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Kinematical Lie algebras and symplectic symmetric spaces I Lie algebraic aspects

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Abstract

The aim of this note is to present a close relation between kinematical Lie algebras and symmetric spaces in a symplectic context: to every kinematical Lie algebra is canonically associated a symplectic symmetric space. For non-flat symmetric spaces, this correspondence is one to one onto a specific class of symplectic symmetric spaces whose structure we describe in detail. In particular, the transvection Lie algebra of such a symmetric space is either three-graded or of the Poincaré type. The denomination "Poincaré type" refers to symplectic symmetric spaces characterized by a property that generalizes the fact that the classical Poincaré group $SO_o(1, D) \ltimes \mathbb{R}^{D+1}$ turns out to be the transvection group of an unexpected purely symplectic symmetric space structure on the cotangent bundle of the hyperbolic space $SO_o(1, D)/SO(D)$ (i.e. the mass-shell orbit). The Lie triple system associated with every such symplectic symmetric space is of Jordan type (in the sense of W. Bertram), i.e. it is a homotope of the Lie triple system associated with a Hermitian symmetric space. However, the class of those Jordan-Lie triple systems associated with a classical kinematical Lie algebra is not stable under the natural operations of homotopies and dualities defined on Jordan-Lie triple systems. In order to restore stability, we need to introduce a natural generalization of the notion of kinematical Lie algebras, which is the framework where the present work is formulated. The last section of this work presents some remarks on the coadjoint orbits that are naturally associated with symplectic symmetric spaces.

Keywords Symplectic symmetric spaces · Kinematical Lie algebras · involutive Lie algebras

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1 Introduction

1.1 Physical aspects

Of central importance in physics is the notion of (equivalence class of) inertial reference systems or inertial frames. To some extent, the set of reference frames together with the group of transformations relating them determine the global geometry of space-time. The laws of physics are invariant under the transformations relating the inertial frames of a given equivalence class of such frames. In the Newtonian formulation of mechanics, two inertial frames are related by a transformation of the Galilean group. With the works of Lorentz, Poincaré and Einstein, it was understood that two inertial frames can more accurately be related by what is now called a Lorentz, or Poincaré, transformation, a transformation which leaves the speed of light invariant. The Lie algebras of these transformation groups are maximal, in the sense that, in a space-time with three space dimensions and one time direction, they contain three generators of “translation", three generators of “boost", three rotation generators that form a basis of $\mathfrak{s} \cong \mathfrak{so}(3)$, and one time-translation generator. Altogether, this gives ten generators, ten being the number of linearly independent Killing vectors of the maximally symmetric (pseudo)Riemannian spaces of dimension four [21].

In two seminal papers, Bacry, Lévy-Leblond and Nuyts [16, 17] determined the various possible relativity principles that a four-dimensional, isotropic space-time could accommodate, thereby defining the various possible classes of inertial frames. In other words, they asked whether there could exist theoretically possible relativity principles that differ from the Galilean or the Lorentzian ones discussed above. The various possible transformation groups expressing the equivalence relations among inertial frames within each possible class were qualified as *kinematical* in [16, 17], and the main result of these two papers was the classification of the kinematical algebras pertaining to four-dimensional, isotropic space-times. It was also observed in [16, 17] that, to every kinematical Lie algebra $\mathfrak{g}$ they classified, there always exists a six-dimensional subalgebra $\tilde{\mathfrak{h}} \subset \mathfrak{g}$, suggesting the existence of a four-dimensional homogeneous space $G/\tilde{H}$ describing a possible isotropic space-time, the rotation algebra $\mathfrak{so}(3)$ always being contained in $\tilde{\mathfrak{h}}$. As a vector space, any kinematical Lie algebra $\mathfrak{g}$ appearing in the classification of [16, 17] can be decomposed as $\mathfrak{g} = \mathfrak{s} \oplus \mathcal{P} \oplus \mathcal{Z}$, where $\mathfrak{s} \cong \mathfrak{so}(3)$, and $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1$. The subspace $\mathcal{Z}$ is a line, while both $\mathcal{P}_0$ and $\mathcal{P}_1$ are $\mathfrak{so}(3)$ modules in the vector representation.

The classification of Bacry, Lévy-Leblond, and Nuyts was reconsidered later [1, 24, 25] in higher- and lower-dimensional space-times, replacing the isotropy algebra $\mathfrak{so}(3)$ by $\mathfrak{so}(D)$, with the number of space dimensions $D \geq 1$ and the dimension of $\mathcal{P}_0$ and $\mathcal{P}_1$ being equal to D, for the vector representation of $\mathfrak{so}(D)$. The detailed relations between the various Klein pairs $(\mathfrak{g}, \tilde{\mathfrak{h}})$ and the various possible isotropic space-times of the form $G/\tilde{H}$ were given in [27]. In the latter reference it was stressed that not all Klein pairs effectively give raise to homogeneous spaces, neither is it necessary that they should be geometrically realizable. See also [23, 26, 35] for a summary and related results.

In the present paper, we show that, already for the cases $\mathfrak{so}(D)$ with $D > 3$, there is a symplectic symmetric space [7, 12] that governs the structure of any of these classical kinematical Lie algebras. Besides, it appears necessary to further extend the notion of kinematical Lie algebra to what we call *generic* kinematical Lie algebra that encompass the cases where $\mathfrak{s}$ can be any Lie algebra, not necessarily $\mathfrak{so}(D)$. The reason why such an extension should necessarily appear is spelled out in next section 1.2. The notion of generic kinematical Lie algebra defines remarkable families of Lie algebras and symmetric spaces that persist in dimensions $D \leqslant 3$ and that were specifically considered by Figueroa-O'Farrill and collaborators in the classical case $\mathfrak{s} \cong \mathfrak{so}(D)$.

The classification of the kinematical algebras is not a purely academic question. Although Minkowski's space-time is a well-adapted arena where to describe physical processes where particles can travel at velocities approaching the speed of light, the Newtonian space-time is perfectly adapted (and used every day) to host physical phenomena where the velocities of particles are negligible compared to the speed of light. On the other hand, modern cosmology (see, for example, [40] for a textbook) indicates that the geometry of our early, inflationary universe can be approximated by a de Sitterian space-time, with a positive, constant curvature. Finally, in the context of string theory, J. Maldacena [34] formulated a conjecture that gives a prime importance to anti-de Sitterian (AdS) space-times of various dimensions. From a practical point of view, depending on the nature of the physical system under study, one given structure of space-time will be preferred to another.

1.2 Mathematical aspects

1.2.1 Symplectic symmetric spaces

Symplectic symmetric spaces were introduced by one of us et al. in the mid-nineties [7, 12]. The idea is to define a class of symplectic manifolds admitting a *preferred* symplectic linear connection. There are many ways to do this, e.g. requiring a compatibility with a (pseudo)Riemannian metric, considering critical solutions of a variational problem on symplectic connections, etc. (see, for example, [8], and references therein). Imposing the symplectic manifold to admit a large [7] set of symmetries is one of them. This choice was originally motivated, on the first hand, by the positive solution due to Sekigawa and Van Hecke [37] of a conjecture of S. Kobayashi, asserting that a compact Kähler manifold whose (local) geodesic symmetries (w.r.t. the Kähler metric) are symplectic (w.r.t. the Kähler two-form) must be a (local) Hermitian symmetric space, and on the second hand by a conjecture due to A. Weinstein in the context of invariant star-product on Hermitian symmetric spaces [39].
A symplectic symmetric space is a Fedosov manifold, i.e. a symplectic manifold (M, ω) equipped with a torsion-free symplectic linear connection ∇ (i.e. $\nabla\omega = 0$) enjoying the property that every ∇-geodesic symmetry centred at any point of M is well-defined as a global affine transformation of (M, ∇). Locally, this last property amounts to saying that the curvature (3, 1)-tensor R^∇ of the torsion-free connection

∇ is parallel: $\nabla R^{\nabla} = 0$; beware the fact that this last equation is not equivalent nor implied by a Bianchi identity. Affine symmetric spaces can be described and studied within several (equivalent) geometrical contexts depending on the user's needs and preferences. There are three main approaches. The first one is "differential geometrical", i.e. based on the datum of the connection. The second one is Lie theoretical, i.e. where the symmetric manifold is considered as a homogeneous space G/H, and the third one, adopted in the present article, consists in O. Loos approach to symmetric spaces within the Jordan algebraic context [33]. In this last framework, a symmetric space appears as a manifold with multiplication, generalizing the notion of Lie group; see Remark 2.1 below.

1.2.2 Semisimple symplectic symmetric spaces, their homotopes and dualities

In contrast with Riemannian symmetric spaces (i.e. Riemannian manifolds whose centred geodesic symmetries are globally defined and all isometrical), symplectic symmetric spaces are generally not homogeneous spaces of semisimple Lie groups. And conversely, a simple Lie group does generally not cover any symplectic symmetric space; for the structure and classification of semisimple symplectic symmetric spaces, see [14]. However, those that are semisimple constitute an important class of symplectic symmetric spaces. A first reason being that the class of semisimple symplectic symmetric spaces is acted on by several dualities. The compact–noncompact duality for Hermitian symmetric spaces is one of them (see, for example, [28]). Another one, called Hermitian–Cayley type, is the duality between Hermitian symmetric spaces and the causal symmetric spaces of Cayley type [22, 29]. To some extent, one can consider the class of hyper-Kähler symmetric spaces as the fixed points of those dualities. A second reason of importance of the semisimple class is that it naturally lies in the framework of Jordan algebra theory, which allows for considering *homotopies* between spaces as special kinds of deformations within the specific class of the *Jordan–Lie triple systems*. Roughly speaking, given a simple symplectic symmetric space, the set of its homotopes naturally consists in an algebraic variety [2, 10], called in Bertram's terminology the *structure variety*. The interior points of such a structure variety are simple symmetric spaces, while the ones on the boundary are not anymore (see [2, 3] for structure and full classification). A continuous path in the structure variety from an interior point to a boundary point can be thought of as some kind of generalized Inonu-Wigner contraction.

Nevertheless, the above-mentioned dualities do not always correspond to geometric constructions. An example of a duality which can geometrically be realized is the one of compact–noncompact Hermitian symmetric spaces. For those, the duality is based on holomorphic holonomy-equivariant embeddings generalizing the $U(1)$-equivariant holomorphic embedding of the hyperbolic plane into the Riemann sphere. The Hermitian–Cayley-type duality is not as clearly associated to such a geometrical construction. All this brings us to kinematical Lie algebras.

1.2.3 Kinematical Lie algebras and the Hermitian–Cayley-type duality

As a byproduct of the main result of the present work, a generic kinematical Lie algebra corresponds to a homotope of a Hermitian symmetric space. By *generic* we mean the following. In space-time dimension greater than or equal to five, the fact that the isotypical component of the natural representation of the rotation Lie algebra in its anti-symmetric square is empty separates the isomorphism classes of kinematical Lie algebras into distinguished families that will described below. By *generic*, we mean a kinematical Lie algebra belonging to one of those families. In smaller space-time dimensions, the aforementioned generic families subsist, but there are also other types [27], due to the fact that the above property of the isotypical component does not hold anymore. These extra, lower-dimensional classes, are not considered in the present work. The four-dimensional case will be reconsidered at the light of the present context in a forthcoming work. The complete classification of kinematical Lie algebras with isotropy algebra $\mathfrak{so}(D)$ with generic space dimension D was given in [1, 24, 25]. This being said, the other elements and their natural Jordan-duals (in the sense of the previous subsection) of the structure variety containing those Hermitian symmetric spaces associated with the usual kinematical Lie algebras are lost when restricting the Levi factor of the holonomy Lie algebra to be isomorphic to the rotation Lie algebra $\mathfrak{so}(D)$.

For this reason: in order to define a class of symmetric spaces containing the above-mentioned kinematical Hermitian symmetric spaces and which is stable under Jordan–Lie homotopy and symmetric space dualities, we need to relax the condition on the Levi factor of the holonomy to be isomorphic to the rotations by only requiring the emptiness condition spelled out above on the isotypical component of the natural representation of the Lie algebra that replaces the rotation algebra of the usual case. Precisely, we formulate our

Definition 1.1 A **generic**[1] **kinematical Lie algebra** is a triple $(\mathfrak{g}, \mathfrak{s}, V)$ where

1. $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,
2. $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$, and
3. V is a faithful absolutely simple[2] finite-dimensional $\mathfrak{s}$-module over $\mathbb{K}$ such that
    (a) the *weak isotypical component* (see below) of V in $\Lambda^2(V)$ is empty.
    (b) The action of $\mathfrak{s}$ on V preserves a non-degenerate scalar product (not necessarily positive definite) on V.

The triple is moreover conditioned by the requirement that $\mathfrak{g}$ admits a vector space decomposition

$$\mathfrak{g} = \mathcal{Z} \oplus \mathfrak{s} \oplus \mathcal{P} \tag{1}$$

where

[1] the adjective "generic" refers to two things: first, to the fact that the dimension D of V is generic, e.g. greater or equal to four in the classical case where $\mathfrak{s}$ is isomorphic to the rotation Lie algebra $\mathfrak{so}(D)$; and second to the fact that the Lie algebra $\mathfrak{s}$ is not defined as being isomorphic to the rotation Lie algebra.

[2] Recall that a simple module M over an algebra A on a ground field $\mathbb{K}$ is called absolutely simple if the only A-commuting $\mathbb{K}$-endomorphisms of M are in the ground field $\mathbb{K}$: $\mathtt{End}_{\mathbb{K}}(M)/\mathtt{End}_A(M) \simeq \mathbb{K}$.

4. $\mathcal{P}$ is stable under $\mathfrak{s}$ and isomorphic to the reducible $\mathfrak{s}$-module $V \oplus V$, and
5. $\mathcal{Z}$ is a line which centralizes $\mathfrak{s}$ in $\mathfrak{g}$.

By *weak isotypical component* we mean

Definition 1.2 Let W be a (not necessarily semisimple) $\mathfrak{s}$-module and L an irreducible $\mathfrak{s}$-module. The **weak isotypical component** $W_{(L)}$ of L in W is defined as the vector sum in W of all submodules of W that are isomorphic to L.

In the sequel, we will abusively use the adjective "isotypical" for "weakly isotypical".

Remarks 1.1 1. In the context of Definition 1.2, the $\mathfrak{s}$-module $W_{(L)}$ does not necessarily admit a supplementary $\mathfrak{s}$-submodule in W.
2. The requirement on V to be absolutely simple (as opposed to only simple) is not essential, but it simplifies the discussion.

Although the deformational aspects of the present study will be deferred to a further article, we anticipate that the main result of the present work enables to tie the notion of (necessarily generalized) kinematical Lie algebra to a specific class of symplectic symmetric spaces that is stable under homotopy and duality in the context of Jordan triple systems. We will precise this in a forthcoming work.

