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8D conformal gravity with Einstein sector, and its relation to the Q-curvature

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Abstract: We first streamline the construction of the unique six-dimensional conformal gravity action found by Lü, Pang and Pope, that admits Einstein metrics as solutions to the field equations. We then prove that there exists a unique eight-dimensional conformal gravity action that admits Einstein metrics as solutions to the field equations, and explicitly build the corresponding action. Finally, we relate these results to Branson's Q-curvature and the Fefferman-Graham obstruction tensor, to conjecture that on every even-dimensional space there exists a unique — up to boundary terms — conformally-invariant gravity theory that is extremised by Einstein metrics.

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Contents

1 Introduction

Conformal gravity is a privileged model for an extension of Einstein's general relativity theory, since on top of the usual diffeomorphism symmetries, it is invariant under Weyl rescalings of the metric. As a classical theory of gravity, four-dimensional conformal gravity, also called Weyl gravity, was investigated in great details by several authors, see in particular [1–10]. A review of the relevance of Weyl gravity throughout the last decades can be found in [11] to which we refer for more references.

Viewed as a quantum theory, conformal gravity was shown by Stelle [12] to be renormalizable, albeit non-unitary. This triggered an important body of works, where in particular Einstein gravity was argued to emerge from quantum corrections to Weyl gravity [13, 14]; for a review, see e.g. [15]. The non-unitarity of conformal gravity has been discussed in many references, see e.g. [16, 17], where precisely the non-unitary sector can be decoupled from the space of solutions to conformal gravity by imposing appropriate boundary conditions, thereby leaving the space of solutions of Einstein's equations with a cosmological constant [18]. Indeed, while the solutions to Einstein's equations with a cosmological constant — namely, Einstein manifolds — are also solutions to the field equations of conformal gravity, the converse is not true. Nevertheless, as we mentioned above, Maldacena [18] showed that the non-Einstein metrics of four-dimensional conformal gravity can be eliminated by imposing an appropriate Neumann boundary condition on the asymptotic (anti) de Sitter (A)dS spacetime metric, which constitutes a very interesting and concrete relation between Weyl and Einstein gravity theories.

Whether this connection holds in higher spacetime dimensions D was tested in six-dimensional conformal gravity [19], where the action actually is a two-parameter family of actions, due to the fact that in $6D$ there exist three linearly independent scalar densities that are strictly Weyl-invariant [20, 21]; see e.g. [22] for a review, and below in the body of the paper. Instead, in four dimensions there is only one Weyl-invariant scalar density, the one leading to Weyl gravity. It was found in [23] that, up to an overall factor in front of the action functional, there is a *unique* linear combination of the three local conformal invariants in $6D$ for which Einstein metrics are solutions to the corresponding variational

JHEP02(2026)101

problem. In the present technical note, we want to see whether this property extends to eight dimensions $D = 8$.

For the construction of the Lagrangian density, one has to start from the list of possible Weyl-invariant scalar densities in $8D$, also called local (or pointwise) conformal invariants, which were classified in [24] by using the Weyl-covariant calculus developed in [25]. These purely algebraic tools were also used in [26] to determine the general structure of global conformal invariants on manifolds of arbitrary dimension. It was already known from [27, 28] that, on closed manifolds of even dimensions $D = 2m$, global conformal invariants are given by the integral over the manifold of the Euler density plus a linear combination of the local conformal invariants in that dimension, plus total derivatives. On manifolds of dimension $D = 4m - 1$, $m \in \mathbb{N}^+$, further global conformal invariants were found in [26], thereby completing the results of [28].

In section 2 we briefly review the Weyl-conformal calculus developed in [25]. Then, in section 3 we review the theories of conformal gravity in four and six dimensions, that admit an Einstein sector. In section 4 we then discuss the notion of Q-curvature and illustrate it explicitly in dimensions two, four and six. Then, in section 5 we construct the most general conformal gravity theory in eight dimensions, that admits an Einstein sector, and find that the result is unique, up to boundary terms and an additive constant proportional to the Euler characteristic. We relate this action to the eight-dimensional Q-curvature, to find that, up to boundary terms and an additive constant, our action coincide with the (normalised) integrated Q-curvature. We end the note in section 6 with a general discussion of both the Q-curvature and the Fefferman-Graham obstruction tensor in arbitrary even dimension $D = 2m$. We conjecture that there is only one conformal gravity action in even dimension, that admits an Einstein sector. It coincides with the integrated Q-curvature, up to normalisation, boundary terms, and additive constant proportional to the Euler characteristic $\chi(M_{2m})$ of the manifold.

2 Weyl-covariant tensor calculus

The problem of classifying all the Weyl-invariant scalar densities built out of a metric in arbitrary (even) dimension is famously difficult, see e.g. [20, 29–33] and refs. therein. The problem is very simple in four dimensions for which the square of the Weyl tensor gives the solution, whereas it is already much more complicated in six dimensions [21]. In eight dimensions, the classification of the Weyl-invariant scalar densities built out of a metric tensor was obtained in [24]. This classification relies on the Weyl-covariant tensor calculus developed in [25] that we will briefly review in this section, as it is also instrumental in the classification of the Weyl-invariant action functionals that admit Einstein metrics as solutions to the variational problem. We use the conventions and notation of [34, 35], where the classification of Weyl anomalies in arbitrary dimension was obtained.

First of all, we recall that the Weyl tensor is the traceless part of the Riemann curvature tensor. In components, we have

$$W^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} - 2\left(\delta^{\mu}{}_{[\rho}K_{\sigma]\nu} - g_{\nu[\rho}K_{\sigma]}{}^{\mu}\right) , \tag{2.1}$$

JHEP02(2026)101

in terms of the components of the Riemann tensor and of the Schouten tensor

$$K_{\mu\nu} = \frac{1}{D-2}\left(R_{\mu\nu} - \frac{1}{2(D-1)} g_{\mu\nu}\right) . \tag{2.2}$$

Under infinitesimal Weyl rescalings of the metric

$$\delta_\sigma g_{\mu\nu} = 2\,\sigma(x)\, g_{\mu\nu} , \tag{2.3}$$

the components of Weyl tensors are invariant: $\delta_\sigma W^\mu{}_{\nu\alpha\beta} = 0$. Denoting by $\Delta_\mu{}^\nu$ the $\mathrm{GL}(D)$ generators that act on tensors through $\Delta_\mu{}^\nu\, T^\alpha_\beta = \delta^\nu_\beta T^\alpha_\mu - \delta^\alpha_\mu T^\nu_\beta$, the symbol $\nabla_\mu = \partial_\mu - \Gamma_{\mu\nu}{}^\rho\,\Delta_\rho{}^\nu$ denotes the usual torsion-free metric-compatible (Levi-Civita) covariant derivative associated with the Christoffel symbols $\Gamma_{\mu\nu}{}^\rho$, in terms of which $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma_{\nu\sigma}{}^\mu + \ldots$. The commutator of covariant derivatives gives $[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$ and, in general, $[\nabla_\mu, \nabla_\nu] = R_{\mu\nu\rho}{}^\sigma\,\Delta_\sigma{}^\rho$. The components of the Cotton tensor are given by $C_{\alpha\rho\sigma} = 2\,\nabla_{[\sigma} K_{\rho]\alpha} \equiv \nabla_\sigma K_{\rho\alpha} - \nabla_\rho K_{\sigma\alpha}$. The Weyl-covariant derivative constructed in [25] is given by

$$\mathcal{D}_\mu = \nabla_\mu + K_{\mu\alpha}\,\mathbf{\Gamma}^\alpha , \tag{2.4}$$

where we refer to this work for the definition of the generators $\mathbf{\Gamma}^\alpha$; see also below for a few examples. The important property of the Weyl-covariant derivative $\mathcal{D}$ is that its curvature vanishes if and only if the metric is conformally flat. Explicitly, one has [25]

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = W_{\mu\nu\rho}{}^\sigma\,\Delta_\sigma{}^\rho - C_{\alpha\mu\nu}\,\mathbf{\Gamma}^\alpha . \tag{2.5}$$

The first term on the right-hand side is the same as in the expression for the commutator of the Levi-Civita covariant derivative, except that now the Weyl tensor replaces the Riemann curvature tensor. The second term on the right-hand side brings the Cotton tensor, which is the conformal field strength in 3D, where the Weyl tensor identically vanishes. In dimensions $D > 3$, the Cotton tensor can be written as a covariant divergence of the Weyl tensor, viz., $C_{\alpha\rho\sigma} = -\frac{1}{D-3}\,\nabla_\mu W^\mu{}_{\alpha\rho\sigma}$.

