x - \xi_\epsilon}{\epsilon} \right) - \epsilon^2 \mathbf{Z} \left( \frac{x - \xi_\epsilon}{\epsilon} \right) + \phi_\epsilon \left( \frac{x - \xi_\epsilon}{\epsilon} \right),$$

where  $\|\phi_\epsilon\| = \mathcal{O}(\epsilon^3)$ . The mass of  $\mathbf{u}_\epsilon$  is (see (2.3))

$$\begin{aligned} \mu_\epsilon &:= \epsilon^{-2} \sum_{i=1}^k \int_{\mathbb{R}^N} \left( \sigma_i U \left( \frac{x - \xi_\epsilon}{\epsilon} \right) - \epsilon^2 Z_i \left( \frac{x - \xi_\epsilon}{\epsilon} \right) + \phi_{i,\epsilon} \left( \frac{x - \xi_\epsilon}{\epsilon} \right) \right)^2 dx \\ &= \epsilon^{-2+N} \left( \sum_{i=1}^k \sigma_i^2 \int_{\mathbb{R}^N} U^2(x) dx + \mathcal{O}(\epsilon^2) \right) \\ &= \epsilon^{-2+N} (\mu_0 + \mathfrak{o}(1)) \quad (\text{see (1.11)}). \end{aligned} \tag{3.1}$$

Since  $\mathbf{u}_\epsilon$  must have a prescribed mass as in (2.3), we have to find  $\epsilon = \epsilon(\mu)$  such that

$$\mu_\epsilon = \mu.$$

As the map  $\epsilon \mapsto \mathbf{u}_\epsilon$  is continuous, so is  $\epsilon \mapsto \mu_\epsilon$ . Moreover, (3.1) implies that  $\mu_\epsilon \rightarrow +\infty$  if  $N = 1$  (resp.  $\mu_\epsilon \rightarrow 0$  if  $N = 3$ ) as  $\epsilon \rightarrow 0$ . The Intermediate Value Theorem implies that a set of the form  $(R, +\infty)$  (resp.  $(0, r)$ ) is in the image of  $\epsilon \mapsto \mu_\epsilon$ .  $\square$

**3.2. The critical case.** Let  $N = 2$ . It is important to point out that a refinement of the ansatz is needed! Indeed, if we expand more carefully (3.1), we can determine the coefficient of the next order of  $\epsilon$ :

$$\mu_\epsilon = \mu_0 - 2\epsilon^2 \underbrace{\sum_{i=1}^k \sigma_i \int_{\mathbb{R}^2} U(x) Z_i(x) dx}_{=:\Xi(\xi_0)} + \mathfrak{o}(\epsilon^2) = \mu.$$

This does not allow to determine which values  $\mu_\epsilon$  takes because  $\Xi(\xi_0) = 0$  as proved in Remark 3.3.

**Remark 3.2.** Let  $(f_1, \dots, f_k) \in (L^2(\mathbb{R}^N))^k$  and  $\mathbf{R} = (R_1, \dots, R_k)$  a solution to the linear system

$$-\Delta R_i + R_i - \sum_{j=1}^k \beta_{ij} (U_j^2 R_i + 2U_i U_j R_j) = f_i. \tag{3.2}$$

Consider  $z := \sum_{i=1}^k \sigma_i R_i$ . Multiplying the  $i$ -th equation by  $\sigma_i$  and summing them up, we find that

$$-\Delta z + z - U^2 \sum_{i=1}^k \sigma_i \sum_{j=1}^k \beta_{ij} \sigma_j^2 R_i - 2U^2 \sum_{i=1}^k \sum_{j=1}^k \beta_{ij} \sigma_i^2 \sigma_j R_j = \sum_{i=1}^k \sigma_i f_i.$$

Making use of (1.8) and of the fact that  $\beta_{ij} = \beta_{ij}$ , this boils down to

$$-\Delta z + z - 3U^2 z = \sum_{i=1}^k \sigma_i f_i \quad \text{in } \mathbb{R}^N. \quad (3.3)$$

Therefore, the  $L^2$  scalar product of  $\mathbf{U}$  with  $\mathbf{R}$  is

$$\sum_{i=1}^k \int_{\mathbb{R}^N} U_i R_i = \int_{\mathbb{R}^N} \sum_{i=1}^k \sigma_i U R_i = \int_{\mathbb{R}^N} U(x) z(x) \, dx. \quad (3.4)$$