1.3 Structure of the present work

In its essence, this article is interdisciplinary as it realizes a bridge between a theoretical physics notion and a differential geometric one. In order to reach a wider audience, in particular our colleagues in the theoretical physics community, we decide to write, in a first part of this article, a concise and essentially self-contained introduction to symmetric spaces and their symplectic version. In particular, up to basics in differential geometry, we made an effort to present proofs of the results that are relevant to this work.

1.3.1 Basics on symmetric spaces and symplectic symmetric spaces

Section 2 consists in an introduction to symmetric spaces and their symplectic analogues. All the material in this introduction has been well established for several decades, the main references being [6, 7, 12, 28, 31–33].
We start by the notion of symmetric space within the approach of O. Loos. We then explain the correspondence between symmetric spaces and their tangent analogues: involutive Lie algebras (iLa's). This correspondence generalizes to symmetric space Lie's third theorem.
We then proceed by introducing the main actor of this work, the notion of symplectic symmetric space, and describe, in the same lines of ideas as in the first step, their tangent analogues: symplectic iLa's (siLa's). We detail the correspondence (analogous to Lie's third theorem) that integrates siLa's to symplectic symmetric spaces. We also recall how a symplectic symmetric space can, up to automorphism, uniquely be decomposed into elementary pieces. This result is analogue to the de Rham decomposition theorem in Riemannian geometry[20].

1.3.2 Generic kinematical Lie algebras correspond to symplectic symmetric spaces

Sect. 3 constitutes the heart of the present work: we first show that, to every generic kinematical Lie algebra $(\mathfrak{g}, \mathfrak{s}, V)$ as defined in Definition 1.1 is canonically associated a simply connected symplectic symmetric space (M, ω, ∇) of dimension $\dim \mathcal{P}$ on which the simply connected Lie group G admitting $\mathfrak{g}$ as Lie algebra acts by automorphisms, i.e. by symplectic diffeomorphisms of (M, ω) that preserve the connection ∇. As a byproduct of the construction, we prove that every generic kinematical Lie algebra $(\mathfrak{g}, \mathfrak{s}, V)$ satisfies $[\mathcal{Z}, \mathcal{P}] \subset \mathcal{P}$, i.e. $\mathcal{Z}$ acts on $\mathcal{P}$.
We then concentrate on describing the fine structure of the symplectic symmetric spaces associated with our generic kinematical Lie algebras. By what was presented in Sect. 2, it is sufficient to consider the generic kinematical Lie algebras for which the associated simply connected symplectic symmetric space is indecomposable and non-flat (i.e. $R^{\nabla} \neq 0$), which we assume from now on.
For those, it turns out that the action of $\mathcal{Z}$ on $\mathcal{P}$ is either (complex) semisimple or nilpotent (no mixing). We entirely describe the structure in the case the action of $\mathcal{Z}$ on $\mathcal{P}$ is (complex) semisimple. For instance, those generic kinematical Lie algebras that are such that $\mathfrak{s}$ is semisimple are *exactly* the simple transvection Lie algebras of simple symplectic symmetric spaces. These spaces are classified in [14] and there structure is described: they are Hermitian symmetric spaces, para-Hermitian causal symmetric spaces of Cayley type, or hyper-Kähler symmetric spaces.
We describe the spaces for which the action of $\mathcal{Z}$ on $\mathcal{P}$ is nilpotent under the condition that $\mathfrak{s}$ is contained in a Levi factor of $\mathfrak{g}$; this "Levi" condition, for instance, holds when $\mathfrak{s}$ is compact semisimple. A simply connected symplectic symmetric space associated with such a generalized kinematical Lie algebra is called of the *Poincaré type*. We prove that every such symplectic symmetric space is equivariantly symplectomorphic to the cotangent bundle of a simple symmetric space. This result classifies those spaces. In particular, when $\mathfrak{s}$ is compact: the Poincaré types are exactly the cotangent bundles of Cartan's Riemannian symmetric spaces. Combining these results with the $\mathcal{Z}$-semisimple action yields a complete classification in all the cases where $\mathfrak{s}$ is semisimple.

1.3.3 Hamiltonian aspects

Section 4 is twofold: it first provides a concise presentation of standard basic facts on homogeneous symplectic spaces, and, secondly, it presents some features of the dynamical aspects of our present work. The original results of this section consist in two facts. First, the action of the group (necessarily Lie) generated by the geodesic symmetries on the (non-flat, indecomposable) symplectic symmetric space associated with a generalized kinematical Lie algebra is strongly Hamiltonian in the sense that it defines an equivariant moment map. And second, the Lie algebra of this Lie group also underlies a structure of generic kinematical Lie algebra.

1.3.4 Conclusions and perspectives

In the last Sect. 5, we summarize what has been done in the present article and suggest some of its developments.

2 Basics on symmetric spaces and symplectic symmetric spaces

2.1 Symmetric spaces

We first recall some basic facts about (affine) symmetric spaces. The references for the results appearing in this section are [6, 12, 31, 33]. Roughly speaking, a (locally) symmetric space is an affine manifold whose geodesic symmetries are affine transformations. More precisely:

Definition 2.1 [33] A **symmetric space** is a pair (M, s), where M is a smooth connected manifold, and where $s : M \times M \to M$ is a smooth map such that

(i) for all x in M, the partial map $s_x : M \to M : y \mapsto s_x(y) := s(x, y)$ is an involutive diffeomorphism of M called the **symmetry** at x.
(ii) For all x in M, x is an isolated fixed point of s_x.
(iii) For all x and y in M, one has $s_x s_y s_x = s_{s_x(y)}$ (**Fundamental Jordan identity**).

Definition 2.2 Two symmetric spaces (M, s) and (M', s') are **isomorphic** if there exists a diffeomorphism $\varphi : M \to M'$ such that $\varphi s_x = s'_{\varphi(x)}\varphi$. Such a φ is called an **isomorphism** of (M, s) onto (M', s'). When $(M, s) = (M', s')$, one talks about **automorphisms**. The group of all automorphisms of the symmetric space (M, s) is denoted by $Aut(M, s)$.

Example 2.1 An important class of symmetric spaces is the one of Lie groups. Given a Lie group $\mathfrak{G}$ with group law $(x, y) \mapsto xy$, the map

$$s : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G} : (x, y) \mapsto xy^{-1}x$$

defines a structure of symmetric space on the group manifold $\mathfrak{G}$.

Proposition 2.1 *On a symmetric space (M, s), there exists one and only one affine connection ∇ which is invariant under the symmetries.*
It is explicitly given by the following formula:

$$(\nabla_X Y)_x = \frac{1}{2}\left[X\,,\; Y + s_{x_\star} Y\right]_x \tag{2}$$

at every point x of M and for all tangent smooth vector fields X and Y on M.
The linear connection ∇ enjoys the properties of being torsion-free such that its curvature $(3, 1)$-tensor R^∇ is parallel.

Moreover, the automorphism group of the symmetric space (M, s) *coincides with the group of the affine transformations of the connection* ∇*:*

$$Aut(M, s) \; = \; Aff(M, \nabla) \; .$$

In particular, the automorphism group $Aut(M, s)$ *is a Lie group of transformations of* M.

Using basic facts in differential geometry (see, for example, [31]), the proof of the above proposition is straightforward: it suffices to check that formula (2) indeed defines a covariant derivative in the tangent bundle of $T(M)$, which is a routine computation. One readily verifies the announced properties using the explicit formula (see [12] and [4]).

Remark 2.1 The existence and uniqueness of a symmetry-invariant affine connection on every symmetric space were proved by Loos in [33] volume I. For this reason, this canonical connection on such a symmetric space will be hereafter called the **Loos connection**. Note, however, that the explicit formula for it was first given in [4].

Example 2.2 In the case of the flat symmetric space $\mathbb{R}^n$ equipped with its Euclidean affine symmetries, the affine group is $\mathrm{Aff}(\mathbb{R}^n) \; = \; \mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$. One observes that in this case, the subgroup $\mathbb{R}^n$ also transitively acts on the space. And moreover, it is characteristic of the flat connection in the sense that there is only one connection on $\mathbb{R}^n$ that is invariant under the translation group $\mathbb{R}^n$.

This observation is actually general. First observe

Proposition 2.2 *The automorphism group* $Aut(M, s)$ *transitively acts on* M.

Proof Consider any two points x and y in M. By connectedness, there exists a continuous path $\gamma : [0, 1] \to M$ joining them: $\gamma(0) = x$ and $\gamma(1) = y$. For every point z of the curve $C := \gamma([0, 1])$, choose a ∇-normal neighbourhood U_z and extract (by compactness of C) a finite open cover $\{U_i := U_{\gamma(t_i)}\}_{t_1=0<t_2<\ldots<t_N=1}$ of C in $\cup_{z\in C} U_z$. In every intersection $U_i \cap U_{i+1}$ choose a point p_i and consider geodesic arcs $\eta^{p_i}_{\gamma(t_i)}$ joining $\gamma(t_i)$ to p_i and $\eta^{\gamma(t_{i+1})}_{p_i}$ joining p_i to $\gamma(t_{i+1})$. The union of all these arcs then defines a broken geodesic line joining x to y. Denote by m^+_i the mid-point of the arc $\eta^{p_i}_{\gamma(t_i)}$ and by m^-_i the mid-point of the arc $\eta^{\gamma(t_{i+1})}_{p_i}$. Then, the affine transformation $s_{m^-_N} \circ s_{m^+_N} \ldots s_{m^-_1} \circ s_{m^+_1}$ sends x onto y. □

In a second step, we introduce the analogue of the translation group in the flat case:

Definition 2.3 The **transvection group** (or **displacement group**) $G(M, s)$ of a symmetric space (M, s) is defined as the subgroup of the automorphism group $Aut(M, s)$ generated by $\{s_x \circ s_y \, ; \; x, y \in M\}$.

As a byproduct of Proposition 2.2 and its proof, one then gets (see [33] vol. I pp. 88 Thm 2.8.):

Theorem 2.1 *The transvection group $G(M, s)$ of a symmetric space is a Lie group of transformations of M. It is characterized by the property to be the smallest subgroup of the automorphism group that acts transitively over M and that is stable under the conjugation by a (and therefore all) symmetry s_o ($o \in M$) in $Aut(M, s)$:*

$$\tilde{\sigma} : Aut(M, s) \to Aut(M, s) : g \mapsto s_o \circ g \circ s_o \, .$$

Remarks 2.1 (i) The notion of *transvection* generalizes the one of translation: in Euclidean space, the composition of two centred symmetries is a translation.

(ii) Observe that for every transvection $g \in G(M, s)$ and every point $x \in M$, one has $g s_x g^{-1} = s_{g(x)}$. Indeed, at the level of generators, the fundamental Jordan identity (Definition 2.1 item (iii)) implies $s_y s_z s_x s_z s_y = s_y s_{s_z x} s_y = s_{s_y s_z x}$.

Remark 2.2 Using formula (2) it is easy to check that a tensor field on a symmetric space which is invariant by the symmetries is necessarily parallel w.r.t. the Loos connection.

As a conclusion of the present paragraph, we observe the canonical correspondence $(M, s) \mapsto G(M, s)$ between symmetric spaces and *involutive Lie groups*, i.e. pairs $(G, \tilde{\sigma})$ where G is a Lie group and where $\tilde{\sigma}$ is an involutive automorphism of G.
In the next section, we will see how from the above discussion emerges a generalization (c.f. Example 2.1) of Lie's third theorem for symmetric spaces.

2.2 Involutive Lie algebras and de Rham's decompositions

Definition 2.4 An **involutive Lie algebra** (abbreviated "iLa") is a pair $(\mathfrak{g}, \sigma)$ where $\mathfrak{g}$ is a finite-dimensional real Lie algebra and where

$$\sigma : \mathfrak{g} \to \mathfrak{g}$$

is an involutive ($\sigma^2 = \mathtt{id}_{\mathfrak{g}}$) automorphism of the Lie algebra $\mathfrak{g}$.

Associated with σ, one has a vector decomposition of $\mathfrak{g}$ into its ± 1-eigenspaces:

$$\mathfrak{g} \; =: \; \mathfrak{h} \, \oplus \, \mathcal{P} \quad \text{with} \quad \sigma \; = \; \mathtt{id}_{\mathfrak{h}} \, \oplus (-\mathtt{id}_{\mathcal{P}}) \, . \tag{3}$$

One then observes

Lemma 2.1 *Given an iLa with associated decomposition (3), one has the inclusions*

$$[\mathfrak{h} \, , \, \mathfrak{h}] \subset \mathfrak{h} \, ,$$
$$[\mathfrak{h} \, , \, \mathcal{P}] \subset \mathcal{P} \, ,$$
$$[\mathcal{P} \, , \, \mathcal{P}] \subset \mathfrak{h} \, .$$

Reciprocally, if a Lie algebra $\mathfrak{g}$ admits a vector decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathcal{P}$ satisfying the above inclusions, then it is underlain by an iLa structure σ on $\mathfrak{g}$.

Definition 2.5 Given two iLa's $(\mathfrak{g}_i\,,\ \sigma_i)$ $(i = 1, 2)$, a **morphism** between them is a Lie algebra homomorphism

$$\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$$

which intertwines the involutions:

$$\varphi \circ \sigma_1 = \sigma_2 \circ \varphi\ .$$

When φ is bijective, one refers to it as an **isomorphism**.

Definition 2.6 An iLa $(\mathfrak{g}, \sigma)$ is called a **transvection iLa** if the additional conditions hold

1. $[\mathcal{P}, \mathcal{P}] = \mathfrak{h}$, and
2. the action of $\mathfrak{h}$ on $\mathcal{P}$ is faithful.

Every iLa is associated with a "canonical" transvection iLa: we have the following lemma whose proof is obvious.

Lemma 2.2 *Let $(\mathfrak{g}, \sigma)$ be an iLa with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathcal{P}$. Then*

(i)

$$\hat{\mathfrak{g}} := [\mathcal{P}, \mathcal{P}] \oplus \mathcal{P}\ , \quad \hat{\sigma} := \mathtt{id}_{[\mathcal{P},\mathcal{P}]} \oplus (-\mathtt{id}_{\mathcal{P}})$$

is a sub-iLa of $(\mathfrak{g}, \sigma)$.

(ii) The centralizer $\mathfrak{n}$ of $\mathcal{P}$ in $[\mathcal{P}, \mathcal{P}]$ is an ideal of $\hat{\mathfrak{g}}$ and the associated exact sequence

$$\mathfrak{n} \longrightarrow \hat{\mathfrak{g}} \xrightarrow{\pi} \underline{\mathfrak{g}} := \hat{\mathfrak{g}}/\mathfrak{n}$$

naturally induces on the quotient $\underline{\mathfrak{g}} := \hat{\mathfrak{g}}/\mathfrak{n}$ a structure of iLa.