Similarly to the fact that the tensors in (pseudo)Riemann geometry are given by the metric tensor, the Riemann tensor, all its covariant derivatives and traces thereof using the (inverse)metric tensor, the set of *W-tensors* is given by the Weyl tensor, all its Weyl-covariant derivatives and their non-trivially vanishing traces. We introduce super indices and the notation

$$\{W_{\Omega_0}, W_{\Omega_1}, \ldots, W_{\Omega_k}, \ldots\} = \{W^\mu{}_{\nu\rho\sigma}, \mathcal{D}_{\alpha_1} W^\mu{}_{\nu\rho\sigma}, \ldots, \mathcal{D}_{\alpha_k}\mathcal{D}_{\alpha_{k-1}} \ldots \mathcal{D}_{\alpha_1} W^\mu{}_{\nu\rho\sigma}, \ldots\} .$$

The defining property of the W-tensors is that they transform, under infinitesimal Weyl rescalings of the metric, with the first derivative of the Weyl parameter only [25]:

$$\delta_\sigma W_{\Omega_i} = \partial_\alpha \sigma\, [T^\alpha]_{\Omega_i}{}^{\Omega_{i-1}}\, W_{\Omega_{i-1}} . \tag{2.6}$$

We will also use the notation $W^\mu{}_{\nu\rho\sigma,\alpha_1} := \mathcal{D}_{\alpha_1} W^\mu{}_{\nu\rho\sigma}$, $W^\mu{}_{\nu\rho\sigma,\alpha_1\alpha_2} := \mathcal{D}_{\alpha_2}\mathcal{D}_{\alpha_1} W^\mu{}_{\nu\rho\sigma}$, etc.

JHEP02(2026)101

By introducing the tensor $\mathcal{P}^{\alpha\nu}_{\mu\beta} := -g^{\alpha\nu}g_{\mu\beta} + \delta^{\alpha}_{\mu}\delta^{\nu}_{\beta} + \delta^{\alpha}_{\beta}\delta^{\nu}_{\mu}$, we can present the first few W-tensors as follows:

$$
\begin{aligned}
W_{\Omega_0} &= W^{\mu}{}_{\nu\rho\sigma}\,,\\
W_{\Omega_1} &= W^{\mu}{}_{\nu\rho\sigma,\alpha_1} = \mathcal{D}_{\alpha_1}\,W^{\mu}{}_{\nu\rho\sigma} = \nabla_{\alpha_1}\,W^{\mu}{}_{\nu\rho\sigma}\,,\\
W_{\Omega_2} &= \nabla_{\alpha_2}\,W^{\mu}{}_{\nu\rho\sigma,\alpha_1} - K_{\alpha_2\lambda}\,\mathcal{P}^{\lambda\delta}_{\epsilon\alpha_1}\,\Delta_{\delta}{}^{\epsilon}\,W^{\mu}{}_{\nu\rho\sigma} = \mathcal{D}_{\alpha_2}\mathcal{D}_{\alpha_1}W^{\mu}{}_{\nu\rho\sigma} = W^{\mu}{}_{\nu\rho\sigma,\alpha_1\alpha_2}\,,\\
W_{\Omega_3} &= \nabla_{\alpha_3}\,W^{\mu}{}_{\nu\rho\sigma,\alpha_1\alpha_2} - K_{\lambda\alpha_3}\,(\delta^{\gamma}_{\alpha_1}\,\mathcal{P}^{\lambda\delta}_{\epsilon\alpha_2}\,\Delta_{\delta}{}^{\epsilon} + \delta^{\gamma}_{\alpha_2}\,\mathcal{P}^{\lambda\delta}_{\epsilon\alpha_1}\,\Delta_{\delta}{}^{\epsilon} - \mathcal{P}^{\lambda\gamma}_{\alpha_1\alpha_2})\,W^{\mu}{}_{\nu\rho\sigma,\gamma}\,,\\
W_{\Omega_4} &= \nabla_{\alpha_4}\,W^{\mu}{}_{\nu\rho\sigma,\alpha_1\alpha_2\alpha_3} - K_{\lambda\alpha_4}\times\\
&\qquad\times(\delta^{\gamma_1}_{\alpha_1}\,\delta^{\gamma_2}_{\alpha_2}\,\mathcal{P}^{\lambda\delta}_{\epsilon\alpha_3}\,\Delta_{\delta}{}^{\epsilon} + \delta^{\gamma_1}_{\alpha_1}\,\delta^{\gamma_3}_{\alpha_2}\,\mathcal{P}^{\lambda\delta}_{\epsilon\alpha_2}\,\Delta_{\delta}{}^{\epsilon} + \delta^{\gamma_1}_{\alpha_2}\,\delta^{\gamma_2}_{\alpha_3}\,\mathcal{P}^{\lambda\delta}_{\epsilon\alpha_1}\,\Delta_{\delta}{}^{\epsilon}+\\
&\qquad\quad-\delta^{\gamma_1}_{\alpha_2}\,\mathcal{P}^{\lambda\gamma_2}_{\alpha_1\alpha_3} - \delta^{\gamma_1}_{\alpha_1}\,\mathcal{P}^{\lambda\gamma_2}_{\alpha_2\alpha_3} - \delta^{\gamma_1}_{\alpha_3}\,\mathcal{P}^{\lambda\gamma_2}_{\alpha_1\alpha_2})\,W^{\mu}{}_{\nu\rho\sigma,\gamma_1\gamma_2}\,.
\end{aligned}
$$

3 Conformal gravity with Einstein sector in 4D and 6D

In four dimensions there is a single conformal invariant with mass dimension four, which is the square of the Weyl tensor. The equations of motion of the corresponding action set to zero the Bach tensor defined in dimension $D > 3$ by

$$B_{\mu\nu} = \frac{1}{3-D}\,\nabla^{\beta}\nabla^{\alpha}W_{\alpha\mu\nu\beta} - K^{\alpha\beta}\,W_{\alpha\mu\nu\beta} \equiv \frac{1}{3-D}\,\mathcal{D}^{\beta}\mathcal{D}^{\alpha}W_{\alpha\mu\nu\beta}\,. \tag{3.1}$$

On Einstein manifolds, the Ricci tensor is proportional to the metric, and so is the Schouten tensor, showing that the Bach tensor vanishes on Einstein manifolds. By using the differential Bianchi identity for the Riemann tensor, it is also easy to see that the Bach tensor is symmetric. It is evidently traceless, which can be viewed as the Noether identity for the Weyl-invariance of $4D$ conformal gravity. Therefore, all the solutions of four-dimensional Einstein gravity (with or without cosmological constant) are solutions of four-dimensional conformal gravity. As we mentioned in the introduction, the converse is not true, and Maldacena [18] showed what boundary conditions to impose on the metric in asymptotically anti-de Sitter (AdS) manifolds in order to kill the unwanted degrees of freedom, leaving only those of Einstein gravity. Thus, upon using such boundary conditions, four-dimensional conformal gravity is equivalent to ordinary four-dimensional Einstein gravity in asymptotically AdS spacetime.