**Remark 3.3.** It holds true that  $\Xi(\xi_0) = 0$ . Indeed we apply the previous remark with  $f_i = V_i(\xi_0)U_i$  (see (2.4)). Therefore by (3.4)

$$\Xi(\xi_0) = \sum_{i=1}^k \sigma_i \int_{\mathbb{R}^2} U(x) Z_i(x) \, dx = \int_{\mathbb{R}^2} U(x) z(x) \, dx,$$

where  $z$  solves

$$-\Delta z + z - 3U^2 z = \left( \sum_{\ell=1}^k \sigma_\ell^2 V_\ell(\xi_0) \right) U \quad \text{in } \mathbb{R}^2.$$

A direct computation shows that

$$z(x) = -\frac{1}{2} \left( \sum_{\ell=1}^k \sigma_\ell^2 V_\ell(\xi_0) \right) (U(x) + \nabla U(x) \cdot x). \quad (3.5)$$

Indeed, letting  $u_\lambda(x) := \lambda U(\lambda x)$ , it is easy to check that  $u_\lambda(x)$  satisfies

$$-\Delta u_\lambda + \lambda^2 u_\lambda = u_\lambda^3 \quad \text{in } \mathbb{R}^2. \quad (3.6)$$

Differentiating (3.6) with respect to  $\lambda$  and taking  $\lambda = 1$ , we deduce that the function  $v = \frac{\partial u_\lambda}{\partial \lambda}|_{\lambda=1} = U + \nabla U \cdot x$  satisfies

$$-\Delta v + v - 3U^2 v = -2U \quad \text{in } \mathbb{R}^2.$$

Equality (3.5) then results for the fact that the kernel of  $v \mapsto -\Delta v + v - 3U^2 v$  is spanned by  $\partial_i U$ ,  $i = 1, 2$ , and so the operator is injective on radial functions. Next, integrating by parts (for example, using  $\operatorname{div}(\frac{1}{2}U^2 x) = U^2 + U \nabla U \cdot x$ ), it is immediate to check that

$$\int_{\mathbb{R}^2} (U + \nabla U \cdot x) U(x) \, dx = 0.$$

We agree that a refinement of the ansatz is necessary. Without loss of generality, we may assume the common critical point of the potentials is  $\xi_0 = 0$  and also that each potential vanishes at  $\xi_0$ , i.e. in a neighbourhood of the origin

$$V_i(x) = (a_1^{(i)} x_1^2 + a_2^{(i)} x_2^2) + \mathcal{O}(|x|^3) \quad \text{with } a_1^{(i)}, a_2^{(i)} \in \mathbb{R}, \text{ for every } i = 1, \dots, k. \quad (3.7)$$

Set

$$\alpha := \int_{\mathbb{R}^N} U(x) z_0(x) \, dx, \quad (3.8)$$

where  $z_0$  is the radial solution to

$$-\Delta z_0 + z_0 - 3U^2 z_0 = (x_1^2 + x_2^2)U \text{ in } \mathbb{R}^2. \quad (3.9)$$

In Section 4, there is a numerical evidence that  $\alpha > 0$ .

**Theorem 3.4.** *There exists  $\delta > 0$  such that if either  $\Delta\Gamma(0) > 0$  and  $\mu \in (\mu_0 - \delta, \mu_0)$ , or  $\Delta\Gamma(0) < 0$  and  $\mu \in (\mu_0, \mu_0 + \delta)$ , problem (2.8) has a solution  $(\epsilon_\mu, u_\mu)$  such that  $u_\mu$  concentrates at the origin and  $\epsilon_\mu^{-4}(\mu_0 - \mu) \rightarrow \alpha \Delta\Gamma(0)$  as  $\mu \rightarrow \mu_0$ .*

*Proof.* We briefly sketch the main steps of the proof, which relies on the same arguments used in the non-critical case.