(iii) The iLa defined in item (ii) is a transvection iLa.

Definition 2.7 The holonomy of the iLa $(\mathfrak{g}, \sigma)$ is defined as the Lie algebra $[\mathcal{P}, \mathcal{P}]/\mathfrak{n}$. The iLa $(\mathfrak{g}, \sigma)$ is called **flat** when $[\mathcal{P},\ \mathcal{P}] \subset \mathfrak{n}$. Equivalently, it can be shown that the Loos connection of the corresponding symmetric space is flat.

Definition 2.8 Given two iLa's $(\mathfrak{g}_i\ \sigma_i)$ $(i = 1, 2)$, their **direct sum** is defined as the natural iLa structure on the direct sum of the Lie algebras $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, i.e. $(\mathfrak{g}_1 \oplus \mathfrak{g}_2,\ \sigma_1 \oplus \sigma_2)$. An iLa is called **decomposable** if it is isomorphic to a direct sum of two non-trivial iLa's. It is called **indecomposable** otherwise.

It is not at all clear that two decompositions into indecomposable pieces of a given iLa are *isomorphic*, i.e. that there exists an automorphism of the iLa swapping them. In the Riemannian case, it is a consequence of the de Rham decomposition theorem [20]. Happily, it holds in the pure affine case as well [7, 12]:

Theorem 2.2 *Let $(\mathfrak{g}, \sigma)$ be a transvection iLa. And let*

$$(\mathfrak{g}\,,\ \sigma) = \oplus_{i=1}^{r}(\mathfrak{g}_i\,,\ \sigma_i) = \oplus_{j=1}^{\overline{r}}(\overline{\mathfrak{g}}_j\,,\ \overline{\sigma}_j)$$

be two decompositions into indecomposable iLa's. Then, $r = \bar{r}$ *and there exist a permutation* $\tau \in Sym(r)$ *and an automorphism* φ *of* $(\mathfrak{g}, \sigma)$ *such that for all* $i \in \{1, ..., r\}$*:*

$$\varphi(\mathfrak{g}_i) = \bar{\mathfrak{g}}_{\tau(i)} .$$

Now, as announced earlier, we have the

Theorem 2.3 *The correspondence* $(M, s) \mapsto G(M, s)$ *induces a bijection between the isomorphism classes of simply connected symmetric spaces and the isomorphism classes of transvection involutive Lie algebras.*

Proof Associating a simply connected symplectic symmetric space to an iLa $(\mathfrak{g}, \sigma)$, although elementary, fully uses standard techniques in differential geometry: since the bijection (Lie's third theorem) that associates a connected simply connected Lie group G to a Lie algebra $\mathfrak{g}$ is functorial, to the pair $(\mathfrak{g}, \sigma)$ is associated a pair $(G, \tilde{\sigma})$ with $\sigma := \tilde{\sigma}_{\star e}$. The subgroup $G^{\tilde{\sigma}}$ of group elements that are fixed by $\tilde{\sigma}$ is closed. Indeed, it is the pre-image of the identity e of G under the continuous map $G \to G : g \mapsto g^{-1}\tilde{\sigma}(g)$. The subgroup $G^{\tilde{\sigma}}$ is therefore an embedded Lie subgroup of G and so is its connected component $H := G_0^{\tilde{\sigma}}$ containing the identity e. The set G/H of left lateral classes (the "coset" space) of H in G therefore carries a unique structure of smooth manifold, homogeneous under the obvious action of G on the classes: $g_0(gH) := g_0 gH$. The first terms of the long exact sequence in homotopy $...\pi_1(G) \to \pi_1(G/H) \xrightarrow{\partial} \pi_0(H) \to ...$ associated with the principal bundle $H \to G \to G/H$ yield, since G is simply connected and H connected, the triviality of the fundamental group $\pi_1(G/H) = \{1\}$. In other words, the manifold G/H is simply connected. One then readily checks that the formula

$$s_{gH}(g'H) = g\,\tilde{\sigma}(g^{-1}g')H$$

defines a (G-equivariant) structure $s : G/H \times G/H \to G/H$ of symmetric space on the smooth manifold G/H.

In the above construction, we did not require the iLa to be transvection. This will be used for injectivity of the correspondence.

Let now (M, s) be a simply connected symmetric space and denote by G its Lie group of transvections. Choose a base point o in M and denote by H its stabilizer in G. The natural projection $\pi : G \to M : g \mapsto g(o)$ then defines an H-principal bundle and induces a linear projection $\pi_{\star e} : \mathfrak{g} \to T_o(M)$ where $\mathfrak{g}$ denotes the Lie algebra of G and $T_o(M)$ the tangent space to M at point o. Consider now the subgroup $G^{\tilde{\sigma}} \subset G$ of fixed points under the involutive automorphism $\tilde{\sigma} : G \to G : g \mapsto s_o g s_o$. The Lie algebra $\mathfrak{g}$ then decomposes as a direct sum of vector space $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathcal{P}$ with $\tilde{\sigma}_{\star e} =: \sigma =: \mathtt{id}_{\mathfrak{g}^+} \oplus (-\mathtt{id}_{\mathcal{P}})$. It then turns out that the Lie algebra $\mathfrak{g}^+$ of $G^{\tilde{\sigma}}$ coincides with the Lie algebra $\mathfrak{h}$ of the stabilizer H of o. Indeed, on the one hand, if Z is any element of $\mathfrak{g}$, one has $\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 \exp(-t(\sigma(Z) + Z))(o) = Z^\star_o - Z^\star_o = 0$ (where $Z^\star$ is the fundamental vector field associated with $Z \in \mathfrak{g}$) because[3] $s_{o_{\star o}} = -\mathtt{id}_{T_o(M)}$. Hence, the inclusion $\mathfrak{g}^+ \subset \mathfrak{h}$. On the other hand, for all $h \in H$ and $x \in M$, one

[3] This follows from the fact that o is an isolated fixed point of s_o

has $\tilde{\sigma}(h)(x) = s_o h s_o(x) = s_o h s_o h^{-1} h(x) = s_o s_{h(o)} h(x) = s_o^2 h(x) = h(x)$. Hence, $H \subset G^{\tilde{\sigma}}$.
The fact that the obtained iLa $(\mathfrak{g} = \mathfrak{h} \oplus \mathcal{P}, \sigma)$ is transvection easily follows from the fact that the transvection group G is the smallest subgroup of automorphisms of (M, s) which is stable under the conjugation by s_o.
At last, the canonical isomorphism of symmetric spaces $G/H \to M : gH \mapsto g(o)$ yields the announced bijection at the level of the isomorphism classes. □

Remark 2.3 At this level, one may observe that the Loos connection is the covariant derivative in the tangent bundle associated with the canonical connection in the H-structure $G \to G/H \simeq M$. Indeed, the proof of Theorem 2.3 shows that the homogeneous space G/H is reductive (the subspace $\mathcal{P}$ is Ad_H-invariant in $\mathfrak{g}$) in a canonical way. In particular, it is equipped with a canonical connection[4] one form (see, for example, [31]) $\alpha \in \Omega^1(G, \mathfrak{h})$ defined by

$$\alpha_g(\Xi) := \mathtt{pr}_{\mathfrak{h}} \left(L_{{g^{-1}}_{\star g}}(\Xi) \right)$$

where $\Xi \in T_g(G)$, where L denotes the left-translation action on G and where $\mathtt{pr}_{\mathfrak{h}}$ denotes the projection from $\mathfrak{g}$ onto $\mathfrak{h}$ parallel to the subspace $\mathcal{P}$.
Within that context, the Loos connection ∇ coincides with the covariant derivative in the tangent bundle $T(G/H) \simeq G \times_H \mathcal{P}$ associated with the canonical connection one form α.

Combining Theorems 2.2 and 2.3, we immediately get the following geometric decomposition theorem (analogue to the de Rham decomposition theorem in Riemannian geometry) which provides a well-defined categorical meaning to the Cartesian product in the class of symmetric spaces:

Theorem 2.4 *Let (M, s) be a simply connected symmetric space. And let*

$$(M, s) = \times_{i=1}^{r} (M_i, s_i) = \times_{j=1}^{\overline{r}} (\overline{M}_j, \overline{s}_j)$$

be two decompositions in indecomposable symmetric spaces. Then, $r = \overline{r}$ and there exists a permutation $\tau \in Sym(r)$ and an automorphism transformation $\phi \in Aut(M, s) = Aff(M, \nabla)$ of (M, s) such that for every $i = 1, ..., r$:

$$\phi(M_i) = \overline{M}_{\tau(i)} .$$

2.3 Symplectic symmetric spaces

In this section, we consider a class of symplectic manifolds. Recall that a symplectic manifold is a pair (M, ω) where M is a connected smooth manifold and where ω is a non-degenerate closed two-form on M. A symplectic diffeomorphism, or symplectomorphism, between two symplectic manifolds (M, ω) and (M', ω') is a

[4] sometimes called the Nomizu connection.

diffeomorphism $\varphi : M \to M'$ such that $\varphi^\star(\omega') = \omega$. In the case $(M, \omega) = (M', \omega')$, one speaks about symplectic transformation. The group of symplectic transformations is denoted by $Sp(M, \omega)$ or simply $Sp(\omega)$ when no confusion is possible.

Definition 2.9 [6, 12] A **symplectic symmetric space** is a triple (M, ω, s), where (M, s) is a symmetric space in the sense of Loos and where $\omega \in \Omega^2(M)$ is a non-degenerate two-form on M that is invariant under the symmetries: for every $x \in M$, one has

$$s_x^\star \omega = \omega .$$

Definition 2.10 Two symplectic symmetric spaces (M, ω, s) and (M', ω', s') are **isomorphic** if there exists a symplectic diffeomorphism $\varphi : (M, \omega) \to (M', \omega')$ such that $\varphi s_x = s'_{\varphi(x)} \varphi$. Such a φ is called an **isomorphism** of (M, ω, s) onto (M', ω', s'). When $(M, \omega, s) = (M', \omega', s')$, one talks about **automorphisms**. The group of all automorphisms of the symplectic symmetric space (M, ω, s) is denoted by $Aut(M, \omega, s)$.

Remark 2.2 yields

Proposition 2.3 *On a symplectic symmetric space (M, ω, s), the two-form ω is parallel with respect to the Loos connection:*

$$\nabla \omega = 0 .$$

In particular, ω is closed, hence symplectic.
Every symplectic symmetric space is therefore a homogeneous symplectic space of its transvection group.

In particular, the transvection group $G(M, s)$ of the underlying symmetric space (transitively) acts by symplectomorphisms on the symplectic manifold (M, ω). The homogeneous symplectic space underlying a symplectic symmetric space is therefore a symplectic cover of a coadjoint orbit. This aspect is considered in Sect. 4.

Examples 2.1 In dimension two, it turns out that the coadjoint orbit covered by a symplectic symmetric space itself admits a structure of symplectic symmetric space. We give here, up to isomorphism, the complete list of those two-dimensional coadjoint orbits. Four of them are simply connected and two are not.

1. The flat symplectic plane $(\mathbb{R}^2 = \{(q, p)\}\, , \omega = \lambda\, \mathrm{d}p \wedge \mathrm{d}q)$ $(\lambda \in \mathbb{R}_0)$ equipped with the Euclidean centred symmetries:

$$s_x y := 2x - y .$$

   The associated Loos connection if flat, hence metric. The transvection group is the translation group $(\mathbb{R}^2, +)$. It is a generic coadjoint orbit of the three-dimensional Heisenberg group. It is a Hermitian symmetric space, i.e. it admits a compatible complex structure: the one defined by the natural identification of $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

2. The two-sphere:

$$S^2_R \ := \ \{x \in \mathbb{R}^3 \ s.t. \ \|x\| \ = \ R\} \quad (R > 0)$$

equipped with its standard area form ω^R (induced by the embedding in the Euclidean space $\mathbb{R}^3$). The symmetry s_x at a point $x \in S^2_R$ is the axial rotation of π radians around the vector line directed by $\vec{0x}$. The Loos connection is the Levi–Civita connection w.r.t. the first fundamental form β induced by the embedding in the Euclidean 3D space. It is a generic coadjoint orbit of the rotation group $SO(3)$ which is its transvection group. It is a Käher symmetric space: the complex structure is the field of endomorphisms J of the tangent bundle $T(S^2_R)$ defined by $\beta(Ju, v) := \omega^R(u, v)$.

3. The hyperbolic plane $\mathbb{D}^2_R$ of negative curvature $-R^2$ ($R > 0$) realized as a connected component of an elliptic sphere in Minkowski space-time $\mathbb{R}^{(1,2)}$ with scalar product η of signature $(- + +)$:

$$\mathbb{D}^2_R \ := \ \{x \in \mathbb{R}^{(1,2)} \mid \eta(x \, , \, x) \ = \ -R^2\}_0 \ .$$

The symplectic structure ω^R is the area form that induced the natural volume form on the ambient Minkowski space. The symmetry s_x at a point $x \in \mathbb{D}^2_R$ is the axial Minkowski rotation of π radians around the vector line directed by $\vec{0x}$. The Loos connection is the Levi–Civita connection w.r.t. the first fundamental form β induced by the embedding in the Minkowski space. It is a generic coadjoint orbit of the Lorentz group $SO_0(1, 2)$ which is its transvection group. It is a Käher symmetric space: the complex structure is the field of endomorphisms J of the tangent bundle $T(\mathbb{D}^2_R)$ defined by $\beta(Ju, v) := \omega^R(u, v)$.