In [23] the most general six-dimensional conformal gravity theory was found, such that all Einstein manifolds are solutions to the equations of motion. This is less trivial than the four-dimensional case, since in six dimensions there are three independent Weyl invariant scalar densities, with mass dimension six, built with the Weyl tensor and its covariant derivatives, so that the six-dimensional conformal gravity depends on two free parameters, up to an overall constant. In the following, we review this result.

Using the Weyl-covariant tensor calculus introduced in [25], it is not a difficult task to build a basis of Weyl-invariant scalar densities in $6D$. One finds the following three Weyl-invariant scalar densities:

$$\mathcal{I}_1 = \sqrt{|g|}\,\Big(W_{\alpha}{}^{\rho\sigma}{}_{\beta}\,W_{\mu\rho\sigma\nu}\,W^{\mu\alpha\beta\nu}\Big)\,, \tag{3.2}$$

$$\mathcal{I}_2 = \sqrt{|g|}\,\Big(W_{\alpha\beta}{}^{\mu\nu}\,W_{\mu\nu\rho\sigma}\,W^{\rho\sigma\alpha\beta}\Big)\,, \tag{3.3}$$

$$\widetilde{\mathcal{I}}_3 = \sqrt{|g|}\,\Big(\tfrac{1}{2}\,\mathcal{D}_{\alpha}W_{\mu\nu\rho\sigma}\,\mathcal{D}^{\alpha}W^{\mu\nu\rho\sigma} + \tfrac{8}{9}\,\mathcal{D}_{\alpha}W^{\alpha\beta\gamma\delta}\,\mathcal{D}^{\mu}W_{\mu\beta\gamma\delta} + W^{\mu\nu\rho\sigma}\,\mathcal{D}_{\alpha}\mathcal{D}^{\alpha}W_{\mu\nu\rho\sigma}\Big)\,. \tag{3.4}$$

Therefore, up to boundary terms, the most general action for six-dimensional conformal gravity can be written as

$$S_6[g_{\mu\nu}] = \int \mathrm{d}^6x\, (w_1\, \mathcal{I}_1 + w_2\, \mathcal{I}_2 + w_3\, \widetilde{\mathcal{I}}_3)\,, \tag{3.5}$$

where the coefficients w_i, $i = 1, 2, 3$, are arbitrary (non-simultaneously vanishing) real constants.

We now compute the variation of the action $S_6[g_{\mu\nu}]$, in order to determine for which choice of coefficients $\{w_i\}$ an Einstein metric can be solution to the Euler-Lagrange equations of motion. Discarding terms that identically vanish on an Einstein manifold, we find

$$\begin{aligned}
\tfrac{1}{\sqrt{|g|}}\,\delta S_6 = \Big[&-\tfrac{2}{15}\,(69\,w_1 - 144\,w_2 + 206\,w_3)\,K_\mu{}^\mu\, W_\alpha{}^{\gamma\delta\epsilon}\, W_{\beta\gamma\delta\epsilon} \\
&+\tfrac{3}{5}\,(19\,w_1 - 44\,w_2 + 56\,w_3)\, W_\alpha{}^{\gamma\delta\epsilon}\, W_\beta{}^\nu{}_\delta{}^\mu\, W_{\gamma\mu\epsilon\nu} + \\
&+\tfrac{3}{10}\,(19\,w_1 - 44\,w_2 + 56\,w_3)\, W_\alpha{}^{\gamma\delta\epsilon}\, W_{\beta\gamma}{}^{\mu\nu}\, W_{\delta\mu\epsilon\nu} + \\
&+\tfrac{1}{10}\,(-39\,w_1 + 84\,w_2 - 116\,w_3)\, K_\mu{}^\mu\, \mathcal{D}^\nu W_\alpha{}^{\gamma\delta\epsilon}\, \mathcal{D}_\gamma W_{\beta\nu\delta\epsilon} + \\
&+\tfrac{1}{20}\,(-9\,w_1 + 24\,w_2 - 26\,w_3)\, K_\mu{}^\mu\, \mathcal{D}_\beta W_{\gamma\delta\epsilon\nu}\, \mathcal{D}_\alpha W^{\gamma\delta\epsilon\nu}\Big]\, (\delta g^{\alpha\beta} - \tfrac{1}{6}\, g_{\mu\nu}\, \delta g^{\mu\nu}\, g^{\alpha\beta})\,.
\end{aligned} \tag{3.6}$$

One of the main tasks leading to the above expression was to write it in a basis of linearly independent structures, so that the expression vanishes if and only if its coefficients vanish. This happens if and only if

$$w_2 = \tfrac{1}{20}\, w_1, \quad w_3 = -\tfrac{3}{10}\, w_1\,, \tag{3.7}$$

so that, up to an overall constant — choose $w_1 = \frac{20}{3}$ — the action $\tilde{S}_6[g_{\mu\nu}]$ for the six-dimensional conformal gravity theory with an Einstein sector is unique, equal to

$$\tilde{S}_6[g_{\mu\nu}] = \int \mathrm{d}^6x\, (\tfrac{20}{3}\, \mathcal{I}_1 + \tfrac{1}{3}\, \mathcal{I}_2 - 2\, \widetilde{\mathcal{I}}_3)\,. \tag{3.8}$$

This result is consistent with the combination $4\,\mathcal{I}_1 + \mathcal{I}_2 - \frac{1}{3}\,\mathcal{I}_3$ found [23], where $I_3 = \frac{\mathcal{I}_3}{\sqrt{|g|}}$ is the last invariant in eq. (1.1) of [23], since $\widetilde{I}_3 = \frac{\widetilde{\mathcal{I}}_3}{\sqrt{|g|}}$ with $\widetilde{\mathcal{I}}_3$ given in (3.4) can also be written in the following way:

$$\begin{aligned}
\widetilde{I}_3 = \nabla^\alpha\, (\tfrac{1}{2}\, W^{\beta\gamma\delta\epsilon}\, W_{\beta\gamma\delta\epsilon,\alpha} - \tfrac{8}{9}\, W_\alpha{}^{\beta\gamma\delta}\, W_\beta{}^\epsilon{}_{\gamma\delta,\epsilon}) + \tfrac{4}{3}\, I_1 - \tfrac{1}{3}\, I_2 + \\
+ \tfrac{1}{6}\, W^{\alpha\beta\gamma\delta}\, W_{\alpha\beta\gamma\delta,\epsilon}{}^\epsilon + \tfrac{8}{3}\, W_\alpha{}^{\gamma\delta\epsilon}\, W_{\beta\gamma\delta\epsilon}\, K^{\alpha\beta} - W_{\beta\gamma\delta\epsilon}\, W^{\beta\gamma\delta\epsilon}\, K_\alpha{}^\alpha\,,
\end{aligned} \tag{3.9}$$

where the second line is equal to the bulk terms in $\frac{1}{6}\, I_3$ of [23], so that we have $\widetilde{\mathcal{I}}_3 = \frac{4}{3}\,\mathcal{I}_1 - \frac{1}{3}\,\mathcal{I}_2 + \frac{1}{6}\,\mathcal{I}_3 + \partial_\mu \mathcal{V}^\mu$, and from it follows the equality of Lagrangian densities, up to total derivatives:

$$\tfrac{20}{3}\, \mathcal{I}_1 + \tfrac{1}{3}\, \mathcal{I}_2 - 2\, \widetilde{\mathcal{I}}_3 = 4\, \mathcal{I}_1 + \mathcal{I}_2 - \tfrac{1}{3}\, \mathcal{I}_3 - 2\, \partial_\mu \mathcal{V}^\mu\,. \tag{3.10}$$

Finally, for completeness we note that the conformal invariant given in Prop. 3.4 of [20] is given by

$$\begin{aligned}
\mathcal{I}_3^{(\mathrm{FG})} = \sqrt{|g|}\, \Big(&16\, C_{\alpha\beta\gamma}\, C^{\alpha\beta\gamma} + 16\, W_\mu{}^{\alpha\beta\gamma}\, W_{\nu\alpha\beta\gamma}\, K^{\mu\nu} + \nabla_\varepsilon\, W_{\alpha\beta\gamma\delta}\, \nabla^\varepsilon\, W^{\alpha\beta\gamma\delta} \\
&+ 16\, W_{\alpha\beta\gamma\delta}\, \nabla^\beta\, C^{\alpha\gamma\delta}\Big)\,.
\end{aligned} \tag{3.11}$$

One can explicitly verify that the relation with $\tilde{\mathcal{I}}_3$ is

$$\mathcal{I}_3^{(\mathrm{FG})} = 2\,(\tilde{\mathcal{I}}_3 - 4\,\mathcal{I}_1 + \mathcal{I}_2). \tag{3.12}$$

4 Relation with Branson's Q-curvature

The notion of Q-curvature was introduced by Branson when studying the regularisation of the functional determinant of elliptic operators [36]. It emerges in many other mathematical contexts [37] and, in particular, plays an important rôle in conformal geometry [38], see also the book [39] and refs. therein.