We look for a solution to (2.8), where we choose  $\xi = \epsilon^2\tau$  as

$$\mathbf{u} = \underbrace{\mathbf{U} - \epsilon^4 \mathbf{Q}}_{=: \mathbf{W}} + \phi, \quad (3.10)$$

where  $\mathbf{U} = (U_1, \dots, U_k)$  is the non-degenerate synchronized solution of the limit system (1.7), the remainder term  $\phi \in K^\perp$  (see (2.11)) and the second order correction term  $\mathbf{Q} := (Q_1, \dots, Q_k)$  solves the linear system

$$\begin{aligned} -\Delta Q_i + Q_i - \sum_{j=1}^k \beta_{ij} (U_j^2 Q_i + 2U_i U_j Q_j) \\ = \left( a_1^{(i)} x_1^2 + a_2^{(i)} x_2^2 \right) U_i \text{ in } \mathbb{R}^2, \quad i = 1, \dots, k. \end{aligned} \quad (3.11)$$

The proof proceeds as in the previous case. Here, it is only a matter of noting a couple of crucial facts. First of all, the size of the error  $\mathcal{E}_{\epsilon, \tau}$  is

$$\|\mathcal{E}_{\epsilon, \tau}\| \lesssim \epsilon^5,$$

because (see also (2.21))

$$\begin{aligned} \mathcal{E}_i = \mathbf{i}^* \left\{ \epsilon^8 \sum_{j=1}^k \beta_{ij} \left( 2U_j Q_i Q_j + U_i Q_j^2 - \epsilon^4 Q_i Q_j^2 \right) \right\} \\ + \mathbf{i}^* \left\{ \epsilon^6 V_i(\epsilon x + \epsilon^2 \tau) Q_i - \epsilon^2 \left[ V_i(\epsilon x + \epsilon^2 \tau) - \epsilon^2 (a_1^{(i)} x_1^2 + a_2^{(i)} x_2^2) \right] U_i \right\}. \end{aligned} \quad (3.12)$$

Next, the component of the error along the element of the kernel is (see also (2.26))

$$\begin{aligned} -\langle \mathcal{E}_{\epsilon, \tau}, \Phi_j \rangle &= -\sum_{\ell=1}^k \sigma_\ell \left\langle \mathcal{E}_\ell, \frac{\partial U}{\partial x_j} \right\rangle \\ &= \epsilon^2 \sum_{\ell=1}^k \sigma_\ell^2 \int_{\mathbb{R}^2} V_\ell(\epsilon x + \epsilon^2 \tau) U \frac{\partial U}{\partial x_j} dx + \mathbf{o}(\epsilon^5) \\ &= -\frac{1}{2} \epsilon^3 \sum_{\ell=1}^k \sigma_\ell^2 \int_{\mathbb{R}^2} \frac{\partial V_\ell}{\partial x_j}(\epsilon x + \epsilon^2 \tau) U^2(x) dx + \mathbf{o}(\epsilon^5) \end{aligned}$$



$$\begin{aligned}
 &= -\epsilon^5 \left( \sum_{\ell=1}^k \sigma_\ell^2 a_j^{(\ell)} \tau_j \int_{\mathbb{R}^2} U^2 dx + \frac{1}{4} \sum_{\ell=1}^k \sum_{i=1}^N \sigma_\ell^2 \frac{\partial^3 V_\ell}{\partial x_i^2 \partial x_j} (0) \int_{\mathbb{R}^2} x_i^2 U^2 dx \right) + \mathfrak{o}(\epsilon^5) \\
 &= -\epsilon^5 \left( \underbrace{\tau_j \gamma \sum_{\ell=1}^k \sigma_\ell^2 a_j^{(\ell)}}_{=\frac{1}{2} \frac{\partial^2 \Gamma}{\partial x_j^2} (0)} + \frac{1}{4\gamma} \sum_{i=1}^N \frac{\partial^3 \Gamma}{\partial x_i^2 \partial x_j} (0) \int_{\mathbb{R}^2} x_i^2 U^2 dx \right) + \mathfrak{o}(\epsilon^5) \\
 &= -\frac{1}{2} \epsilon^5 \left( \tau_j \frac{\partial^2 \Gamma}{\partial x_j^2} (0) + \frac{\tilde{\gamma}}{2N\gamma} \frac{\partial \Delta \Gamma}{\partial x_j} (0) + \mathfrak{o}(1) \right),
 \end{aligned}$$

where we have used that  $\int_{\mathbb{R}^2} x_i^2 U^2 dx$  is independent of  $i$  and so

$$\int_{\mathbb{R}^2} x_i^2 U^2 dx = \frac{1}{N} \sum_{i=1}^N \int_{\mathbb{R}^2} x_i^2 U^2 dx = \frac{\tilde{\gamma}}{N} \quad \text{with } \tilde{\gamma} := \int_{\mathbb{R}^2} |x|^2 U^2(x) dx.$$