4. The "massive" generic coadjoint orbit $\mathcal{O}^m$ of the Poincaré group $G := O_0(1, 1) \ltimes \mathbb{R}^{(1,1)}$ in dimension $(1, 1)$. It is realized as a connected component of a generic sphere in $\mathbb{R}^3$ equipped with a degenerate scalar product of rank two whose isometry group is isomorphic to the Poincaré group. This degenerate ambient space $\mathbb{R}^3$ is isometric to the dual $\mathfrak{g}^\star$ of the Lie algebra $\mathfrak{g}$ of the Poincaré group equipped with the (dual) Killing metric. Within that realization, the orbit $\mathcal{O}^m$ stands as a cylinder over a branch of hyperbola (a mass-shell orbit of mass m). The symplectic structure ω^m is then given by the Kostant–Kirillov–Souriau two-form on the coadjoint orbit $\mathcal{O}^m$. Concerning the symmetric space structure and associated Loos connection, they can be described as follows. Observe first that the connected component $\mathbb{L}$ of the group of affine transformations of the real line (the $ax + b$ group) is naturally contained as a Lie subgroup in the Poincaré group: $\mathbb{L} \simeq O_0(1, 1) \ltimes N$ where N is a light ray in the vector subgroup $\mathbb{R}^{(1,1)}$. It turns out that the Lie group $\mathbb{L}$ simply transitively acts (by restriction of the coadjoint action) on the orbit $\mathcal{O}^m$. The affine group $\mathbb{L}$ is exponential in the sense that its exponential map realizes a global diffeomorphism between its Lie algebra $\mathfrak{L}$ and $\mathbb{L}$. Denoting by $H \in \mathfrak{L}$ an infinitesimal generator of $O_0(1, 1)$ and by $E \in \mathfrak{L}$ an infinitesimal generator of N, one gets, for any choice of base point o in $\mathcal{O}^m$, a global coordinate system on our

orbit:

$$\phi : \mathbb{R}^2 \to \mathcal{O}^m \subset \mathfrak{g}^\star : (a, n) \mapsto \mathrm{Ad}^\flat_{\exp(aH)\exp(nE)}(o) \ .$$

This coordinate system turns out to be a global Darboux chart on $\mathcal{O}^m$:

$$\phi^\star \omega^m \;=\; m\,\mathrm{d}a \wedge \mathrm{d}n \ .$$

Note that the map ϕ expresses an explicit realization of the orbit $(\mathcal{O}^m, \omega^m)$ as the cotangent bundle (equipped with its canonical Liouville symplectic structure) of the above-mentioned mass-shell orbit. The symmetric space structure can now be explicitly described within the coordinate system as a "deformation" of the flat structure: one has

$$s_{(a,n)}(a', n') \;=\; \left(2a \,-\, a' \,,\, 2\cosh(2(a - a'))\, n \,-\, n'\right) \ .$$

Formula (2) easily yields the Loos connection which is not a metric connection, i.e. it cannot be identified with the Levi–Civita connection for any non-degenerate metric on $\mathcal{O}^m$.

5. The generic coadjoint orbit $\mathcal{O}^R$ of the displacement group of the Euclidean plane $G := SO(2) \ltimes \mathbb{R}^2$. It is realized as a connected component of a generic sphere in $\mathbb{R}^3$ equipped with a degenerate scalar product of rank two whose group of isometries is isomorphic to $SO(2) \ltimes \mathbb{R}^2$. This degenerate ambient space $\mathbb{R}^3$ is isometric to the dual $\mathfrak{g}^\star$ of the Lie algebra $\mathfrak{g}$ of displacement group equipped with the (dual) Killing metric. Within that realization, the orbit $\mathcal{O}^R$ stands as a cylinder over a circle of radius R. The symplectic structure ω^R is then given by the Kostant–Kirillov–Souriau two-form on the coadjoint orbit $\mathcal{O}^R$. It is the Euclidean analogue of the above massive orbit of the Poincaré group (item 4.). As for the Poincaré case, the canonical Loos linear connection on $\mathcal{O}^R$ is not a metric connection.
6. Analogue to the hyperbolic plane (item 3.), the generic hyperbolic orbit AdS_2^R of positive curvature R^2 of the Lorentz group $SO_0(1, 2)$:

$$\mathrm{AdS}_2^R \;:=\; \{x \in \mathbb{R}^{(1,2)} \mid \eta(x \,,\, x) \;=\; R^2\} \ .$$

In the same way as for the hyperbolic plane, the anti-de Sitter space AdS_2^R can be realized as a coadjoint orbit of the Lorentz group equipped with its first fundamental form (of Lorentz signature in this case) induced by the (dual) Killing form on $\mathfrak{so}(1, 2)^\star$. The symmetries are defined in the same way as for the elliptic orbit (item 3.). Within this coadjoint orbit realization, the symplectic structure is the canonical one. The transvection group is the Lorentz group $SO_0(1, 2)$. The canonical Loos connection is the Levi–Civita connection associated with the Lorentz first fundamental form. It is therefore metric. However, in contrast with the hyperbolic plane, the anti-de Sitter space AdS_2^R is para-Hermitian in the sense of Kaneyuki [30] (c.f. Definition 2.12 below).

It is remarkable that each of those two-dimensional symplectic symmetric spaces is canonically associated with some kinematical Lie algebra of space-time dimension two. The present article shows that this is not accidental. In fact, all the structures described below are somehow already apparent in this two-dimensional situation.

2.4 Symplectic involutive Lie algebras

We started with the notion of involutive Lie algebra corresponding to the infinitesimal structure of a (affine) symmetric space [31].
Now, we pass to the symplectic context. The tangent version of a *symplectic* symmetric space (see [7, 12]) is the following one.

Definition 2.11 A **symplectic iLa** is a triple $(\mathfrak{g}, \sigma, \Omega)$ where $(\mathfrak{g}, \sigma)$ is an iLa and where Ω is an $\mathfrak{h}$-invariant non-degenerate bilinear two-form on $\mathcal{P}$. In particular, the pair $(\mathcal{P}, \Omega)$ is a *symplectic vector space*.

The dimension of $\mathcal{P}$ defines the **dimension** of the triple. Two such triples $(\mathfrak{g}_i, \sigma_i, \Omega_i)$ $(i = 1, 2)$ are **isomorphic** if there exists a Lie algebra isomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\psi \circ \sigma_1 = \sigma_2 \circ \psi$ and $\psi^*\Omega_2 = \Omega_1$.

Analogously as for the other types of symmetric spaces, we have

Theorem 2.5 *[12] There is a natural bijective correspondence between the isomorphism classes of simply connected symplectic symmetric spaces (M, ω, s) and the isomorphism classes of transvection symplectic iLa's $(\mathfrak{g}, \sigma, \Omega)$.*

The above bijection is easily obtained by combining Theorem 2.3 with the following basic observation:

Lemma 2.3 *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and H be a closed subgroup (hence Lie and embedded into G). Let $\mathfrak{h}$ be the Lie algebra of H. Then:*

(i) for every H-invariant (q, p)-multilinear-tensor $T \in \mathfrak{g}/\mathfrak{h}^{\otimes q} \otimes ((\mathfrak{g}/\mathfrak{h})^\star)^{\otimes p}$, the following formula defines a smooth G-invariant tensor field $\tilde{T}$ on the smooth manifold G/H:

$$\left\langle \tilde{T}_{gH}\,,\ \xi_1 \otimes \ldots \otimes \xi_q \otimes v_1 \otimes \ldots \otimes v_p \right\rangle$$
$$:= \left\langle T\,,\ (g_{\star H})^\star \xi_1 \otimes \ldots \otimes (g_{\star H})^\star \xi_q \otimes (g^{-1})_{\star gH}(v_1) \otimes \ldots \otimes (g^{-1})_{\star gH}(v_p) \right\rangle \tag{4}$$

where $\xi_i \in T^\star_{gH}(G/H)$ $(i = 1, ..., q)$ and $v_j \in T_{gH}(G/H)$ $(j = 1, ..., p)$ and where we canonically identify the tangent space $T_H(G/H)$ to the quotient vector space $\mathfrak{g}/\mathfrak{h}$.

(ii) Every smooth G-invariant tensor field on the smooth manifold G/H is of the above form.

Proof The Lie group H acts on $\mathfrak{g}/\mathfrak{h}$ by natural projection: $h(X + \mathfrak{h}) := \mathtt{Ad}_h(X) + \mathfrak{h}$ $(X \in \mathfrak{g}\,,\ h \in H)$ as the vector subspace $\mathfrak{h}$ is stable under the adjoint action restricted to H. One then readily checks that the element $\tilde{T}$ is well-defined as soon as it is invariant under this action of H naturally extended to tensors. Smoothness is then obvious by the very definition of the smooth structure on G/H. Item (ii) is immediate from evaluation to the base point $H \in G/H$. □

In the symplectic context, one also has a decomposition à la de Rham:

Theorem 2.6 *Let* $(\mathfrak{g}, \sigma, \Omega)$ *be a transvection siLa. And let*

$$(\mathfrak{g}\,,\,\sigma,\Omega)\;=\;\oplus_{i=1}^{r}(\mathfrak{g}_i\,,\,\sigma_i\,,\Omega_i)\;=\;\oplus_{j=1}^{\overline{r}}(\overline{\mathfrak{g}}_j\,,\,\overline{\sigma}_j\,,\overline{\Omega}_j)$$

be two decompositions into indecomposable siLa's (the direct sum of siLa's is defined in the obvious way). Then, $r = \overline{r}$ *and there exist a permutation* $\tau \in Sym(r)$ *and an automorphism* φ *of* $(\mathfrak{g}, \sigma, \Omega)$ *such that for all* $i \in \{1, ..., r\}$*:*

$$\varphi(\mathfrak{g}_i) \;=\; \overline{\mathfrak{g}}_{\tau(i)}\ .$$

In the class of symplectic symmetric spaces, one finds those whose canonical Loos connection is metric. For instance, Kähler (so-called Hermitian) symmetric spaces belong to that class. More generally, one has

Definition 2.12 Let $(\mathfrak{g}, \sigma, \Omega)$ be a symplectic triple. One says that it is **pseudo-Hermitian** (resp. **para-Hermitian**) if there exists an $\mathfrak{h}$-commuting symplectic endomorphism J of $\mathcal{P}$ such that its square is opposite (resp. equal) to the identity operator on $\mathcal{P}$. Formally, for every $Z \in \mathfrak{h}$:

$$\mathtt{ad}_Z|_{\mathcal{P}} \circ J \;=\; J \circ \mathtt{ad}_Z|_{\mathcal{P}} \quad\text{and}\quad J^2 \;=\; \pm\,\mathtt{id}_{\mathcal{P}}\ .$$

3 Generic kinematical Lie algebras are symplectic involutive

In this section, the ground field $\mathbb{K}$ of our generic kinematical Lie algebras is either the reals or the complex field. The first preliminary result is the following theorem which proves that to a generic kinematical Lie algebra is naturally associated a symmetric space.

Remark 3.1 The proof of Theorem 3.1 does not use the requirement on the simple $\mathfrak{s}$-module V to be absolutely simple.

Theorem 3.1 *Let* $(\mathfrak{g}, \mathfrak{s}, V)$ *be a generic kinematical Lie algebra. Then, the endomorphism* σ *of* $\mathfrak{g}$ *defined by*

$$\sigma \;:=\; \mathtt{id}_{\mathcal{Z}\oplus\mathfrak{s}}\ \oplus(-\mathtt{id}_{\mathcal{P}})$$

is an involutive automorphism of the Lie algebra $\mathfrak{g}$*. In other words, the pair* $(\mathfrak{g}, \sigma)$ *is an iLa.*

Proof The hypothesis on the isotypical component of V in $\Lambda^2(V)$ implies, by duality, that there is no non-trivial $\mathfrak{s}$-equivariant projection from $\Lambda^2(V^\star)$ onto $V^\star$. Using an $\mathfrak{s}$-invariant non-degenerate scalar product on V, one concludes that there is no non-trivial $\mathfrak{s}$-equivariant projection from $\Lambda^2(V)$ onto V. This entails that $\mathfrak{s}$ cannot non-trivially intersect the *weak* isotypical component of V in $\mathfrak{g}$. Indeed, suppose it does. Then this intersection contains a non-trivial simple $\mathfrak{s}$-module V_1 isomorphic to V, in particular V_1 is an ideal of $\mathfrak{s}$ (because it is contained in $\mathfrak{s}$). So the restriction of the Lie bracket of $\mathfrak{s}$ to V_1 yields an $\mathfrak{s}$-equivariant linear map r from $\Lambda^2(V_1)$ to V_1. But $V_1 \cong V$ is $\mathfrak{s}$-irreducible; therefore, the image of r must be isomorphic to either V or trivial. The hypothesis on isotypical components implies that V_1 is an Abelian ideal in $\mathfrak{s}$. As this Abelian ideal is isomorphic to V as an $\mathfrak{s}$-module, this would contradict the hypothesis of faithfulness of V as a module of $\mathfrak{s}$.

In the same way, considering a decomposition of $\mathcal{P}$ into isomorphic simple $\mathfrak{s}$-modules (each of them isomorphic to V):

$$\mathcal{P} \;=\; \mathcal{P}_0 \;\oplus\; \mathcal{P}_1 \;,$$

for all $i, j = 0, 1$, the $\mathfrak{s}$-module $[\mathcal{P}_i, \mathcal{P}_j]$ cannot intersect the isotypical component of V in $\mathfrak{g}$.

Indeed, let us first choose two isomorphisms of $\mathfrak{s}$-modules:

$$\phi_k : V \to \mathcal{P}_k : v \mapsto v_k \quad (k = 0, 1) \;.$$

Consider the decomposition into $\mathfrak{s}$-submodules:

$$\Lambda^2(\mathcal{P}) \;=\; \Lambda^2(\mathcal{P}_0) \;\oplus\; \Lambda^2(\mathcal{P}_1) \;\oplus\; \mathcal{P}_0 \wedge \mathcal{P}_1$$

where $\mathcal{P}_0 \wedge \mathcal{P}_1$ is the linear subspace of $\Lambda^2(\mathcal{P})$ generated by the elements of the form $p_0 \wedge p'_1$ with $p_0 \in \mathcal{P}_0$ and $p'_1 \in \mathcal{P}_1$.

Denote by

$$C : \Lambda^2(\mathcal{P}) \to \mathfrak{g}$$

the homomorphism of $\mathfrak{s}$-modules given by

$$C(p \wedge p') \;:=\; [p, p'] \;.$$

Now, given i and j, define the homomorphism of $\mathfrak{s}$-modules

$$\Delta_{ij} : \Lambda^2(V) \to \Lambda^2(\mathcal{P}) : v \wedge w \mapsto v_i \wedge w_j \;.$$

Denote by

$$\mathtt{pr}_i : \mathfrak{g} \to \mathcal{P}_i$$

the natural projection parallel to $\mathfrak{h} \oplus \mathcal{P}_{\bar{i}}$ with $\bar{i} \neq i$. And finally consider the homomorphisms of $\mathfrak{s}$-modules

$$\Delta_{ijk} : \Lambda^2(V) \to V$$

defined by

$$\Delta_{ijk} \;:=\; \phi_k^{-1} \circ \mathtt{pr}_k \circ C \circ \Delta_{ij} \;.$$

The hypothesis on the isotypical component of V in $\Lambda^2(V)$ implies that these maps Δ_{ijk} are all identically zero. Therefore, observing that

$$\phi_k^{-1}\left(\mathtt{pr}_k[v_i, w_j]\right) \;=\; \Delta_{ijk}(v \wedge w) \;,$$

one obtains the inclusion

$$[\mathcal{P}\,,\,\mathcal{P}]\;\subset\;\mathfrak{h}\;:=\;\mathcal{Z}\oplus\mathfrak{s}\;.$$

Let us now analyse the space $[\mathcal{Z},\mathcal{P}]$. If, for some $j=0,1$, the $\mathfrak{s}$-submodule $[\mathcal{Z},\mathcal{P}_j]$ is non-trivial, it must be isomorphic to V as an $\mathfrak{s}$-module. What has been observed in the first preceding paragraph combined with a dimensional argument, $[\mathcal{Z},\mathcal{P}_j]$ therefore lives in the isotypical component of V in $\mathfrak{g}$, that is, it lives in $\mathcal{P}$. Hence, the structure of involutive Lie algebra. □

We end this section by introducing a map $\Omega:\mathcal{P}\times\mathcal{P}\to\mathbb{K}$, with no restriction on the ground field ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$), which will be crucial in the rest of this article, together with the following

Lemma 3.1 *The $\mathbb{K}$-bilinear map*

$$\Omega:\mathcal{P}\times\mathcal{P}\to\mathfrak{h}/\mathfrak{s}\;\simeq\;\mathbb{K}:(X,Y)\mapsto[X,Y]\;+\;\mathfrak{s}$$

is $\mathfrak{h}$-equivariant.