One may introduce the Q-curvature by studying how to complete the powers of the Laplacian — in this section we assume the manifold to be Riemannian, but the signature will be irrelevant to the discussion — to obtain a conformally covariant operator. An operator $\mathcal{O}$ is said to be conformally covariant if it transforms under infinitesimal Weyl transformation in the following way:

$$\delta_\sigma\,\mathcal{O}\,\varphi = \beta\,\sigma\,\mathcal{O}\,\varphi\,, \quad \text{if } \delta_\sigma\,\varphi = \alpha\,\sigma\,\varphi\,, \quad \text{for some constants } \alpha\,,\beta\,. \tag{4.1}$$

In D dimensions the transformation of the Laplacian $\triangle = g^{\mu\nu}\,\nabla_\mu\nabla_\nu$ is

$$\delta_\sigma\,\triangle\varphi = -(2-\alpha)\,\sigma\,\triangle\varphi + \alpha\,\triangle\sigma\,\varphi + (2\,\alpha - 2 + D)\;\nabla_\mu\,\sigma\,\nabla^\mu\,\varphi\,, \tag{4.2}$$

if φ transforms as in (4.1). There is no choice of α to make it conformally covariant. But one can notice that the Laplacian of the Weyl parameter is included in the transformation of the trace of the Schouten tensor:

$$\delta_\sigma\,K_\mu{}^\mu = -\triangle\sigma - 2\,\sigma\,K_\mu{}^\mu\,, \tag{4.3}$$

so that, for some constant β,

$$\begin{aligned}\delta_\sigma\,(\triangle + \beta\,K_\mu{}^\mu)\,\varphi = &-(2-\alpha)\,\sigma\,(\triangle + \beta\,K_\mu{}^\mu)\,\varphi +\\ &+(\alpha-\beta)\,\triangle\sigma\,\varphi + (2\,\alpha - 2 + D)\,\nabla_\mu\,\sigma\,\nabla^\mu\,\varphi\,.\end{aligned} \tag{4.4}$$

Thus, is it sufficient to choose $\alpha = \beta = -\frac{D-2}{2}$ to get a conformally covariant operator. The resulting operator in D dimensions, usually called *Yamabe operator* [40], reads

$$Y_D = \triangle + \tfrac{D-2}{2}\,K_\mu{}^\mu = \triangle + \tfrac{D-2}{4(D-1)}\,R\,, \tag{4.5}$$

whose transformation is

$$\delta_\sigma\,Y_D\,\varphi = -\tfrac{D+2}{2}\,\sigma\,Y_D\,\varphi\,, \quad \text{if } \delta_\sigma\,\varphi = -\tfrac{D-2}{2}\,\sigma\,\varphi\,. \tag{4.6}$$

Notice that in the critical dimension $D = 2$, the Laplacian is automatically conformally covariant, and the integral of the density $\sqrt{|g|}\,K$ (or equivalently, the Einstein-Hilbert action) is conformally invariant, since the Laplacian contribution $\triangle\sigma$ in the conformal transformation of $K_\mu{}^\mu$ contributes through a total derivative.

Consider now the more ambitious task of conformally completing the square of the Laplacian. The result in four dimensions was found by Fradkin and Tseytlin in [41], and also by Riegert [42]; in arbitrary dimensions, it was found by Paneitz in [43]. By dimensional analysis, one can start from the following ansatz:[1]

$$\begin{aligned} P_D\,\varphi = {} & \triangle^2\varphi + \beta_1\,\nabla^\nu\,K_\mu{}^\mu\,\nabla_\nu\,\varphi + \beta_2\,\triangle K_\mu{}^\mu\,\varphi + \beta_3\,K_{\mu\nu}\,K^{\mu\nu}\,\varphi + \\ & + \beta_4\,K_\mu{}^\mu\,K_\nu{}^\nu\,\varphi + \gamma_1\,K^{\mu\nu}\,\nabla_\mu\,\nabla_\nu\,\varphi + \gamma_2\,K_\mu{}^\mu\,\triangle\varphi\,. \end{aligned} \tag{4.7}$$

By explicit evaluation, one obtains

$$\begin{aligned} \delta_\sigma\,P_D\,\varphi + (4-\alpha)\,\sigma\,P_D\,\varphi = {} & (-2\,(\beta_1+\beta_2) + \alpha\,\gamma_2)\,K_\mu{}^\mu\,\triangle\sigma\;\varphi \\ & + (\alpha\,\beta_2 + (D-6)\,\beta_2)\,\nabla_\mu K_\nu{}^\nu\,\nabla^\mu\sigma\;\varphi \\ & + (6-\beta_1+\gamma_1+2\,\alpha\,(\gamma_2-2) + D\,(\gamma_2-1) - 2\,\gamma_2)\,K_\mu{}^\mu\,\nabla_\nu\sigma\,\nabla^\nu\;\varphi \\ & + (-2\,\beta_3 + \alpha\,\gamma_1)\,K^{\mu\nu}\,\nabla_\mu\nabla_\nu\sigma\;\varphi \\ & + (2\,(\alpha-1)-\gamma_2)\,\triangle\sigma\,\triangle\varphi + 2\,(D-4+2,\alpha)\,\nabla^\mu\sigma\,\triangle\nabla_\mu\varphi \\ & + (D-2+4\,\alpha-\beta_1)\,\nabla^\mu\varphi\,\triangle\nabla_\mu\sigma + (\alpha-\beta_2)\,\varphi\,\triangle^2\sigma \\ & + ((D-2)(D-6+4\,\alpha-\beta_1) + 2\,(\alpha-1)\,\gamma_1)\,K^{\mu\nu}\,\nabla_\mu\sigma\,\nabla_\nu\sigma \\ & + (2\,(D-2) + 4\,\alpha - \gamma_1)\,\nabla_\mu\nabla_\nu\sigma\,\nabla^\mu\nabla^\nu\varphi\,. \end{aligned} \tag{4.8}$$

The right-hand side vanishes if and only if

$$\alpha = \beta_2 = -\tfrac{D-4}{2}, \quad \beta_1 = 6-D, \quad \beta_3 = 4-D, \quad \beta_4 = \tfrac{D(D-4)}{4}, \quad \gamma_1 = 4, \quad \gamma_2 = 2-D. \tag{4.9}$$

Replacing these values for the constants in the ansatz (4.7), and manipulating a little bit, one finds the *Paneitz operator*

$$\begin{aligned} P_D = {} & \nabla_\mu\,(\nabla^\mu\,\nabla^\nu + 4\,K^{\mu\nu} - 4\,(D-2)\,g^{\mu\nu}\,K_\rho{}^\rho)\,\nabla_\nu \\ & + \tfrac{D-4}{2}\,(-2\,K_{\mu\nu}\,K^{\mu\nu} + \tfrac{D}{2}\,K_\mu{}^\mu\,K_\nu{}^\nu - \triangle K_\mu{}^\mu)\,. \end{aligned} \tag{4.10}$$