Since  $\frac{\partial^2 \Gamma}{\partial x_j^2} (0) \neq 0$  for any  $j = 1, 2$ , we can argue exactly as in the previous part to get the existence of a solutions concentrating at  $\xi_0 = 0$  as

$$\mathbf{u}_\epsilon = \mathbf{u} \left( \frac{x - \xi_\epsilon}{\epsilon} \right) - \epsilon^4 \mathbf{Q} \left( \frac{x - \xi_\epsilon}{\epsilon} \right) + \phi_\epsilon \left( \frac{x - \xi_\epsilon}{\epsilon} \right),$$

whose mass is

$$\begin{aligned}
 \mu_\epsilon &= \epsilon^{-2} \sum_{i=1}^k \int_{\mathbb{R}^2} \left( \sigma_i U \left( \frac{x - \xi_\epsilon}{\epsilon} \right) - \epsilon^4 Q_i \left( \frac{x - \xi_\epsilon}{\epsilon} \right) + \phi_{i,\epsilon} \left( \frac{x - \xi_\epsilon}{\epsilon} \right) \right)^2 dx \\
 &= \mu_0 - 2\epsilon^4 \underbrace{\sum_{i=1}^k \int_{\mathbb{R}^2} \sigma_i U(x) Q_i(x) dx}_{=: \Upsilon} + \mathfrak{o}(\epsilon^4) \quad (\text{see (1.11)}). \tag{3.13}
 \end{aligned}$$

Since  $u_\epsilon$  must have a prescribed mass equal to  $\mu$  we have to find  $\epsilon = \epsilon(\mu)$  such that

$$\mu_\epsilon = \mu. \tag{3.14}$$

As the map  $\epsilon \mapsto \mathbf{u}_\epsilon$  is continuous, so is  $\epsilon \mapsto \mu_\epsilon$ . Moreover, it goes to  $\mu_0$  as  $\epsilon \rightarrow 0$ . Thanks to Remark 3.5 (below), its image must contain an interval of the form  $(\mu_0 - \delta, \mu_0)$  if  $\alpha \Delta \Gamma(0) > 0$  or  $(\mu_0, \mu_0 + \delta)$  if  $\alpha \Delta \Gamma(0) < 0$ . That concludes the proof.  $\square$

**Remark 3.5.** It holds true that  $\Upsilon = \frac{1}{2} \alpha \Delta \Gamma(0)$  (see (3.8)). We apply Remark 3.2 with  $f_\ell = \sigma_\ell (a_1^{(\ell)} x_1^2 + a_2^{(\ell)} x_2^2) U$  (see (3.11)). Therefore by (3.4)

$$\Upsilon = \sum_{i=1}^k \int_{\mathbb{R}^N} U_i(x) Q_i(x) dx = \int_{\mathbb{R}^N} U(x) z^*(x) dx,$$

where  $z^*$  solves

$$-\Delta z + z - 3U^2 z = \sum_{\ell=1}^k \sigma_\ell^2 (a_1^{(\ell)} x_1^2 + a_2^{(\ell)} x_2^2) U \quad \text{in } \mathbb{R}^2.$$

Now, we can write

$$z^* = \left( \sum_{\ell=1}^k \sigma_\ell^2 a_1^{(\ell)} \right) z_1^* + \left( \sum_{\ell=1}^k \sigma_\ell^2 a_2^{(\ell)} \right) z_2^*,$$

where  $z_i^*$  solves

$$-\Delta z_i^* + z_i^* - 3U^2 z_i^* = x_i^2 U \quad \text{in } \mathbb{R}^2.$$

We observe that  $z_2^*(x_1, x_2) = z_1^*(x_2, x_1)$  and so

$$\int_{\mathbb{R}^N} U(x) z_1^*(x) dx = \int_{\mathbb{R}^N} U(x) z_2^*(x) dx = \frac{1}{2} \int_{\mathbb{R}^N} U(x) z_0(x) dx,$$