Proof Since $\mathcal{Z}$ is central in $\mathfrak{h}$, for all $H\in\mathfrak{h}$ and $X,Y\in\mathcal{P}$, the iLa structure implies

$$\Omega([H,X]\,,\,Y)\;+\;\Omega(X\,,\,[H,Y])\;=\;[[H,X]\,,\,Y]\;+\;[X\,,\,[H,Y]]\;+\;\mathfrak{s}\;.$$

Then, by using the Jacobi identity and that fact that $\mathcal{Z}$ is central in $\mathfrak{h}$, one finds $[[H,X],\,Y]\;+\;[X,\,[H,Y]]\equiv-[H,[X,Y]]\in[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{s}$. By definition of Ω, we therefore obtained that $\Omega([H,X],\,Y)\;+\;\Omega(X,\,[H,Y])\;=0$, which proves the $\mathfrak{h}$-invariance of Ω. □

The iLa structure being established, we can now pass to the additional symplectic structure. We will first consider the case of a complex generic kinematical Lie algebra.

3.1 The complex case

In this paragraph, we consider the case where the base field $\mathbb{K}=\mathbb{C}$ is the one of complex numbers. It means that the Lie algebras and modules are all over the complex numbers.

Remark 3.2 The proofs of the results in this paragraph do not use the requirement on the simple $\mathfrak{s}$-module V to be absolutely simple.

Theorem 3.2 *Let $(\mathfrak{g},\mathfrak{s},V)$ be a complex generic kinematical Lie algebra. Then*

(i) if the (complex bilinear) map

$$\Omega:\mathcal{P}\times\mathcal{P}\to\mathfrak{h}/\mathfrak{s}:(X,Y)\mapsto[X,Y]+\mathfrak{s}$$

has a trivial radical, then the triple $(\mathfrak{g},\sigma,\Omega)$ is a (complex) siLa (where we (non-canonically) identify $\mathfrak{h}/\mathfrak{s}$ to the field of complex numbers $\mathbb{C}$).

(ii) If the above map Ω admits a non-trivial radical, then the symmetric space associated with the iLa $(\mathfrak{g}, \sigma)$ is decomposable.
(iii) When decomposable, the symmetric space is flat. In particular, since it is even-dimensional and flat, it admits a structure of symplectic symmetric space.

Proof The radical of Ω

$$D \;:=\; \mathrm{rad}(\Omega) \;:=\; \{X \in \mathcal{P} \mid \Omega(X, Y) \;=\; 0\;,\; \forall\, Y \in \mathcal{P}\}$$

is an $\mathfrak{h}$-submodule of $\mathcal{P}$—for $X \in \mathrm{rad}(\Omega)$ and $H \in \mathfrak{h}$, one has $\Omega([H, X], Y) = -\Omega(X, [H, Y])$ which vanishes because $[H, Y] \in \mathcal{P}$ and $X \in \mathrm{rad}(\Omega)$, hence $[H, X] \in \mathrm{rad}(\Omega)$ whenever $X \in \mathrm{rad}(\Omega)$—such that

$$[D\,,\,\mathcal{P}] \;\subset\; \mathfrak{s}\;, \tag{5}$$

by definition of Ω. As a result, the quotient space $\mathcal{P}/D$ is an $\mathfrak{h}$-module.
As an $\mathfrak{s}$-submodule, D must either be reduced to zero, equal to $\mathcal{P}$ or isomorphic to V.

1. Suppose $D \;\cong\; V$. In that case, the $\mathfrak{h}$-module $\mathcal{P}/D$ is isomorphic to V. Let us consider the $\mathfrak{s}$-commuting action of $\mathcal{Z} \subset \mathfrak{h}$ on the $\mathfrak{h}$-module $\mathcal{P}/D$. Since the latter is a simple $\mathfrak{s}$-module isomorphic to V, Schur's lemma implies that the latter action must be proportional to the identity. On the other hand, since the $\mathfrak{h}$-module $\mathcal{P}/D$ is symplectic—for any $X \in \mathcal{P}/D$, the condition $\Omega(X, Y) = 0\; \forall\; Y \in \mathcal{P}/D$ implies that $X = 0$, so that $\Omega|_{\mathcal{P}/D\times\mathcal{P}/D}$ is non-degenerate; moreover, the the $\mathfrak{h}$-invariance of Ω is carried to $\mathcal{P}/D \times \mathcal{P}/D$—, we conclude that the proportionality constant must be zero, since the identity map cannot be symplectic. In particular, for every $\mathfrak{s}$-module W supplementary to D in $\mathcal{P}$, one must have

   $$[\mathcal{Z}\,,\,W] \;\subset\; D\;.$$

   Still in the situation where D is isomorphic to V, let $d \in D$. Then, since $[W, D]$ lives in $\mathfrak{s}$—recall (5)—, the identity

   $$0 \;=\; [w_1,\,[w_2,\,d]] \;+\; [d,\,[w_1,\,w_2]] \;+\; [w_2,\,[d,\,w_1]]$$

   implies that the middle term lives in D while both the other ones live in W. Therefore, we have the relation:

   $$[[W\,,\,W]\,,\,D] \;=\; \{0\}\,. \tag{6}$$

   On the other hand, since D is an $\mathfrak{h}$ (hence $\mathfrak{s}$) module and because $\mathcal{Z}$ commutes with $\mathfrak{s}$, Schur's lemma implies that the action of $\mathcal{Z}$ on $D \simeq V$ is scalar. Considering the components

   $$[W\,,\,W] \;\subset\; [W\,,\,W]_{\mathcal{Z}} \;\oplus\; [W\,,\,W]_{\mathfrak{s}}$$

   according to the above decomposition of $\mathfrak{h}$, we conclude, from the inclusion (6), that the action of $[W,\; W]_{\mathfrak{s}}$ on D must be scalar too, in the sense that there exists a one form α on $[W, W]_{\mathfrak{s}}$ such that for every $T \in [W, W]_{\mathfrak{s}}$, one has

$$\mathtt{ad}_T|_D \;=\; \alpha(T)\,\mathtt{id}_D\;.$$

Now, as $\mathfrak{s}$-modules, D, W and $\mathcal{P}/D$ are isomorphic. And on the last one, there exists, by construction, an $\mathfrak{s}$-invariant symplectic structure. So, such an $\mathfrak{s}$-invariant symplectic structure exists on the three of them (possibly different from the restriction of Ω). Therefore, for every T, one must have $\alpha(T) = 0$, i.e. α must be identically zero.

Now, by the fact that $D \cong V$ is acted upon faithfully by $\mathfrak{s}$, the relation $\mathtt{ad}_{[W,W]_{\mathfrak{s}}}|_D = 0$ implies that $[W, W]_{\mathfrak{s}} = \{0\}$, hence $[W, W] \subset \mathcal{Z}$. Since $\mathcal{Z}$ is one-dimensional, one has either $[W, W] = \mathcal{Z}$ or $[W, W] = \{0\}$. But since W is symplectic w.r.t. Ω, we get the equality:

$$[W\,,\,W] \;=\; \mathcal{Z}$$

implying from (6) that

$$[\mathcal{Z}, D] \;=\; \{0\}\;.$$

We recall that we had obtained $[\mathcal{Z}, W] \subset D$ earlier.

We will now prove, still under the hypothesis that D is isomorphic to V, that $[\mathcal{P}, \mathcal{P}] \subset \mathcal{Z}$. In order to do so, we first remark that D is Abelian. Indeed, Jacobi tells us that

$$[[D\,,\,D]\,,\,W] \;\subset\; [[W, D]\,,\,D] \;\subset\; D\;.$$

Since $[D, D] \subset \mathfrak{s}$, one also has $[[D,\, D],\, W] \subset W$. Hence

$$[[D\,,\,D]\,,\,W] \;\subset\; D \cap W \;=\; \{0\}\;.$$

From the faithfulness of W as a $\mathfrak{s}$-module, one must have $[D, D] = \{0\}$.

To further proceed in proving the inclusion $[\mathcal{P}, \mathcal{P}] \subset \mathcal{Z}$, we now consider the following contraction of our iLa by considering the new bracket $[.,.]_0$ on $\mathfrak{g}_0 := \mathfrak{g}$ defined by (with obvious notations):

$$\begin{aligned}
[W\,,\,W]_0 \;&:=\; \{0\}\\
[\mathcal{Z}\,,\,\mathfrak{g}_0]_0 \;&:=\; \{0\}\\
[\mathfrak{g}_0\,,\,D]_0 \;&:=\; [\mathfrak{g}\,,\,D]\\
[\mathfrak{s}\,,\,\mathfrak{g}_0]_0 \;&:=\; [\mathfrak{s}\,,\,\mathfrak{g}]\;.
\end{aligned}$$

The space $\mathfrak{g}_0 \;=\; \mathfrak{h} \;\oplus\; \mathcal{P}$ again underlies an iLa. Indeed, consider $d \in D$ and $w_1, w_2 \in W$. Then, using (5) and (6):

$$\begin{aligned}
0 &= [w_1,\,[w_2,\,d]] \;+\; [d,\,[w_1,\,w_2]] \;+\; [w_2,\,[d,\,w_1]]\\
&= [w_1,\,[w_2,\,d]] \;+\; [w_2,\,[d,\,w_1]]\\
&= [w_1,\,[w_2,\,d]_0] \;+\; [w_2,\,[d,\,w_1]_0]
\end{aligned}$$

which equals

$$\begin{aligned}&[w_1\,,\,[w_2\,,\,d]_0]_0\;+\;[w_2\,,\,[d\,,\,w_1]_0]_0\\&\quad=\;[w_1\,,\,[w_2\,,\,d]_0]_0\;+\;[w_2\,,\,[d\,,\,w_1]_0]_0\;+\;[d\,,\,[w_1\,,\,w_2]_0]_0\end{aligned}$$

since $[W, D]$ lives in $\mathfrak{s}$. The remaining Jacobi identities are immediate.
Now, since $\mathfrak{k}_0 := [\mathcal{P},\,\mathcal{P}]_0 \subset \mathfrak{s}$ and $\mathfrak{s}$ acts faithfully on $\mathcal{P}$, the space $\mathfrak{k}_0\,\oplus\,\mathcal{P} \subset \mathfrak{g}_0$ is the transvection iLa of a (pseudo-)Riemannian symmetric space. The decomposition $\mathcal{P}\;=\;W \oplus D$ is stable under the action of the holonomy Lie algebra $\mathfrak{k}_0$; hence, as a consequence of the de Rham-Wu decomposition theorem ([41]), this symmetric space is reducible, entailing

$$[D, W]\;=\;[D, W]_0\;=\;\mathfrak{k}_0\;=\;\{0\}\;.$$

This means that the pseudo-Riemannian symmetric space at hand is flat.
We now resume to our analysis of the kinematical algebra $\mathfrak{g}$ and cease considering $\mathfrak{g}_0$. We choose a generator Z_0 of $\mathcal{Z}$ and consider the $\mathfrak{s}$-intertwiner

$$\rho\;:=\;\mathtt{ad}_{Z_0}|_W\;:W \to D\;.$$

Consider the symplectic vector space (W, Ω^W) defined by

$$[w, w']\;:=\;\Omega^W(w, w')\,Z_0\;.$$

Then, the Jacobi identity tells us that

$$0\;=\;\oint_{1,2,3}[w_1\,,\,[w_2\,,\,w_3]]\;=\;-\oint_{1,2,3}\Omega^W(w_2, w_3)\,\rho(w_1)$$

which non-degeneracy of Ω^W forces

$$\rho\;\equiv\;0\,,$$

which shows that $[\mathcal{Z}, W] = \{0\}$. Above, we had found that $[\mathcal{Z}, D] = \{0\}$; hence, we have that $[\mathcal{Z},\mathcal{P}]\;=\;\{0\}$. The Lie algebra $\mathfrak{g}$, in the case $\mathrm{rad}(\Omega)\;\cong\;V$ is therefore the semidirect product of $\mathfrak{s}$ acting on a decomposable flat iLa whose one direct factor is a Heisenberg Lie algebra $(\mathcal{Z}\,\oplus W)$:

$$\mathfrak{g}\;=\;\mathfrak{s} \ltimes (D \oplus (\mathcal{Z}\,\oplus W))\;.$$

2. In case the radical $\mathtt{rad}(\Omega)$ is the entire space $\mathcal{P}$, i.e. $\mathfrak{K}\;:=\;[\mathcal{P},\mathcal{P}]\;\subset\;\mathfrak{s}$, the space $\mathfrak{K} \oplus \mathcal{P}$ is the transvection iLa of a holonomy reducible (pseudo-)Riemannian symmetric space [42]. In particular, the holonomy Lie algebra $\mathfrak{K}$ splits into a direct sum of ideals:

$$\mathfrak{K}\;=\;\mathfrak{K}_0\;\oplus\;\mathfrak{K}_1$$

that are such that, say, $[\mathfrak{K}_0, \mathcal{P}_1] = \{0\}$, contradicting the faithfulness of $\mathcal{P}_1 \simeq V$ as soon as $\mathfrak{K}$ is non-trivial. Therefore, since $\mathfrak{s}$ acts faithfully on $\mathcal{P}$, we conclude that $\mathfrak{K} = [\mathcal{P}, \mathcal{P}] = \{0\}$. We therefore have

$$\mathfrak{g} = \mathfrak{s} \ltimes (\mathcal{P} \oplus \mathcal{Z}) \ .$$

The only constraint we have on $\mathcal{Z}$ is that it is a line in the centralizer of $\mathfrak{s}$, in the endomorphisms of $\mathcal{P}$.
All the possibilities for the action of $\mathcal{Z}$ on $\mathcal{P}$ are possible.

3. It remains to analyse the case where $(\mathcal{P}, \Omega)$ is symplectic, i.e. the iLa underlies a symplectic symmetric space modelled on $(\mathcal{P}, \Omega)$. We assume here that the associated symmetric space is decomposable.