We recognise the same structure as in the Yamabe operator Y_D: on the first line there is the Laplacian squared (improved in such a way as to take the form $\nabla_\mu\,\mathcal{S}^{\mu\nu}\,\nabla_\mu$, where $\mathcal{S}^{\mu\nu}$ is rank-two symmetric tensor operator), while the second line, which vanishes in four dimensions, gives a purely multiplicative (i.e., non-differential) operator. Let us denote it by $\mathcal{Q}_{4,D}$, where 4 is the order of $\triangle^2$:

$$\mathcal{Q}_{4,D} := \sqrt{|g|}\,(-2\,K_{\mu\nu}\,K^{\mu\nu} + \tfrac{D}{2}\,K_\mu{}^\mu\,K_\nu{}^\nu - \triangle K_\mu{}^\mu)\,, \tag{4.11}$$

the density factor $\sqrt{|g|}$ being included for future convenience. In analogy with the Yamabe operator case, $\mathcal{Q}_{4,D}$ is expected to be conformally invariant, when integrated on a closed manifold of dimension $D = 4$:

$$\delta_\sigma \int \mathrm{d}^4x\,\mathcal{Q}_4 = \delta_\sigma \int \mathrm{d}^4x\,\sqrt{|g|}(-2\,K_{\mu\nu}\,K^{\mu\nu} + 2\,K_\mu{}^\mu\,K_\nu{}^\nu - \triangle K_\mu{}^\mu) = 0\,, \tag{4.12}$$

where $\mathcal{Q}_4 := \mathcal{Q}_{4,4}$. A simple way to see this is to notice that

$$\tfrac{1}{\sqrt{-g}}\,\mathcal{Q}_4 = -2\,K_{\mu\nu}\,K^{\mu\nu} + 2\,K_\mu{}^\mu\,K_\nu{}^\nu - \triangle K_\mu{}^\mu \tag{4.13}$$

$$\begin{aligned} & = -\tfrac{1}{16}\,(32\,(K_{\mu\nu}\,K^{\mu\nu} - K_\mu{}^\mu\,K_\nu{}^\nu) - 4\,W_{\mu\nu\rho\sigma}\,W^{\mu\nu\rho\sigma}) - \tfrac{1}{4}\,W_{\mu\nu\rho\sigma}\,W^{\mu\nu\rho\sigma} - \triangle K_\mu{}^\mu \\ & = -\tfrac{1}{16}\,\varepsilon^{\mu\nu\rho\sigma}\,\varepsilon_{\alpha\beta\gamma\delta}\,R_{\mu\nu}{}^{\alpha\beta}\,R_{\rho\sigma}{}^{\gamma\delta} - \tfrac{1}{4}\,W_{\mu\nu\rho\sigma}\,W^{\mu\nu\rho\sigma} - \triangle K_\mu{}^\mu\,, \end{aligned} \tag{4.14}$$

[1] Recall that $\nabla^\mu\,K_{\mu\nu} = \nabla_\nu\,K$, as a consequence of the differential Bianchi identity.

JHEP02(2026)101

where the first term on the last line is the four-dimensional Euler invariant (*Gauss-Bonnet invariant*), which is topological. Therefore, when integrated, only the second term could contribute to the Weyl transformation, but it is manifestly conformally invariant in $4D$ when multiplied by $\sqrt{|g|}$ to make it a scalar density.

This story can be generalised in the following way. One considers the mth power $\triangle^m$ of the Laplacian; its conformal completion $P_{2m,D}$ in D dimensions was discussed by Graham, Jenne, Mason, and Sparling, in [44]. As argued by Branson, it takes the form

$$P_{2m,D} = \nabla_\mu \mathcal{S}_D^{\mu\nu} \nabla_\nu + \tfrac{D-2m}{2} \tfrac{1}{\sqrt{|g|}} \mathcal{Q}_{2m,D} \,, \tag{4.15}$$

where $\mathcal{S}_D^{\mu\nu}$ is a rank-two symmetric tensor operator, such that $\nabla_\mu \mathcal{S}_D^{\mu\nu} \nabla_\nu = \triangle^m + \dots$, where the ellipsis stands for lower derivative terms, and $\mathcal{Q}_{2m,D}$ is a purely multiplicative (non-differential) operator defined by the explicit expression of $P_{2m,D}$. The conformal transformation of $P_{2m,D}$ is required to be

$$\delta_\sigma P_{2m,D}\, \varphi = -\tfrac{D+2m}{2}\, \sigma\, P_{2m,D}\, \varphi \,, \quad \text{if} \;\; \delta_\sigma\, \varphi = -\tfrac{D-2m}{2}\, \sigma\, \varphi \,. \tag{4.16}$$

In $D = 2\,m$, $\mathcal{Q}_{2m} := \mathcal{Q}_{2m,2m}$ is the Q-*curvature* in $2m$ dimensions. Its integral is conformally invariant:

$$\delta_\sigma \int \mathrm{d}^{2m} x\, \mathcal{Q}_{2m} = 0 \,. \tag{4.17}$$

If $m = 1$, $P_{2,D} = Y_D$ is the Yamabe operator, and $\mathcal{Q}_{2,D} = -\sqrt{|g|}\, K_\mu{}^\mu$, so that the two-dimensional Q-curvature is

$$\mathcal{Q}_2 = -\sqrt{|g|}\, K_\mu{}^\mu = -\tfrac{1}{2} \sqrt{|g|}\, R \,. \tag{4.18}$$

If $m = 2$, $P_{4,D} = P_D$ is the Paneitz operator, and the four dimensional Q-curvatures is given by (4.13). The property (4.14) for the decomposition of a global conformal invariant generalises to arbitrary even dimensions[2] [26, 28]:

$$\mathcal{Q}_{2m} = \alpha_D\, \mathcal{E}_{2m}(R) + \mathcal{I} + \partial_\mu \mathcal{V}^\mu, \tag{4.19}$$

where α_D is a constant, $\mathcal{E}_{2m}(R)$ is the Euler density in dimension $D = 2m$,

$$\mathcal{E}_{2m}(R) = \sqrt{|g|}\, \varepsilon^{\mu_1 \dots \mu_{2m}}\, \varepsilon_{\alpha_1 \dots \alpha_{2m}}\, R_{\mu_1 \mu_2}{}^{\alpha_1 \alpha_2} \dots R_{\mu_{2m-1} \mu_{2m}}{}^{\alpha_{2m-1} \alpha_{2m}} \,, \tag{4.20}$$

and where the second term $\mathcal{I}$ is a *local* (i.e. pointwise) conformally invariant density. The general decomposition (4.19) implies that the functional derivative $\frac{1}{\sqrt{|g|}} \frac{\delta S_{2m}}{\delta g^{\mu\nu}}$ of the functional

[2]In [26], a confusion in the motivation behind the works leading to [28] is explained. The conjecture made by Deser and Schwimmer [45], taken as a motivation in [28], does *not* concern global conformal invariants. Instead, it concerns the general structure of conformal (or Weyl) anomalies in quantum field theory, a different notion as compared to global conformal invariants. The conjecture [45] of Deser and Schwimmer for the classification of conformal anomalies was solved in [34, 35] by using cohomological techniques. In particular, it was proven that conformal anomalies are trivial in odd spacetime dimensions. The same cohomological tools were used in [26] to provide an alternative derivation and completion of the main result of [28] concerning the general structure of global conformal invariants in arbitrary dimension. In particular, in [26] were found the global conformal invariants in dimensions $4m - 1$, $m \in \mathbb{N}^+$.