where  $z_0$  solves (3.9). Then

$$\begin{aligned} \int_{\mathbb{R}^N} U(x) z^*(x) dx &= \left( \sum_{\ell=1}^k \gamma \sigma_\ell^2 a_1^{(\ell)} \right) \int_{\mathbb{R}^N} U(x) z_1^*(x) dx + \left( \sum_{\ell=1}^k \gamma \sigma_\ell^2 a_2^{(\ell)} \right) \int_{\mathbb{R}^N} U(x) z_2^*(x) dx \\ &= \frac{\alpha}{2\gamma} \gamma \underbrace{\sum_{\ell=1}^k \sigma_\ell^2 (a_1^{(\ell)} + a_2^{(\ell)})}_{=\frac{1}{2} \Delta \Gamma(0)} \end{aligned}$$

and the claim follows.

#### 4. NUMERICAL EVIDENCE FOR THE ASSUMPTION $\alpha \neq 0$

Set

$$\alpha_N := \int_{\mathbb{R}^N} U(x) S(x) dx, \quad (4.1)$$

where  $U$  is the positive radial solution of

$$-\Delta U + U = U^p \quad \text{in } \mathbb{R}^N$$

and  $S$  is the radial solution of

$$-\Delta S + S - pU^{p-1}S = |x|^2 U \quad \text{in } \mathbb{R}^N.$$

In the previous section  $\alpha = \alpha_2$ . In this section, we would like to provide numerical evidence that  $\alpha_N > 0$  for the  $L^2$ -critical exponent, i.e.

$$p = 1 + \frac{4}{N} \quad \Rightarrow \quad \int_0^\infty U(r) S(r) r^{N-1} dr > 0. \quad (4.2)$$

This was proved in [20, Remark 3.5] for  $N = 1$ . Here we numerically estimate the integral in (4.2) for larger values of  $N$ . To do this,  $U(0)$  is estimated using a bisection procedure to find the right initial condition that lies between the set of  $U(0)$  such that  $\forall r > 0$ ,  $U(r) > 0$  and those such that  $U$  has at least one root. To determine  $S(0)$ , we impose that  $S(r_0) = 0$  for a “large”  $r_0$ . Such  $S(0)$  is easy to compute since  $S(r_0)$  is an affine

function of  $S(0)$ . On Fig. 1, you can see the result of these computations as the graphs of the functions  $p \mapsto \int_0^\infty U S r^{N-1} dr$  for  $N \in \{1, \dots, 8\}$ . The large dot on each curve indicates the point of the graph for  $p = 1 + 4/N$ . These provide clear evidence that (4.2) holds.

From these graphs, we also conjecture that

$$\alpha_N \rightarrow 0 \quad \text{as } p \rightarrow 2^* - 1 = \frac{N+2}{N-2}.$$