   Let us first consider the case where $\mathcal{Z}$ is central in $\mathfrak{g}$. In this case, the holonomy $\mathfrak{K}$ of the transvection Lie algebra is a subalgebra of $\mathfrak{s}$ (because $\mathfrak{s}$ faithfully acts on $\mathcal{P}$). On the other hand, for decomposability, there exists a $\mathfrak{K}$-stable decomposition $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1$ with commuting factors $[\mathcal{P}_0, \mathcal{P}_1] = \{0\}$. Note that $[\mathcal{P}_0, \mathcal{P}_0] \subset \mathfrak{K}$ trivially acts on $\mathcal{P}_1$, contradicting the faithfulness of the action of $\mathfrak{s}$ on $V \simeq \mathcal{P}_1$ if non-trivial. And similarly for $[\mathcal{P}_1, \mathcal{P}_1]$, both factors are therefore flat, i.e. $[\mathcal{P}_0, \mathcal{P}_0] = [\mathcal{P}_1, \mathcal{P}_1] = \{0\}$. Hence, $\mathfrak{K} = \{0\}$.

   Now, let us consider the case where $\mathcal{Z}$ non-trivially acts on $\mathcal{P}$. In that case, the holonomy is of the form:

$$\mathfrak{K} = \mathcal{Z} \oplus \mathfrak{K}_{\mathfrak{s}}$$

   where $\mathfrak{K}_{\mathfrak{s}} \subset \mathfrak{s}$. Under the hypothesis of decomposability, the same argument as above implies $\mathfrak{K}_{\mathfrak{s}} = \{0\}$, which yields, say, $[\mathcal{P}_0, \mathcal{P}_0] = \mathcal{Z}$ hence, necessarily, $[\mathcal{P}_1, \mathcal{P}_1] = \{0\}$, in contradiction with the fact that Ω is symplectic.

□

We end this section by observing the following dichotomy concerning the action of the (complex) line $\mathcal{Z}$ on $\mathcal{P}$:

Proposition 3.1 *The action of the (complex) line $\mathcal{Z}$ on $\mathcal{P}$ is either nilpotent or semisimple. Moreover, when nilpotent, the action of $\mathcal{Z}$ on $\mathcal{P}$ squares to zero.*

Proof Let Z_0 be a generator of the complex line $\mathcal{Z}$ and denote by A its (complex linear) action on $\mathcal{P}$:

$$A := \mathtt{ad}_{Z_0}|_{\mathcal{P}} : \mathcal{P} \to \mathcal{P} \ .$$

The endomorphism A uniquely decomposes (Jordan decomposition) into a sum of two commuting endomorphisms:

$$A =: S + N \tag{7}$$

where S is semisimple and N nilpotent. Remind that, as Ω is $\mathfrak{h}$-invariant, the endomorphism A is symplectic: $A \in \mathfrak{sp}(\mathcal{P}, \Omega)$.

Assume the kernel $\mathcal{N}$ of N to be a proper subspace of $\mathcal{P}$ (i.e. $\{0\} \neq \mathcal{N} \neq \mathcal{P}$). The operator $S|_{\mathcal{N}}$ acts as A on $\mathcal{N}$ and therefore commutes with $\mathfrak{s}$. Hence, it must be scalar:

$$S|_{\mathcal{N}} =: \lambda\,\mathtt{id}_{\mathcal{N}} \quad (\lambda \in \mathbb{C}).$$

Now, if S admits another eigenvalue, say $\mu \neq \lambda$, then denoting by $\mathcal{P}_\mu$ the corresponding eigenspace in $\mathcal{P}$, one has

$$N(\mathcal{P}_\mu) \subset \mathcal{P}_\mu .$$

But N cannot admit a kernel in $\mathcal{P}_\mu$ because that one would be contained in $\mathcal{N} \subset \mathcal{P}_\lambda$. The operator N being nilpotent, this yields a contradiction. Hence, when $\mathcal{N}$ is proper in $\mathcal{P}$, the semisimple part S of A must act as a scalar on the entire $\mathcal{P}$. Now, since $\mathfrak{sp}(\mathcal{P}, \Omega)$ is semisimple, it contains both nilpotent and semisimple components of any of its elements. Therefore, S must be identically zero (because scalar).
At last, observe that the kernel $\mathcal{N}$ of $N = \mathtt{ad}_{Z_0}|_{\mathcal{P}}$ is invariant under the action of $\mathfrak{h}$. If symplectic, the argument in the proof of Proposition 1.2 page 249 of [14] implies that it commutes with its orthogonal subspace in $\mathcal{P}$. Hence, Theorem 3.1 item (ii) implies flatness.
The image $\mathcal{I} := N(\mathcal{P})$ of $N = \mathtt{ad}_{Z_0}|_{\mathcal{P}}$ is $\mathfrak{s}$-invariant. Indeed, if $N(X)$ $(X \in \mathcal{P})$ is an element of $\mathcal{I}$, we have for every $T \in \mathfrak{s}$: $[T, N(X)] = [T, [Z_0, X]] = [Z_0, [T, X]] = N([T, X]) \in \mathcal{I}$. The subspace $\mathcal{I}$ therefore cannot properly intersect $\mathcal{N}$, because this intersection, whose intersection is strictly smaller than the one of $\mathcal{N}$, would then belong to another isotypic component than the one of V in $\mathcal{P}$. This implies $\mathcal{I} = \mathcal{N}$, i.e. $N^2 = 0$. □

Proposition 3.2 *Let us assume the complex symplectic iLa $(\mathfrak{g}, \sigma, \Omega)$ to be indecomposable and non-flat. Assume the endomorphism A to be semisimple. Then:*

(i) the space $\mathcal{P}$ decomposes into a direct sum of two $\mathfrak{h}$-invariant transverse Lagrangian subspaces in duality:

$$\mathcal{P} = L \oplus \overline{L} .$$

(ii) Both L and $\overline{L}$ are Abelian subalgebras of $\mathfrak{g}$.
(iii) Both L and $\overline{L}$ are simple $\mathfrak{s}$-modules.
(iv) The endomorphism A acts on L and $\overline{L}$ with opposite eigenvalues.

Proof Since, in this case, A is semisimple, the space $\mathcal{P}$ decomposes into $\mathcal{Z}$-root spaces:

$$\mathcal{P} := \oplus_{\alpha\in\Phi}\mathcal{P}_\alpha$$

where $\Phi \subset \mathcal{Z}^\star$. Each of these root spaces being $\mathfrak{h}$-invariant, the cardinality of Φ is at most two. In the case it is equal to two, the argument in the proof of Proposition 1.2 page 249 of [14] yields the assertion.
At last, the endomorphism A cannot be scalar as it would not be symplectic in that case. For the same reason, in view of the matrix form of Ω in the decomposition (i), one gets item (iv).

Note also that the map $\overline{L} \to L^\star$ canonically defined by the bilinear two-form Ω admits a non-trivial kernel only if the radical of Ω is non-trivial for both subspaces are isotropic. It is not the case in the present situation; hence, the Lagrangian subspaces L and $\overline{L}$ are in duality. □

Warning: From now on, all the symmetric spaces considered below will be indecomposable and non-flat.

3.2 The real case

Let us now come back to the case where the ground field $\mathbb{K} = \mathbb{R}$ is the reals. In this real case, *we restore the hypothesis on the real* $\mathfrak{s}$*-module* V *to be absolutely simple*. We start by observing

Proposition 3.3 *Let* $(\mathfrak{g}, \mathfrak{s}, V)$ *be a real kinematical Lie algebras with indecomposable and non-flat associated iLa* $(\mathfrak{g}, \sigma)$*. Then, the iLa* $(\mathfrak{g}, \sigma)$ *canonically underlies a symplectic iLa* $(\mathfrak{g}, \sigma, \Omega)$.

Proof We consider the element $\Omega : \mathcal{P} \times \mathcal{P} \to \mathfrak{h}/\mathfrak{s} : (X, Y) \to [X, Y] + \mathfrak{s}$. As V is an absolutely simple real $\mathfrak{s}$-module, we have seen that the complexified iLa $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathcal{P}^{\mathbb{C}}$ associated with the complex generic kinematical $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{s}^{\mathbb{C}}, V^{\mathbb{C}})$ Lie algebra becomes symplectic when equipped with the complexified element $\Omega^{\mathbb{C}}$. This implies that the radical of the real element Ω must be trivial. □

3.2.1 The action of $\mathcal{Z}$ on $\mathcal{P}$ is semisimple

According to the dichotomy resulting from Proposition 3.1, we first investigate the case where the action of $\mathcal{Z}$ on $\mathcal{P}$ to be (possibly complex) semisimple.
As earlier, we consider a generator Z_0 of the $\mathbb{K}$-line $\mathcal{Z}$ and denote by A its action on $\mathcal{P}$:

$$A \;:=\; \mathtt{ad}_{Z_0}|_{\mathcal{P}} \;: \mathcal{P} \to \mathcal{P}\;.$$

The complex linear extension $A^{\mathbb{C}}$ of A on the complexified space $\mathcal{P}^{\mathbb{C}}$ is semisimple, and one has the decomposition into $\mathcal{Z}^{\mathbb{C}}$-root spaces:

$$\mathcal{P}^{\mathbb{C}} \;:=\; \mathcal{P}_\alpha \;\oplus\; \mathcal{P}_{-\alpha}$$

where $\alpha \in (\mathcal{Z}^{\mathbb{C}})^\star$.
Denoting by $X \mapsto \overline{X}$ the conjugation of $\mathfrak{g}^{\mathbb{C}}$ associated with the real form $\mathfrak{g}$, one observes that for every Z in $\mathcal{Z} \subset \mathcal{Z}^{\mathbb{C}}$:

$$\overline{[Z\,,\,X_\alpha]} \;=\; \overline{\alpha(Z)}\,\overline{X_\alpha} \;=\; [\overline{Z}\,,\,\overline{X_\alpha}] \;=\; [Z\,,\,\overline{X_\alpha}]\;.$$

Hence, $\overline{X_\alpha}$ is an A-eigenvector, implying that $\lambda := \alpha(Z_0)$ is either real or purely imaginary.

Lemma 3.2 *In the case λ is real, A is real semisimple and $\mathcal{P}$ admits an $\mathfrak{h}$-invariant para-complex structure that is realized by the action of an element of $\mathcal{Z}$.*

Proof We set $X_\alpha =: u_\alpha + i\, v_\alpha$ with u_α and v_α in $\mathcal{P}$. We then observe:

$$A(X_\alpha) = \lambda(X_\alpha) = \lambda u_\alpha + i\lambda\, v_\alpha = A(u_\alpha) + i\, A(v_\alpha)\;,$$

hence

$$A(u_\alpha) = u_\alpha$$

and similarly for $\mathcal{P}_{-\alpha}$. □

Following the same lines, we get

Lemma 3.3 *In the case λ is purely imaginary, $\mathcal{P}$ admits an $\mathfrak{h}$-invariant complex structure that is realized by the action of an element of $\mathcal{Z}$.*

As an immediate corollary, we have

Proposition 3.4 *Let $(\mathfrak{g}, \mathfrak{s}, V)$ be a real generic kinematical Lie algebra with indecomposable non-flat associated iLa $(\mathfrak{g}, \sigma, \Omega)$. Suppose the action of $\mathcal{Z}$ on $\mathcal{P}$ to be semisimple. Then:*

- *(i) if the action of $\mathcal{Z}$ is not real semisimple, then the symmetric space admits a symplectic-compatible complex structure: the siLa is pseudo-Hermitian.*
- *(ii) If the action of $\mathcal{Z}$ is real semisimple, then the symmetric space admits a symplectic-compatible para-complex structure; the siLa is para-Hermitian.*

The above result classifies the generic kinematical Lie algebras where $\mathfrak{s}$ is semisimple. Indeed, one has

Theorem 3.3 *Let $(\mathfrak{g}, \mathfrak{s}, V)$ be a generic kinematical Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that $\mathfrak{s}$ is semisimple and the action of $\mathcal{Z}$ on $\mathcal{P}$ is semisimple. Assume the associated siLa to be indecomposable and non-flat. Then*

- *(i) $\mathfrak{g}$ is simple.*
- *(ii) The associated siLa $(\mathfrak{g}, \sigma, \Omega)$ is the transvection siLa of a simple symplectic symmetric space.*
- *(iii) The Lie algebra $\mathfrak{s}$ is compact if and only if the symplectic symmetric space is Hermitian.*
- *(iv) The Lie algebra $\mathfrak{s}$ is complex if and only if the symplectic symmetric space is hyper-Kähler.*
- *(v) The Lie algebra $\mathfrak{s}$ is not complex nor compact if and only if the symplectic symmetric space is causal of Cayley type (para-Hermitian).*

Reciprocally, every simple symplectic iLa is the associated siLa with a generalized a generic kinematical Lie algebra such that $\mathfrak{s}$ is semisimple and the action of $\mathcal{Z}$ on $\mathcal{P}$ is semisimple.

Proof The fact that the action of $\mathfrak{h}$ is reductive on $\mathcal{P}$ forces $\mathfrak{g}$ to be a reductive Lie algebra. By indecomposability of the symmetric space, it must be simple [12, 14]. Items (ii)-(v) then follow from [12, 14]. Regarding the last assertion is structural, it immediately follows from the fact that the simple siLa are the simple iLa whose Levi factor of the holonomy act reducibly on $\mathcal{P}$. □

3.2.2 The Levi condition

Motivated by the classical notion of kinematical Lie algebra, i.e. $\mathfrak{s} \simeq \mathfrak{so}(D)$, we observe

Proposition 3.5 *Let* $(\mathfrak{g}, \sigma)$ *be an iLa and* $\mathfrak{s}$ *be a compact and semisimple Lie subalgebra of* $\mathfrak{g}$. *Then it is contained in a* σ*-stable Levi factor of* $\mathfrak{g}$.

The proof follows from a few observations. First, we have

Proposition 3.6 *Let* K *be a connected compact real Lie group. Let* $K = TK_0$ *be a decomposition of* K *into a Lie group direct product of its centre* T *and a compact semisimple Lie group* K_0.
Let S *be a semisimple Lie subgroup of* K *with no centre. Then* S *is contained in* K_0, *which is in particular unique.*

Proof It is sufficient to assume S simple (apply the argument below to every of its simple components). Let $\tau : S \to T$ be the map defined by the global decomposition $K = TK_0$. The decomposition $K = TK_0$ is a direct product of groups; hence, the map τ is a Lie group homomorphism. Therefore, its kernel is a normal subgroup of S and one concludes by simplicity. □

Therefore, one has the

Corollary 3.1 *Every compact semisimple Lie subalgebra* $\mathfrak{s}$ *of a real Lie algebra* $\mathfrak{g}$ *is contained in a Levi factor of* $\mathfrak{g}$.