$S_{2m}[g] = \int \mathrm{d}^{2m}x\, \mathcal{Q}_{2m}$ furnishes a divergenceless, traceless, rank-two symmetric, conformally covariant tensor of weight $2-2m$. In two dimensions it is obviously proportional to the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. In four dimensions, it gives the Bach tensor $B_{\mu\nu}$, the left-hand side of the equations of motion of four-dimensional conformal gravity:

$$\begin{aligned}
\delta \int \mathrm{d}^4x\, \mathcal{Q}_4 &= \delta \left(-\frac{1}{4} \int \mathrm{d}^4x \sqrt{|g|}\, W_{\mu\nu\rho\sigma}\, W^{\mu\nu\rho\sigma} \right) \\
&= - \int \mathrm{d}^4x\, \sqrt{|g|}\, (\nabla^\alpha C_{\mu\nu\alpha} + W_{\mu\alpha\nu\beta}\, K^{\alpha\beta})\, \delta g^{\mu\nu} \\
&= - \int \mathrm{d}^4x\, \sqrt{|g|}\, B_{\mu\nu}\, \delta g^{\mu\nu}.
\end{aligned} \tag{4.21}$$

In the general case, as proved in [33], one gets a higher-dimensional generalisation of the Bach tensor, called the *Fefferman-Graham obstruction tensor* $O^{(2m)}_{\mu\nu}$, introduced in the context of the ambient metric construction of [20] — see also [38] for a concise review of the Q-curvature and its definition in terms of the ambient metric in dimension $2m+2$. Explicitly,

$$\delta \int \mathrm{d}^{2m}x\, \mathcal{Q}_{2m} = - \int \mathrm{d}^{2m}x\, \sqrt{|g|}\, O^{(2m)}_{\mu\nu}\, \delta g^{\mu\nu}\,, \tag{4.22}$$

where $O^{(2)}_{\mu\nu} = -\frac{1}{2}\, G_{\mu\nu}$, and $O^{(4)}_{\mu\nu} = B_{\mu\nu}$.

The Fefferman-Graham obstruction tensor $O^{(2m)}_{\mu\nu}$ is not only divergenceless, traceless and symmetric, but it also enjoys the property that it identically vanishes for metrics that are conformally Einstein, see e.g. [33], also Chapt. 7 of [46], and references therein. Thus, the local conformal invariant $\mathcal{I}$ in the general decomposition (4.19) of the Q-curvature is a combination of the possible $2m$-dimensional local (pointwise) conformal invariants, such that the variational principle based on $\mathcal{Q}_{2m}$ always admits an Einstein sector upon extremization. Since the number of local conformal invariants quickly growths with the dimension, in general there might be several linear combinations of the local conformal invariants that lead to symmetric, divergenceless and traceless tensors vanishing on Einstein metrics, and the integrated Q-curvature could be only one among many global conformal invariants that give rise to such symmetric tensors, upon variational derivative with respect to the (inverse) metric. Equivalently, in an arbitrary space of even dimension $D = 2m$, there could be several tensors that share the properties of the Fefferman-Graham obstruction tensor. We will return to this discussion in section 6 and proceed now with a detailed review of the six-dimensional case.

The six-dimensional Q-curvature can be computed following the above construction. The result is [47] (see also [37]):

$$\begin{aligned}
\mathcal{Q}_6 = \sqrt{|g|}\, \Big(& 8\, \nabla^\mu K^{\nu\rho}\, \nabla_\mu K_{\nu\rho} + 16\, K_{\mu\nu}\, \triangle\, K^{\mu\nu} - 32\, K_{\mu\nu}\, K^\mu{}_\rho\, K^{\nu\rho} \\
& - 16\, K^{\mu\nu}\, K_{\mu\nu}\, K_\rho{}^\rho + 8\, K_\mu{}^\mu\, K_\nu{}^\nu\, K_\rho{}^\rho - 8\, K_\mu{}^\mu\, \triangle\, K_\nu{}^\nu \\
& + \triangle^2\, K_\mu{}^\mu + 16\, W^{\mu\rho\nu\sigma}\, K_{\mu\nu}\, K_{\rho\sigma} \Big)\,.
\end{aligned} \tag{4.23}$$

The explicit expression for its variation, given by the six-dimensional Fefferman-Graham obstruction tensor, is explicitly computed in [33]:

$$\begin{aligned}
O^{(6)}_{\mu\nu} = -\tfrac{1}{2}\, (& \triangle\, B_{\mu\nu} - 2\, W_{\rho\mu\nu\sigma}\, B^{\rho\sigma} - 4\, B_{\mu\nu}\, K_\rho{}^\rho + 8\, \nabla_\sigma\, C_{\mu\nu\rho}\, K^{\rho\sigma} \\
& + 8\, \nabla_\sigma\, C_{\nu\mu\rho}\, K^{\rho\sigma} - 4\, C^\rho{}_\mu{}^\sigma\, C_{\sigma\nu\rho} + 2\, C_\mu{}^{\rho\sigma}\, C_{\nu\rho\sigma} + 4\, \nabla_\sigma\, K_\rho{}^\rho\, W_{\mu\nu}{}^\sigma \\
& + 4\, \nabla_\sigma\, K_\rho{}^\rho\, W_{\nu\mu}{}^\sigma - 4\, W_{\rho\mu\nu\sigma}\, K_\tau{}^\rho\, K^{\sigma\tau})\,,
\end{aligned} \tag{4.24}$$

which is divergenceless, traceless, symmetric, and identically vanishing on Einstein manifolds, as it should be. Consistently with the general structure (4.19), one can show (see also [48–50]) that $\mathcal{Q}_6$ in (4.23) can equivalently be written as

$$\begin{aligned}\mathcal{Q}_6 = & -\tfrac{10}{3}\,\mathcal{I}_1 - \tfrac{1}{6}\,\mathcal{I}_2 + \widetilde{\mathcal{I}}_3 - \tfrac{1}{48}\,\mathcal{E}_6(R) \\ & + \sqrt{|g|}\;\nabla^\mu \Big(5\, W_{\mu\alpha\beta\gamma}\, C^{\alpha\beta\gamma} + 8\, K_\mu{}^\alpha\, \nabla_\alpha\, K_\beta{}^\beta + W^{\alpha\beta\gamma\delta}\, \nabla_\beta\, W_{\mu\alpha\gamma\delta} \\ & - 8\, K^{\alpha\beta}\, \nabla_\beta\, K_{\mu\alpha} + 16\, K^{\alpha\beta}\, \nabla_\mu\, K_{\alpha\beta} - 8\, K_\nu{}^\nu\, \nabla_\mu\, K_\rho{}^\rho + \nabla_\mu \triangle K_\nu{}^\nu\Big)\,, \end{aligned} \tag{4.25}$$

where the six-dimensional Euler density is, according to (4.20),

$$\mathcal{E}_6(R) = \sqrt{|g|}\, \varepsilon^{\mu\nu\rho\sigma\kappa\lambda}\, \varepsilon_{\alpha\beta\gamma\delta\varepsilon\zeta}\, R_{\mu\nu}{}^{\alpha\beta}\, R_{\rho\sigma}{}^{\gamma\delta}\, R_{\kappa\lambda}{}^{\varepsilon\zeta}\,. \tag{4.26}$$

Since the Euler invariant is topological and the last two lines contribute to a total derivative, the integral of the Q-curvature on a closed manifold is proportional to the conformal action $\tilde{S}_6[g_{\mu\nu}]$ (3.8) with Einstein sector, up to an additive constant arising from the integral of the Euler density:

$$\int_{M_6} \mathrm{d}^6x\; \mathcal{Q}_6 = 64\pi^3\, \chi(M_6) + \int_{M_6} \mathrm{d}^6x\, \sqrt{|g|}\, (-\tfrac{10}{3}\,\mathcal{I}_1 - \tfrac{1}{6}\,\mathcal{I}_2 + \widetilde{\mathcal{I}}_3)\,, \tag{4.27}$$

where one uses

$$\int_{M_{2m}} \mathrm{d}^{2m}x\, \mathcal{E}_{2m}(R) = (-1)^m\, (4\,\pi)^m\, m!\, 2^m\, \chi(M_{2m}), \tag{4.28}$$

and one recognises the precise combination which defines the Lü-Pang-Pope six-dimensional conformal gravity action in eq. (3.8). That is, up to boundary terms, one has

$$\tilde{S}_6[g_{\mu\nu}] = 128\pi^3\, \chi(M_6) - 2\int_{M_6} \mathrm{d}^6x\; \mathcal{Q}_6\,. \tag{4.29}$$

5 8D conformal gravity with Einstein sector

In this section, we build the most general conformal gravity action in $8D$ that admits an Einstein sector. We will see that, although the number of local conformal invariants increases dramatically compared to the $4D$ and $6D$ cases, there still is only one linear combination of them that ensures that the theory admits an Einstein sector, and we will see that this reproduces the Q-curvature in eight dimensions.