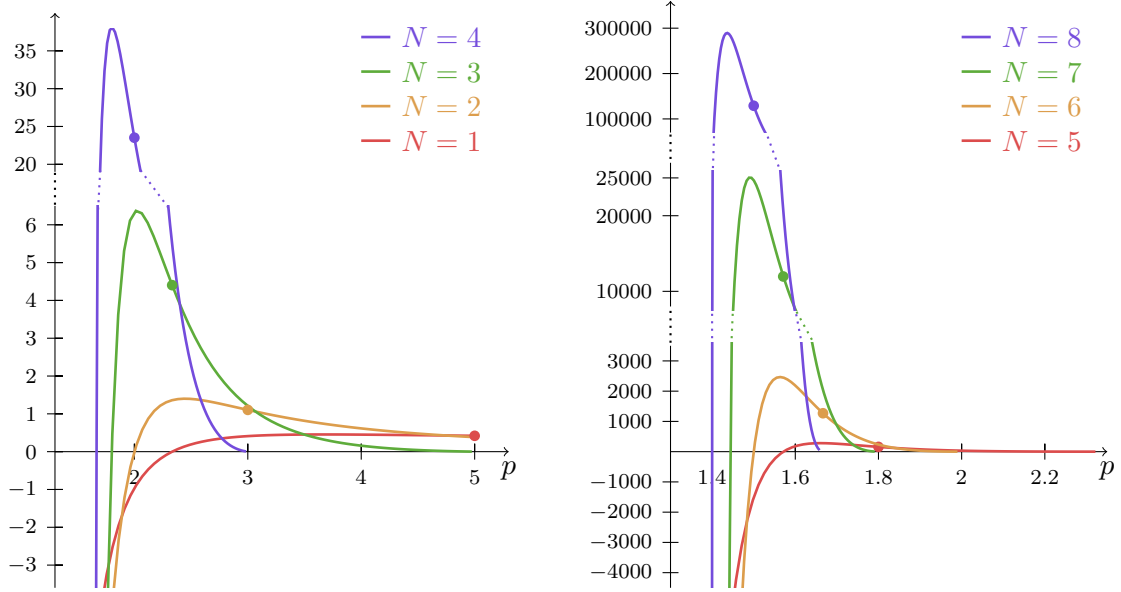


FIGURE 1. Graphs of  $p \mapsto \int_0^\infty U S r^{N-1} dr$  for  $N \in \{1, \dots, 8\}$ .

#### APPENDIX A. A NON-DEGENERATE RESULT

In the following we use some ideas introduced in [21].

Let us consider the eigenvalue problem

$$-\Delta\psi + \psi = \lambda U^2\psi \quad \text{in } \mathbb{R}^N.$$

The classical Fredholm alternative Theorem allows to claim that there exists a sequence of positive eigenvalues  $\{\lambda_m\}_{m \in \mathbb{N}}$  with

$$1 = \lambda_1 < 3 = \lambda_2 < \lambda_3 < \dots < \lambda_m < \lambda_{m+1} < \dots \quad \text{and} \quad \lambda_m \rightarrow +\infty.$$

The eigenspace associated to the first eigenvalue is a 1-dimensional space generated by the positive function  $U$ . Moreover, it is well known that eigenspace associated to the second eigenvalue is a  $N$ -dimensional space generated by  $\psi_i := \frac{\partial U}{\partial x_i}$  for  $i = 1, \dots, N$ .

We observe that system (1.9) can be rewritten as

$$-\Delta\phi_i + \phi_i - (2\beta_{ii}\sigma_i^2 + 1)U^2\phi_i - 2\sigma_i \sum_{\substack{j=1 \\ j \neq i}}^k \beta_{ij}\sigma_j U^2\phi_j = 0 \text{ in } \mathbb{R}^N, \quad i = 1, \dots, k. \quad (\text{A.1})$$

$$-\Delta \mathbf{v} + \mathbf{v} = U^2 \mathcal{M} \mathbf{v} \text{ in } \mathbb{R}^N,$$

with

$$\mathcal{M} := \mathcal{I}d + 2 \mathcal{C} \quad \text{and} \quad \mathcal{C} := \begin{pmatrix} \beta_{11}\sigma_1^2 & \beta_{12}\sigma_1\sigma_2 & \dots & \beta_{1k}\sigma_1\sigma_k \\ \beta_{12}\sigma_1\sigma_2 & \beta_{22}\sigma_2^2 & \dots & \beta_{2k}\sigma_2\sigma_k \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1k}\sigma_1\sigma_k & \beta_{2k}\sigma_2\sigma_k & \dots & \beta_{kk}\sigma_k^2 \end{pmatrix}. \quad (\text{A.2})$$

Let  $\Lambda$  be an eigenvalue of  $\mathcal{M}$  and  $e$  an associated eigenfunction, i.e.

$$\mathcal{M}e = \Lambda e.$$

It is useful to point out that  $\Lambda_\ell$  is an eigenvalue of  $\mathcal{M}$  if and only if  $\Theta_\ell := (\Lambda_\ell - 1)/2$  is an eigenvalue of the matrix  $\mathcal{C}$ . It is immediate to check that  $\Theta = 1$  is an eigenvalue of  $\mathcal{C}$  whose eigenvector is  $(\sigma_1, \dots, \sigma_k)$ . We set  $\Theta_1 = 1$ , which implies  $\Lambda_1 = 3$ .