Proof Let $\mathfrak{K}_0$ be a maximal compact Lie subalgebra of a Levi factor $\mathfrak{L}$ of $\mathfrak{g}$. And let $\mathfrak{K}$ be a maximal compact Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{K}_0$. Let $\mathcal{R}$ be the solvable radical of $\mathfrak{g}$ and consider the projection $\pi_{\mathfrak{L}}$ from $\mathfrak{g}$ onto $\mathfrak{L}$ parallel to $\mathcal{R}$. The map $\pi_{\mathfrak{L}}$ is a Lie algebra homomorphism. Therefore, $\pi_{\mathfrak{L}}(\mathfrak{K})$ is a compact subalgebra in $\mathfrak{L}$, contained in $\mathfrak{K}_0$. Since $\mathfrak{K}$ contains $\mathfrak{K}_0$, one has the equality $\mathfrak{K}_0 = \pi_{\mathfrak{L}}(\mathfrak{K})$ and the splitting

$$\mathfrak{K} \;=\; \mathfrak{K}_0 \oplus (\mathfrak{K} \cap \mathcal{R})\;.$$

Indeed, for every $k \in \mathcal{K}$, one has $k = \pi_{\mathfrak{L}}(k) \oplus \pi_{\mathcal{R}}(k)$. Since $\pi_{\mathfrak{L}}(k)$ belongs to $\mathcal{K}$ as we proved above, we have $\pi_{\mathcal{R}}(k) = k - \pi_{\mathfrak{L}}(k)$ belongs to $\mathcal{K}$ and $\mathcal{R}$. Hence, $\pi_{\mathcal{R}}(\mathcal{K}) = \mathcal{K} \cap \mathcal{R}$.
The Levi decomposition of $\mathfrak{K}$ is

$$\mathfrak{K} \;=\; \mathfrak{K}_0' \oplus \mathfrak{z}_0 \oplus (\mathfrak{K} \cap \mathcal{R})$$

where $\mathfrak{z}_0$ is the centre of $\mathfrak{K}_0$. Hence, $\mathfrak{K} \cap \mathcal{R}$ being a solvable ideal of a compact Lie algebra, $\mathfrak{K}$ must be central in $\mathfrak{K}$. And by the preceding proposition, $\mathfrak{s}$ is, up to conjugation,[5] contained in $\mathfrak{K}_0'$. □

[5] This was first proven by Iwasawa and Cartan, but we refer to [18] for a self-contained presentation.

Proposition 3.5 now follows from the fact that every iLa $(\mathfrak{g}, \sigma)$ admits a σ-stable Levi factor $\mathcal{L}$, necessarily conjugated to $\mathfrak{L}$.
This observation leads us to consider a special class of generic kinematical Lie algebras:

Definition 3.1 A generic kinematical Lie algebra $\mathfrak{g}$ satisfies the **Levi condition** is $\mathfrak{s}$ is contained in a σ-stable Levi factor $\mathcal{L}$ of $\mathfrak{g}$.

3.2.3 The $\mathcal{Z}$-nilpotent case: Poincaré spaces

We now investigate the case where the action of $\mathcal{Z}$ on $\mathcal{P}$ is nilpotent. Remind that in this case, one has $A^2 = 0$ (c.f. the proof of Proposition 3.1).

Definition 3.2 A generic kinematical (real or complex) Lie algebra is called of the **Poincaré type** if

1. it satisfies the Levi condition 3.1.
2. The action of $\mathcal{Z}$ on $\mathcal{P}$ is nilpotent.
3. The underlying symmetric space is not solvable (i.e. its transvection iLa contains a non-trivial Levi factor).
4. The underlying symmetric space is indecomposable.

Proposition 3.7 *Let $(\mathfrak{g}, \sigma, \Omega)$ be a siLa underlying a Poincaré generic kinematical Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then:*

(i) the radical $\mathcal{R}$ is Abelian.
(ii) The subspaces $\mathcal{P}_{\mathcal{R}} := \mathcal{P} \cap \mathcal{R}$ and $\mathcal{P}_{\mathcal{L}} := \mathcal{P} \cap \mathcal{L}$ of $\mathcal{P}$ are Lagrangians in duality.
(iii) The subspaces $\mathcal{P}_{\mathcal{R}}$ and $\mathcal{P}_{\mathcal{L}}$ of $\mathcal{P}$ are $\mathfrak{s}$-modules each of them isomorphic to V.
(iv) One has the equality:

$$[\mathcal{P}_{\mathcal{R}}\,,\,\mathcal{P}_{\mathcal{L}}] \;=\; \mathcal{Z}\;.$$

(v) The action of $\mathcal{Z}$ establishes an isomorphism of $\mathfrak{s}$-modules from $\mathcal{P}_{\mathcal{L}}$ to $\mathcal{P}_{\mathcal{R}}$.

Proof Since the action of $\mathcal{Z}$ is nilpotent, $\mathfrak{g}$ is not semisimple. The subspace $\mathcal{P}_{\mathcal{R}}$ is non-trivial and different from $\mathcal{P}$ (otherwise, the symmetric space would be solvable). Therefore, it is an $\mathfrak{s}$-submodule of $\mathcal{P}$ isomorphic to V.
The subspace $[\mathcal{P}, \mathcal{P}_{\mathcal{R}}]$ is non-trivial (as Ω is non-degenerate) and contained in $\mathfrak{h} \cap \mathcal{R}$. Hence, it cannot intersect $\mathfrak{s}$ and must therefore be equal to $\mathcal{Z}$. The line $\mathcal{Z}$ must therefore live in $\mathcal{R}$, entailing $\mathcal{R} \cap \mathfrak{h} = \mathcal{Z}$ and $\mathfrak{h} \cap \mathcal{L} = \mathfrak{s}$. Moreover, $\mathcal{P}_{\mathcal{R}}$ cannot be symplectic, because of indecomposability (Proposition 4.1 page 311 of [13]). Since $[\mathcal{P}_{\mathcal{R}}, \mathcal{P}_{\mathcal{R}}] \subset \mathfrak{h} \cap \mathcal{R} = \mathcal{Z}$, the space $\mathcal{P}_{\mathcal{R}}$ is therefore an Abelian Lie subalgebra, and, in particular Lagrangian. (Otherwise, its radical would be a proper $\mathfrak{s}$-submodule in contradiction with the irreducibility of V.)
At last, the susbspace $\mathcal{P}_{\mathcal{L}}$ is an $\mathfrak{s}$-submodule of $\mathcal{P}$ also isomorphic to V and supplementary to $\mathcal{P}_{\mathcal{R}}$. Since $[\mathcal{P}_{\mathcal{L}}, \mathcal{P}_{\mathcal{L}}] \subset \mathfrak{h} \cap \mathcal{L} = \mathfrak{s}$, the subspace $\mathcal{P}_{\mathcal{L}}$ is Lagrangian in Ω-duality with $\mathcal{P}_{\mathcal{R}}$.
The action of $\mathcal{Z}$ cannot be trivial. Indeed, since $[\mathcal{P}_{\mathcal{L}}\,,\,\mathcal{P}_{\mathcal{L}}] = \mathfrak{s}$ and $[\mathcal{P}_{\mathcal{L}}\,,\,\mathcal{P}_{\mathcal{R}}] = \mathcal{Z}$, Jacobi yields:

$$[[\mathcal{P}_{\mathcal{L}}\,,\,\mathcal{P}_{\mathcal{L}}], \mathcal{P}_{\mathcal{R}}] \;\subset\; [\mathcal{Z}\,,\,\mathcal{P}_{\mathcal{L}}]\;.$$

Hence, if the right-hand side is trivial, so is the action of $\mathfrak{s}$ on $\mathcal{P}_{\mathcal{R}} \simeq V$. Simplicity of V then implies item (v). □

As an immediate corollary, one observes

Corollary 3.2 *The symplectic manifold underlying the simply connected symplectic symmetric space M associated with a generic kinematical Lie algebra of the Poincaré type is symplectomorphic to the cotangent bundle of a semisimple symmetric space:*

$$M \simeq T^{\star}(\widetilde{\mathbb{L}/S})$$

where $\widetilde{\mathbb{L}/S}$ denotes the simply connected symmetric space associated with the iLa $\mathcal{L} = \mathfrak{s} \oplus \mathcal{P}_{\mathcal{L}}$.
Denoting by $\mathbb{L}$ the connected simply connected Lie group admitting $\mathcal{L}$ as a Lie algebra, the symplectomorphism can be chosen to be $\mathbb{L}$-equivariant.

Remark 3.3 Recall that semisimple symmetric spaces were first classified by Berger in [9].

We end this section by noticing that Theorem 3.3 and Corollary 3.2 classify the non-flat generic kinematical Lie algebras $(\mathfrak{g}, \mathfrak{s}, V)$ such that $\mathfrak{s}$ is compact semisimple:

Proposition 3.8 *Let $(\mathfrak{g}, \mathfrak{s}, V)$ be a generic kinematical Lie algebras such that $\mathfrak{s}$ is semisimple and compact. Assume that the associated simply connected symplectic symmetric space M is non-flat and indecomposable. Then, M is either a Hermitian symmetric space or the cotangent bundle of a Riemannian symmetric space.*

Example 3.1 In order to illustrate Proposition 3.8, we note that the Poincaré kinematical Lie algebra is the prototypical example of such generic kinematical Lie algebras. In this case, $\mathfrak{s}$ is the rotation Lie algebra $\mathfrak{s} = \mathfrak{so}(D)$ naturally represented on $V = \mathbb{R}^D \simeq \mathcal{P}_C L \simeq \mathcal{P}_{\mathcal{R}}$. The Levi factor $\mathcal{L}$ is the Lorentz Lie algebra with Cartan decomposition:

$$\mathcal{L} = \mathfrak{s} \oplus \mathcal{P}_{\mathcal{L}} .$$

Under matrix form, $\mathfrak{so}(D)$ is represented by the lower diagonal anti-symmetric $D \times D$ matrices, while $\mathcal{P}_{\mathcal{L}}$ is represented by the symmetric matrices:

$$\begin{pmatrix} 0 & {}^{\tau}\mathcal{P}_{\mathcal{L}} \\ \mathcal{P}_{\mathcal{L}} & \mathfrak{so}(D) \end{pmatrix} = \mathcal{L} .$$

The Minkowski space $\mathfrak{M}^{D+1}$ on which $\mathcal{L}$ naturally acts is represented by the column vectors

$$\mathfrak{M}^{D+1} = \begin{pmatrix} \mathcal{Z} \\ \mathcal{P}_{\mathcal{R}} \end{pmatrix} .$$

It is, to our opinion, remarkable that the Poincaré group therefore turns out to be the transvection group of a symplectic symmetric space. To our knowledge, this fact has not been observed in the literature yet. Perhaps partly because the Loos connection is

here purely symplectic: it leaves parallel the symplectic form but it is not the Levi-Civita connection for any pseudo-Riemannian metric on the space.
It is worth noticing that the underlying symplectic manifold of our above-defined symplectic symmetric space is symplectomorphic to the cotangent bundle $T^\star(Q)$ of the hyperbolic Riemannian symmetric space $Q := SO_o(1, D)/SO(D)$ (i.e. the mass-shell hyperboloid).

4 Remarks on coadjoint orbits

As mentioned in the introduction, since it is a homogeneous symplectic manifold, any simply connected symplectic symmetric space is a symplectic covering space of a coadjoint orbit of some Lie group. One may therefore wonder which coadjoint orbits arise that way, and if they do, under what conditions are they symplectic symmetric spaces? Generally, there exists no simple criterion, either geometric or algebraic, to decide whether a coadjoint orbit $\mathcal{O}$ of a Lie group $\mathbb{L}$ admits a $\mathbb{L}$-invariant (local or global) symplectic symmetric space structure.
For instance, the long-standing conjecture (1930) of E. Cartan stating that every complex homogeneous bounded domain is a symmetric space is a particular instance of this question in the Kähler case. In 1955, the assertion of the conjecture was proven by Borel and Matsushima in the case where the group of biholomorphic transformations is semisimple. Five years later, Piateskii-Shapiro eventually answered the conjecture by the negative, providing a complete description of the fine structure of homogeneous bounded complex domains (normal j-algebras).
More generally, proving that a coadjoint orbit $\mathcal{O}$ of $\mathbb{L}$ admits an $\mathbb{L}$-invariant local symplectic symmetric space structure implies constructing an involutive Lie algebra $(\mathfrak{g} = \mathcal{K} \oplus \mathcal{P}, \sigma)$ containing $\mathcal{L}$ (the Lie algebra of $\mathbb{L}$) and such that, denoting

$$\pi : \mathfrak{g} \to \mathcal{P}$$

the natural projection parallel to $\mathcal{K}$, one has

(1) the kernel of the restriction to $\mathcal{L}$ of π equals the Lie algebra of the stabilizer $\mathcal{C}$ in $\mathbb{L}$ of a point $o \in \mathcal{O}$.
(2) The action of $\mathcal{K}$ on $\mathcal{P}$ is symplectic under the identification between $\mathcal{P}$ and $T_o(\mathcal{O})$ induced by condition (1).

In the case $\mathcal{C}$ is trivial (i.e. $\mathbb{L}$ is a Fröbenius Lie group), this task is currently out of reach: while the Kähler case follows from Dorfmeister–Nakajima's proof of the long-standing Gindikin–Vinberg conjecture [19], the general case of symplectic Lie groups [5] (i.e. Lie groups carrying a left-invariant symplectic structure) is open.

We now pass to some structure results on the coadjoint orbits involved in the setting of generic kinematical Lie algebra. We start by recalling elementary preliminaries (see, for example, [38]) on symplectic Lie group actions and on symplectic symmetric spaces. First, remind

Definition 4.1 Let G be a Lie group that acts on the left on a symplectic manifold (M,ω) by symplectomorphisms:

$$\tau : G\times M\to M : (g,x)\mapsto \tau_g(x) \qquad \tau_g^\star\omega = \omega\,.$$

Denoting by $\mathfrak{g}$ the Lie algebra of G, one says that the action is **weakly Hamiltonian** if every fundamental vector field of the action is Hamiltonian in the sense that for every element X of the Lie algebra $\mathfrak{g}$, there exists a smooth function $\lambda_X\in C^\infty(M)$ on M such that

$$\mathrm{d}\lambda_X = \iota_{X^\star}\omega$$

where ι denotes the interior product and where $X^\star\in\Gamma^\infty(T(M))$ denotes the fundamental vector field on M ("infinitesimal action"):

$$X^\star_x := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \tau_{\exp(-tX)}(x)\,.$$

When the (dual) moment map:

$$\lambda : \mathfrak{g}\to C^\infty(M)$$

is a homomorphism of Lie algebras (w.r.t. the Poisson bracket on M associated with the symplectic structure ω), one says that the action is **strongly Hamiltonian** (or sometimes simply "Hamiltonian").

Remark 4.1 Remind that lifting to universal covers of G and M and centrally extending by the Chevalley 2-cocycle of $\mathfrak{g}$ (valued in the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$) associated to the dual moment map yields a systematic procedure for associating a strongly Hamiltonian action to any symplectic action.

The following result (due to Kostant) shows the relevance of coadjoint orbits in the class of homogeneous symplectic spaces:

Proposition 4.1 *When strongly Hamiltonian, every G-homogeneous symplectic space (M,ω) is a G-equivariant symplectic cover of a coadjoint orbit $\mathcal{O}$ of G. The covering map is in this case the (geometrical) moment map:*

$$\mathcal{J} : M\to\mathcal{O}\subset\mathfrak{g}^\star : x\mapsto[X\to\lambda_X(x)]$$

where $\mathcal{O}$ is the coadjoint orbit of the element

$$\xi_o : \mathfrak{g}\to\mathbb{R} : X\mapsto\lambda_X(o)$$

for any choice of base point o in M.

In that context, one has [7, 12]

Proposition 4.2 *Let* (M, ω, s) *be a simply connected symplectic symmetric space and* $(\mathfrak{g} = \mathfrak{h} \oplus \mathcal{P}, \sigma, \Omega)$ *its corresponding transvection siLa. Denote by* G *the connected simply connected Lie group admitting* $\mathfrak{g}$ *as Lie algebra, and define the element* $\widehat{\Omega}$ *as the natural extension of* Ω *on* $\mathcal{P}$ *to* $\mathfrak{g}$ *by zero on* $\mathfrak{h}$*:*

$$\widehat{\Omega} \;:=\; 0_{\mathfrak{h}\wedge\mathfrak{h}} \;\oplus\; 0_{\mathfrak{h}\wedge\mathcal{P}} \;\oplus\; \Omega_{\mathcal{P}\wedge\mathcal{P}}\;.$$

Then:

(i) The element $\widehat{\Omega}$ *is a Chevalley 2-cocycle of* $\mathfrak{g}$ *associated with the trivial representation of* $\mathfrak{g}$ *on* $\mathbb{R}$.

(ii) The action of G *on* (M, ω) *is strongly Hamiltonian if and only if* $\widehat{\Omega}$ *is a Chevalley coboundary, i.e. there exists an element* ξ_o *in the dual* $\mathfrak{g}^\star$ *of* $\mathfrak{g}$ *such that*

$$\widehat{\Omega} \;=\; \delta\xi_o\;.$$

In that case the associated moment map $\mathcal{J} : M \to \mathfrak{g}^\star$ *covers the coadjoint orbit* $\mathcal{O}$ *of the element* ξ_o.

(iii) Assume the action of G *on* (M, ω) *to be strongly Hamiltonian. Denoting by* $\mathtt{Ad}^\flat$ *the coadjoint action, let us consider the (normal) group of ineffectiveness:*

$$N \;:=\; \{g \in G \mid \mathtt{Ad}^\flat_g(\xi) \;=\; \xi \;\; \forall \xi \in \mathcal{O}\}\;.$$

Denoting by G_{ξ_o} *the stabilizer of* ξ_o *in* G*, the coadjoint orbit* $\mathcal{O}$ *naturally is a symplectic symmetric space if and only if* G_{ξ_o}/N *is contained in the subgroup of* G/N *of fixed elements under the natural involution of* G/N *induced by* $\tilde{\sigma}$ *(c.f. Theorem 2.1).*

Back to our kinematical Lie algebras, we first note

Proposition 4.3 *Let* $(\mathfrak{g}, \mathfrak{s}, V)$ *be a generic kinematical Lie algebra such that the associated siLa* $(\mathfrak{g}, \sigma, \Omega)$ *is indecomposable and non-flat. Then the transvection Lie algebra* $\underline{\mathfrak{g}}$ *underlying the transvection siLa associated to* $(\mathfrak{g}, \sigma, \Omega)$ *naturally carries a structure of generic kinematical Lie algebra* $(\underline{\mathfrak{g}}, \underline{\mathfrak{s}}, V)$.

Proof First, due to the fact that V is faithful, the action of $\mathfrak{h}$ on $\mathcal{P}$ is effective. Now, the Lie subalgebra $\underline{\mathfrak{h}} := [\mathcal{P}, \mathcal{P}]$ of $\mathfrak{h}$ is necessarily transverse to $\mathfrak{h}$ as Ω has trivial radical. Now, consider the map $\zeta : \underline{\mathfrak{h}} \to \mathcal{Z}$ defined as the restriction to $\underline{\mathfrak{h}}$ of the projection $\mathtt{pr}_{\mathcal{Z}} : \mathfrak{h} \to \mathcal{Z}$ parallel to $\mathfrak{s}$. This map is not identically zero as symplectic. Therefore, its kernel $\underline{\mathfrak{s}} := \ker(\zeta)$ is a codimension one ideal of $\underline{\mathfrak{h}}$ contained in $\mathfrak{s}$. Hence, $\underline{\mathfrak{s}} = \mathtt{pr}_{\mathfrak{s}}(\underline{\mathfrak{h}})$ where $\mathtt{pr}_{\mathfrak{s}}$ denotes the projection of $\mathfrak{h}$ onto $\mathfrak{s}$ parallel to $\mathcal{Z}$, which entails that $\mathcal{Z}$ is contained in $\underline{\mathfrak{h}}$. □

Remark 4.2 Observe that if $\mathfrak{s}$ is simple, then $\mathfrak{g} = \underline{\mathfrak{g}}$, i.e. the siLa associated to our generic kinematical Lie algebra is, under the conditions of Proposition 4.3, necessarily transvection.

Proposition 4.4 *Let* $(\mathfrak{g}, \mathfrak{s}, V)$ *be a generic kinematical Lie algebra on the base field* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. *Assume the associated siLa* $(\mathfrak{g}, \sigma, \Omega)$ *to be non-flat. Denote by* $\underline{G}$ *be the connected simply connected Lie group admitting the transvection algebra* $\underline{\mathfrak{g}}$ *as Lie algebra (c.f. Proposition 4.3). Then the action of* $\underline{G}$ *on the simply connected symplectic symmetric space* (M, ω, s) *associated with the transvection siLa* $(\mathfrak{g}, \sigma, \Omega)$ *is strongly Hamiltonian.*

Proof The Chevalley coboundary of the projection $\xi_o := \mathtt{pr}_{\mathcal{Z}} : \underline{\mathfrak{g}} \to \mathcal{Z}$ parallel to $\underline{\mathfrak{s}} \oplus \mathcal{P}$ consists in the extension by zero on $\underline{\mathfrak{h}} \times \underline{\mathfrak{h}}$ of Ω. □

Remark 4.3 As the considerations of the present work are purely algebraic, we differ the study of the group of ineffectiveness and more generally the geometrical study of the (non-simply connected) symplectic symmetric spaces (and space-times) associated with our kinematical siLa's to a further work.

5 Conclusions and further perspectives

In this work, we defined the notion of generic kinematical Lie algebra and proved that the structure of such a Lie algebra is entirely coded by a symplectic symmetric space, which we precisely described.
However, the overlap with the usual notion of kinematical Lie algebra as introduced in the physics literature is complete only in the case of space dimension $D \geq 4$. In lower dimension, there are cases that are not covered as a consequence of the fact that item 3(a) in Definition 1.1 does not hold for $\mathfrak{s} = \mathfrak{so}(D)$ with $D \leq 3$. This therefore fully justifies to consider the analogous notion where either the $\mathfrak{s}$-module V is isomorphic to the adjoint representation of $\mathfrak{s}$ or $\mathfrak{s}$ is Abelian. We will treat those cases in a future work.
As a matter of fact, generic kinematical Lie algebras are deformations of one another. However, the underlying symmetric spaces governing their structure allow to make precise the way they are deformed into each other. Namely, we are planning to study what are called the *homotopic structure varieties* associated by W. Bertram [2, 3, 10] to complex or para-complex symmetric spaces in the context of Jordan algebra theory.
Some of our short- and mid-term further objectives are

- Analyse the $\mathcal{Z}$-nilpotent case without Levi condition.
- Describe the geometry of the generalized space-times associated with our generic kinematical Lie algebras (i.e. the quotient spaces analogue to the kinematical space-times in the classical context).
- Describe the structure of those generalized space-times associated with flat symplectic symmetric spaces.
- Investigate the harmonic analysis attached to the natural quantization of our symplectic symmetric spaces, along the lines of [11, 15], and, in particular, the role of the generalized space-times in this harmonic analytical context.
- Study the naturally induced geometric actions in the sense of Souriau [36].
- Investigate possible infinite-dimensional generalizations.

6 Appendix

6.1 Elementary representation theory

Lemma 6.1 *Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and $\mathcal{M}$ be a semisimple real $\mathfrak{g}$-module. Let*

$$\mathcal{M} := \bigoplus_{i=1}^{r} \mathcal{M}_i = \bigoplus_{j=1}^{r'} \mathcal{M}'_j$$

be two decompositions into simple $\mathfrak{g}$-submodules. We will reserve the index i, respectively j, for the first decomposition $\mathcal{M} = \bigoplus_{i=1}^{r} \mathcal{M}_i$, respectively, for the second one $\mathcal{M} = \bigoplus_{j=1}^{r'} \mathcal{M}'_j$. Then,

$$r = r'$$

and there exists a permutation $\sigma \in S_r$ and a $\mathfrak{g}$-intertwiner $A \in GL(\mathcal{M})$ such that for every $i \in \{1, ..., r\}$:

$$A(\mathcal{M}_i) = \mathcal{M}'_{\sigma(i)} .$$

Proof For all $i \in \{1, ..., r\}$ and $j \in \{1, ..., r'\}$, let

$$\pi_{ij} : \mathcal{M}_i \to \mathcal{M}'_j$$

the restriction of the projection from $\mathcal{M}$ onto $\mathcal{M}'_j$ parallel to $\oplus_{k \neq j} \mathcal{M}'_k$. The maps π_{ij} are $\mathfrak{g}$-intertwiners. Indeed, for every $X \in \mathfrak{g}$ and $m \in \mathcal{M}$, one has, within the second ($'$) decomposition:

$$[X, m] = [X, \sum_{j=1}^{r'} m'_j] = \sum_{j=1}^{r'} [X, m]_j .$$

Hence, one must have $[X, m'_j] = [X, m]_j$, proving the assertion.

Note also that, viewing all $\mathcal{M}_i$ and $\mathcal{M}'_j$ as a subspaces of $\mathcal{M}$, one has that, for every i, there must exists j, say $j(i)$ such that π_{ij} is an injection. Indeed, let us assume that i_0 is such that for every j, $\mathcal{K}_j := \ker(\pi_{i_0 j}) \neq \{0\}$. Since $\pi_{i_0 j}$ is a $\mathfrak{g}$-intertwiner, the subspace $\mathcal{K}_j \subset \mathcal{M}_{i_0}$ is a $\mathfrak{g}$ submodule, hence for every j: $\mathcal{K}_j = \mathcal{M}_{i_0}$ by simplicity. This implies that $\mathcal{M}_{i_0} = \{0\}$.
Therefore, one can define a function

$$\{1, ..., r\} \to \{1, ..., r'\} : i \mapsto j(i) .$$

Therefore, $r' \leq r$. By symmetry, $r' \leq r$, hence the equality. In particular the above map defines a permutation σ of $\{1, ..., r\}$.
Now, by irreducibility, π_{ij} is either trivial or an isomorphism of simple $\mathfrak{g}$-modules. Therefore, the operator

$$A := \oplus_i \pi_{i\sigma(i)}$$

solves the question. □

Lemma 6.2 *Let W be a $\mathfrak{g}$-module, L a simple $\mathfrak{g}$-module and $W_{(L)}$ the weak isotypic component of L in W. Then, the $\mathfrak{s}$-module $W_{(L)}$ is semisimple.*

Proof Let L_1 and L_2 be two different $\mathfrak{s}$-submodules of $W_{(L)}$ each of them being isomorphic to L. Then, as L is simple, their intersection is trivial. Denote by $L_{12} := L_1 \oplus L_2$ and let L_3 be a third $\mathfrak{s}$-submodule in $W_{(L)}$ not entirely contained in L_{12}. The intersection $L_{12} \cap L_3$ is a proper $\mathfrak{s}$-submodule of L_3, hence trivial. Inductively, one constructs a semisimple $\mathfrak{s}$-submodule $\mathfrak{L}$ of the isotypic component that is maximal in the sense that there is no $\mathfrak{s}$-submodule in $W_{(L)}$ isomorphic to L and not entirely contained in $\mathfrak{L}$. But the isotypic component is the sum of all submodules that are isomorphic to L; hence, each of them needs to be contained in $\mathfrak{L}$. Therefore, $\mathfrak{L}$ and $W_{(L)}$ coincide. □

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra and $\mathcal{P}$ be a completely reducible $\mathfrak{s}$-module.
Consider a simple $\mathfrak{g}$-module V and denote by $\mathcal{P}_{(V)}$ its isotypical component in $\mathcal{P}$. Remind that the isotypical component is the sum in $\mathcal{P}$ of all simple $\mathfrak{g}$-submodules that are isomorphic to V. By the previous lemma 6.1, all simple direct factors of any decomposition of $\mathcal{P}_{(V)}$ into simple $\mathfrak{g}$-modules are isomorphic to V. Now observe

Lemma 6.3 *For every $\mathfrak{g}$-submodule $\mathcal{S}$ of $\mathcal{P}$, one has*

$$\mathcal{S} \cap \mathcal{P}_{(V)} = \mathcal{S}_{(V)} \ .$$

Proof Remind that every submodule of a completely reducible module is completely reducible and start by decomposing $\mathcal{S}$ into simple $\mathfrak{g}$-submodules:

$$\mathcal{S} =: \oplus_{\alpha\in\Delta}\mathcal{S}_\alpha \ .$$

And do the same with $\mathcal{S} \cap \mathcal{P}_{(V)}$:

$$\mathcal{S} \cap \mathcal{P}_{(V)} =: \oplus_{\beta\in\Delta^V}\mathcal{S}^V_\beta \ .$$

Consider the projections:

$$\pi_{\beta\alpha} : \mathcal{S}^V_\beta \to \mathcal{S}_\alpha \ .$$

Schur's lemma tells us that each of those is either an isomorphism of $\mathfrak{g}$-module or identically zero. Hence, by the previous lemma, we may assume the inclusion:

$$\Delta^V \subset \Delta \ .$$

Similarly, consider a decomposition into simple $\mathfrak{g}$-modules:

$$\mathcal{P}_{(V)} =: \oplus_{\gamma\in\Phi}\mathcal{P}_\gamma$$

and consider the projections

$$\pi'_{\alpha\gamma} : \mathcal{S}^V_\beta \to \mathcal{P}_\gamma \ .$$

Since at least one of them must be non-trivial, Schur tells us that every $\mathfrak{g}$-module must be isomorphic to V. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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