**Proposition A.1.** *Assume that, all the eigenvalues  $\Lambda_2, \dots, \Lambda_k$  of  $\mathcal{M}$  do not coincide with any of the eigenvalues  $\{\lambda_m : m \in \mathbb{N}\}$ , i.e.*

$$\Lambda_\ell \notin \{\lambda_1, \dots, \lambda_m, \dots\} \text{ for any } \ell = 2, \dots, k. \quad (\text{A.3})$$

*Then the set of solutions to the linear system (A.1) is  $N$ -dimensional, and is generated by  $\psi_i \mathbf{e}_1$ , where  $\mathbf{e}_1 = (\sigma_1, \dots, \sigma_k) \in \mathbb{R}^k$  is an eigenvector associated with  $\Lambda_1 = 3$  and  $\psi_i := \frac{\partial U}{\partial x_i}$  for  $i = 1, \dots, N$ .*

*Proof.* Let  $\Lambda_\ell$  be an eigenvalue of the matrix  $\mathcal{M}$  and let  $\mathbf{e}_\ell \in \mathbb{R}^k$  be an associated eigenvector. We multiply (A.1) by  $\mathbf{e}_\ell$  and taking into account the symmetry of the matrix  $\mathcal{M}$  we get

$$-\Delta(\mathbf{e}_\ell \cdot \mathbf{v}) + \mathbf{e}_\ell \cdot \mathbf{v} = \Lambda_\ell U^2(\mathbf{e}_\ell \cdot \mathbf{v}) \text{ in } \mathbb{R}^N.$$

Since  $\Lambda_\ell \neq \lambda_m$  for every  $m$ , we deduce that

$$\mathbf{e}_\ell \cdot \mathbf{v} = 0 \quad \text{for any } \ell = 2, \dots, k,$$

which implies (by the orthogonality of eigenvectors associated to different eigenvalues) that

$$\mathbf{v} = \psi(x) \mathbf{e}_1 \text{ for some function } \psi \text{ such that } -\Delta\psi + \psi = 3U^2\psi \text{ in } \mathbb{R}^N$$

and the claim follows.  $\square$

A first consequence is the following example.

**Example A.2.** Let  $k = 2$ . The system (1.7) has a non-degenerate synchronized solution if

$$\text{either } -\sqrt{\mu_1\mu_2} < \beta < \min\{\mu_1, \mu_2\} \text{ or } \beta > \max\{\mu_1, \mu_2\}, \quad (\text{A.4})$$

where  $\mu_i := \beta_{ii}$  and  $\beta := \beta_{12} = \beta_{21}$ .

*Proof.* First of all, we observe that in this case system (1.8) reduces to

$$\begin{cases} \mu_1 \sigma_1^2 + \beta \sigma_2^2 = 1, \\ \beta \sigma_1^2 + \mu_2 \sigma_2^2 = 1, \end{cases} \quad (\text{A.5})$$

which admits the solution

$$\sigma_1^2 := \frac{\beta - \mu_2}{\beta^2 - \mu_1 \mu_2}, \quad \sigma_2^2 := \frac{\beta - \mu_1}{\beta^2 - \mu_1 \mu_2} \quad (\text{A.6})$$

if (A.4) holds true.

Next, to prove that it is non-degenerate, we apply Proposition A.1 showing that assumption (A.3) holds. The matrix  $M := (\alpha_{ij})_{i,j=1,2}$  in (A.2) reduces to

$$\alpha_{11} := 3\mu_1 \sigma_1^2 + \beta \sigma_2^2, \quad \alpha_{12} = \alpha_{21} := 2\beta \sigma_1 \sigma_2, \quad \alpha_{22} := 3\mu_2 \sigma_2^2 + \beta \sigma_1^2. \quad (\text{A.7})$$

Its eigenvalues are

$$\Lambda_1 = \frac{\alpha_{11} + \alpha_{22} + \sqrt{(\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}^2}}{2}, \quad \Lambda_2 = \frac{\alpha_{11} + \alpha_{22} - \sqrt{(\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}^2}}{2}. \quad (\text{A.8})$$

Using the definition of  $\alpha_{ij}$  and that of  $\sigma_1, \sigma_2$ , it is not difficult to check that

$$\Lambda_2 = \frac{6 - 2\beta(\sigma_1^2 + \sigma_2^2) - 2\beta(\sigma_1^2 + \sigma_2^2)}{2} = 3 - 2\beta(\sigma_1^2 + \sigma_2^2)$$

and

$$\Lambda_1 = \frac{6 - 2\beta(\sigma_1^2 + \sigma_2^2) + 2\beta(\sigma_1^2 + \sigma_2^2)}{2} = 3.$$

In particular,  $\Lambda_2 < \Lambda_1 = 3$  for  $\beta > 0$ . Moreover, a direct computation shows that if either  $\beta < \min\{\mu_1, \mu_2\}$ , or  $\beta > \max\{\mu_1, \mu_2\}$ , then  $\Lambda_1 \neq 1$ . Finally, if  $\Lambda_1 = 1$ , then it is immediate to check that

$$1 = 3 - 2\beta \frac{2\beta - \mu_1 - \mu_2}{\beta^2 - \mu_1 \mu_2},$$

which implies either  $\beta = \mu_1$ , or  $\beta = \mu_2$  which is not possible.  $\square$

**Proposition A.3.** *Suppose that the matrix  $\mathcal{B} := (\beta_{ij})_{1 \leq i, j \leq k}$  is invertible and has only positive elements. Then the linearized systems (A.1) has a  $N$ -dimensional set of solutions.*

*Proof.* As observed above, we have to prove that if  $(\beta_{ij})$  is invertible and has positive entries, then the set of solutions to (A.1) is  $N$ -dimensional. By Proposition A.1, this amounts to show that if  $(\beta_{ij})$  is invertible and has positive entries, then the eigenvalues  $\Lambda_2, \dots, \Lambda_k$  of  $\mathcal{M}$  are different from  $\lambda_1 = 1, \lambda_2 = 3, \lambda_m > 3$ .

Let us argue in terms of the matrix  $\mathcal{C}$ . By assumption,  $\mathcal{C}$  has positive entries. Therefore by Perron-Frobenius Theorem we deduce that the eigenvalue  $\Theta_1 = 1$ , which is associated to the eigenvector of positive elements  $(\sigma_1, \dots, \sigma_k)$ , is simple, and any other eigenvalue  $\Theta_\ell$  satisfies  $|\Theta_\ell| < 1$ . Moreover, 0 is not an eigenvalue of the matrix  $\mathcal{C}$ , since a straightforward computation shows that

$$\det \mathcal{C} = -(\sigma_1^2 \cdots \sigma_k^2) \det(\beta_{ij}) \neq 0$$

being  $(\beta_{ij})$  invertible. Therefore,  $\Lambda_1 = 3$  is a simple eigenvalue, and we have that both  $-1 < \Lambda_\ell < 3$  and  $\Lambda_\ell \neq 1$  for any  $\ell = 2, \dots, k$ . This completes the proof.  $\square$

**Example A.4.** Let  $k \geq 2$ . The system (1.7) has a non-degenerate synchronized solution if

$$0 < \mu_1 < \cdots < \mu_k, \quad \beta := \beta_{ij} \text{ for every } i \neq j \text{ and } \beta \text{ is large enough,} \quad (\text{A.9})$$

where  $\mu_i := \beta_{ii}$ .

*Proof.* The existence of a synchronized solution is proved in [1], once we choose  $0 < \mu_1 < \cdots < \mu_k$  and  $\beta_{ij} = \beta$  for every  $i \neq j$  with  $\beta > \mu_k$ . Indeed it is easy to check that

$$\sigma_i := \left[ (\mu_i - \beta) \left( 1 + \beta \sum_{j=1}^k \frac{1}{\mu_j - \beta} \right) \right]^{-1/2} \quad \text{for } i = 1, \dots, k$$

is a solution to system (1.8). To prove that it is non-degenerate, we apply Proposition A.3. Indeed in this case

$$\mathcal{B} = \begin{pmatrix} \mu_1 & \beta & \cdots & \beta \\ \beta & \mu_2 & \cdots & \beta \\ \vdots & \vdots & \ddots & \vdots \\ \beta & \beta & \cdots & \mu_k \end{pmatrix},$$

whose elements are strictly positive. It is also easy to check that it is invertible if  $\beta$  is large, since

$$\det \mathcal{B} \sim \beta^k \det \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} = \beta^k (-1)^{k-1} (k-1) \quad \text{as } \beta \rightarrow +\infty.$$

□

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