There are seven possible parity-even scalars that are quartic in the undifferentiated Weyl tensor. One can choose the following basis [51]:

$$I_6 = W_{\alpha\beta}{}^{\nu\sigma}\, W^{\alpha\beta\gamma\delta}\, W_{\gamma\nu}{}^{\rho\mu}\, W_{\delta\sigma\rho\mu}\,, \tag{5.1}$$

$$I_7 = W_\alpha{}^\nu{}_\gamma{}^\sigma\, W^{\alpha\beta\gamma\delta}\, W_\beta{}^\rho{}_\delta{}^\mu\, W_{\nu\rho\sigma\mu}\,, \tag{5.2}$$

$$I_8 = W_{\alpha\beta}{}^{\nu\sigma}\, W^{\alpha\beta\gamma\delta}\, W_{\gamma\delta}{}^{\rho\mu}\, W_{\nu\rho\sigma\mu}\,, \tag{5.3}$$

$$I_9 = W_{\alpha\beta\gamma}{}^\nu\, W^{\alpha\beta\gamma\delta}\, W_\delta{}^{\sigma\rho\mu}\, W_{\nu\rho\sigma\mu}\,, \tag{5.4}$$

$$I_{10} = W_{\alpha\beta\gamma\delta}\, W^{\alpha\beta\gamma\delta}\, W_{\nu\rho\sigma\mu}\, W^{\nu\sigma\rho\mu}\,, \tag{5.5}$$

$$I_{11} = W_\alpha{}^\nu{}_\gamma{}^\sigma\, W^{\alpha\beta\gamma\delta}\, W_\beta{}^\rho{}_\sigma{}^\mu\, W_{\delta\mu\nu\rho}\,, \tag{5.6}$$

$$I_{12} = W_{\alpha\gamma}{}^{\nu\sigma}\, W^{\alpha\beta\gamma\delta}\, W_\beta{}^\rho{}_\nu{}^\mu\, W_{\delta\mu\sigma\rho}\,. \tag{5.7}$$

The corresponding densities $\mathcal{I}_i = \sqrt{|g|}\, I_i$, $i \in \{6, \ldots, 12\}$, are trivially Weyl-invariant in $8D$. Then, there are five independent non-trivial Weyl-invariant scalar densities $\mathcal{I}_j = \sqrt{|g|}\, I_j$, $j \in \{1, \ldots, 5\}$, in eight-dimensions [24], that involve derivatives of the Weyl tensor. In total, that gives twelve linearly independent, local (i.e. pointwise) conformal invariants in $8D$. As a result of the findings in [52], we find that two of the five non-trivial invariants of [24], namely $\mathcal{I}_4$ and $\mathcal{I}_5$, can be expressed in terms of the other ten, up to total derivatives. Therefore, if one is interested in the problem of integrated densities and consider a closed $8D$ manifold, the two invariants $\mathcal{I}_4$ and $\mathcal{I}_5$ from the list of [24] can be omitted. More in details, we find that the two independent, dimension-eight, Weyl-invariant total derivatives found in [52] can be written in terms of the W-tensors as $\sqrt{|g|}\, \nabla_\mu\, J^\mu_{(i)}(W)$, $i = 1, 2$, where

$$
\begin{aligned}
J^\alpha_{(1)}(W) = &-\tfrac{1}{5}\, W_{\beta\gamma\delta}{}^{\sigma}\, W^{\beta\gamma\delta\epsilon}\, \mathcal{D}_\rho \mathcal{D}^\rho W^\alpha{}_{\epsilon\sigma} + W^{\alpha\beta\gamma\delta}\, W_\beta{}^{\epsilon\sigma\rho}\, \mathcal{D}_\rho W_{\gamma\delta\epsilon\sigma} \\
&- \tfrac{4}{15}\, W^{\alpha\beta\gamma\delta}\, W_\beta{}^\epsilon{}_\gamma{}^\sigma\, \mathcal{D}^\rho W_{\delta\epsilon\sigma\rho} + \tfrac{8}{15}\, W^{\alpha\beta\gamma\delta}\, W_\beta{}^\epsilon{}_\gamma{}^\sigma\, \mathcal{D}^\rho W_{\epsilon\sigma\delta\rho} \\
&+ \tfrac{4}{15}\, W^{\alpha\beta\gamma\delta}\, W_\beta{}^\epsilon{}_\gamma{}^\sigma\, \mathcal{D}^\rho W_{\delta\sigma\epsilon\rho}\,, \qquad (5.8) \\
J^\alpha_{(2)}(W) = &\; W^{\alpha\beta\gamma\delta}\, W_\gamma{}^{\epsilon\sigma\rho}\, \mathcal{D}_\rho W_{\beta\epsilon\delta\sigma} - W^{\alpha\beta\gamma\delta}\, W_\beta{}^{\epsilon\sigma\rho}\, \mathcal{D}_\rho W_{\gamma\delta\epsilon\sigma} \\
&- \tfrac{2}{5}\, W^{\alpha\beta\gamma\delta}\, W_\beta{}^\epsilon{}_\gamma{}^\sigma\, \mathcal{D}^\rho W_{\epsilon\sigma\delta\rho} - \tfrac{2}{5}\, W^{\alpha\beta\gamma\delta}\, W_\beta{}^\epsilon{}_\gamma{}^\sigma\, \mathcal{D}^\rho W_{\delta\sigma\epsilon\rho}\,. \qquad (5.9)
\end{aligned}
$$

For a different proof that the space of local conformal invariants of weight -8 that are divergences on 8-manifolds is 2-dimensional, see [53].

Then, we find the following relations:

$$
I_4 = \tfrac{1}{40}\, I_2 - \tfrac{1}{40}\, I_3 + \tfrac{25}{3}\, I_6 + \tfrac{8}{3}\, I_7 + \tfrac{2}{3}\, I_8 - 7\, I_9 - \tfrac{8}{3}\, I_{11} - \tfrac{58}{3}\, I_{12} + \nabla_\alpha \left(-5\, J^\alpha_{(1)} + 2\, J^\alpha_{(2)}\right), \qquad (5.10)
$$

$$
I_5 = \tfrac{1}{5}\, I_2 + \tfrac{14}{3}\, I_6 + \tfrac{4}{3}\, I_7 + \tfrac{1}{3}\, I_8 - 4\, I_9 - \tfrac{4}{3}\, I_{11} - \tfrac{32}{3}\, I_{12} - 4\, \nabla_\alpha\, J^\alpha_{(1)}\,, \qquad (5.11)
$$

that allow us to omit the two densities $\mathcal{I}_4$ and $\mathcal{I}_5$ from the expression for the Lagrangian density of conformal gravity in $8D$. The remaining three non-trivial invariants of [24] are recalled here, for the sake of completeness:

$$
\begin{aligned}
I_1 = &\; W_{\rho\gamma\mu\sigma}\, W^{\rho\gamma\mu\sigma,\alpha}{}_{\alpha\beta}{}^{\beta} + \tfrac{48}{25}\, W^\beta{}_{\gamma\mu\alpha,\beta} W^{\rho\gamma\mu\alpha}{}_{\rho\nu}{}^{\nu} \\
&+ 2\, W_{\mu\beta\gamma\nu,\alpha}\, W^{\mu\beta\gamma\nu,\alpha\rho}{}_{\rho} + \tfrac{42}{125}\, W_{\gamma\alpha\beta\mu}{}^{\beta\alpha}\, W^{\gamma\nu\rho\mu}{}_{\rho\nu} \\
&+ \tfrac{9}{10}\, W_{\alpha\mu\nu\beta,\gamma}{}^{\gamma}\, W^{\alpha\mu\nu\beta,\rho}{}_{\rho} + \tfrac{3}{5}\, W_{\nu\gamma\mu\rho,\beta\alpha}\, W^{\nu\gamma\mu\rho,\beta\alpha} \\
&+ \tfrac{96}{125}\, W^\gamma{}_{\mu\nu\beta,\gamma\alpha}\, W^{\rho\mu\nu\beta}{}_{\rho}{}^{\alpha} + \tfrac{74}{25}\, W_\beta{}^{\alpha\gamma\mu}\, W_{\nu\alpha\gamma\mu}\, W^\beta{}_{\rho\sigma}{}^{\nu,\sigma\rho} \\
&+ \tfrac{208}{5}\, W_{\mu\beta\gamma\alpha}\, W_\sigma{}^{\nu\rho\alpha}\, W^\mu{}_{\nu\rho}{}^{\sigma,\gamma\beta} - 8\, W_\alpha{}^\gamma{}_\beta{}^\mu\, W^\alpha{}_\nu{}^\beta{}_\rho\, W^\nu{}_\gamma{}^\rho{}_{\mu,\sigma}{}^{\sigma} \\
&+ \tfrac{16}{5}\, W_{\alpha\gamma\mu\rho}\, W_{\beta\nu}{}^{\alpha\gamma}\, W^{\beta\nu\mu\rho}{}_{\sigma}{}^{\sigma} - \tfrac{144}{25}\, W^\gamma{}_\alpha{}^\mu{}_\beta\, W_{\rho\gamma}{}^\nu{}_\mu{}^\rho\, W_\sigma{}^\alpha{}_\nu{}^{\beta,\sigma} \\
&+ \tfrac{104}{5}\, W_\alpha{}^\gamma{}_\beta{}^\mu\, W^\beta{}_\mu{}^{\sigma\nu,\alpha}\, W_{\rho\gamma\sigma\nu}{}^{\rho} - \tfrac{88}{25}\, W_{\alpha\beta\gamma\mu}\, W_{\rho\nu}{}^{\alpha\beta,\rho}\, W_\sigma{}^{\nu\gamma\mu,\sigma}\,, \qquad (5.12) \\
I_2 = &\; W_\beta{}^{\alpha\gamma\mu}\, W_{\nu\alpha\gamma\mu}\, W^\beta{}_{\rho\sigma}{}^{\nu,\sigma\rho} + 5\, W_{\alpha\gamma\mu\rho}\, W_{\beta\nu}{}^{\alpha\gamma}\, W^{\beta\nu\mu\rho}{}_{\sigma}{}^{\sigma} \\
&+ 5\, W_{\alpha\beta\gamma\mu}\, W^{\alpha\beta\rho\sigma}{}_{\nu}\, W^{\gamma\mu}{}_{\rho\sigma}{}^{\nu} + \tfrac{12}{5}\, W_{\alpha\beta\gamma\mu}\, W_{\rho\nu}{}^{\alpha\beta,\rho}\, W_\sigma{}^{\nu\gamma\mu,\sigma}\,, \qquad (5.13) \\
I_3 = &\; W_\beta{}^{\alpha\gamma\mu}\, W_{\nu\alpha\gamma\mu}\, W^\beta{}_{\rho\sigma}{}^{\nu,\sigma\rho} - 20\, W_\alpha{}^\gamma{}_\beta{}^\mu\, W^\alpha{}_\nu{}^\beta{}_\rho\, W^\nu{}_\gamma{}^\rho{}_{\mu,\sigma}{}^{\sigma} \\
&- \tfrac{48}{5}\, W^\gamma{}_\alpha{}^\mu{}_\beta\, W_{\rho\gamma}{}^\nu{}_\mu{}^\rho\, W_\sigma{}^\alpha{}_\nu{}^{\beta,\sigma} - 20\, W^\alpha{}_\mu{}^\gamma{}_\beta\, W^\mu{}_\rho{}^\beta{}_{\sigma,\nu}\, W^\rho{}_\alpha{}^\sigma{}_\gamma{}^\nu\,. \qquad (5.14)
\end{aligned}
$$

Therefore, the most general action for eight-dimensional conformal gravity is an arbitrary linear combination of the previous $3 + 7$ invariants:

$$
S_8[g_{\mu\nu}] = \int \mathrm{d}^8x\, (w_1\, \mathcal{I}_1 + w_2\, \mathcal{I}_2 + w_3\, \mathcal{I}_3 + w_6\, \mathcal{I}_6 + \cdots + w_{12}\, \mathcal{I}_{12})\,. \qquad (5.15)
$$

We compute the variation of $S_8[g_{\mu\nu}]$ with respect to the metric and impose that it should vanish for an Einstein metric, to find that the most general action compatible with an Einstein sector is the following unique one, up to an overall factor:

$$\tilde{S}_8 = -\tfrac{1}{4}\int_{M_8} \mathrm{d}^8x\,(\mathcal{I}_1-\tfrac{23}{25}\,\mathcal{I}_2-\tfrac{21}{25}\,\mathcal{I}_3-39\,\mathcal{I}_6+\tfrac{2}{5}\,\mathcal{I}_7-\tfrac{7}{2}\,\mathcal{I}_8+\tfrac{124}{5}\,\mathcal{I}_9+\tfrac{9}{20}\,\mathcal{I}_{10}-4\,\mathcal{I}_{11}-\tfrac{568}{5}\,\mathcal{I}_{12})\,. \tag{5.16}$$

To reach this result, we used an algebraic method very similar to the one described in [24]. The challenge is to properly take into account the algebraic and differential Bianchi identities on the W-tensors. After variation, we imposed the Einstein manifold condition, which requires the Schouten tensor appearing in the expression of the W-tensors to be proportional to the metric. We then selected a basis of independent structures, so that the whole variation written in the basis identically vanishes if and only if all the coefficients (which are linear combination of the w_i constants) vanish. The computations, although relatively straightforward, are very tedious and could not have been done without the use of Mathematica and the suite of packages xAct [54], including xTras [55].

Relation with the Q-curvature. The eight-dimensional Q-curvature was computed in [47]. We verified that there exists a suitable boundary term $\mathcal{V}^\alpha$ such that

$$\begin{aligned}\mathcal{Q}_8 &= \partial_\alpha\,\mathcal{V}^\alpha - \tfrac{1}{128}\,\mathcal{E}_8(R) \\ &\quad -\tfrac{5}{3}\,(\mathcal{I}_1-\tfrac{23}{25}\,\mathcal{I}_2-\tfrac{21}{25}\,\mathcal{I}_3-39\,\mathcal{I}_6+\tfrac{2}{5}\,\mathcal{I}_7-\tfrac{7}{2}\,\mathcal{I}_8+\tfrac{124}{5}\,\mathcal{I}_9+\tfrac{9}{20}\,\mathcal{I}_{10}-4\,\mathcal{I}_{11}-\tfrac{568}{5}\,\mathcal{I}_{12})\,,\end{aligned} \tag{5.17}$$

where $\mathcal{E}_8(R)$ is the Euler density

$$\mathcal{E}_8(R) = \sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\kappa\lambda\xi\phi}\,\varepsilon^{\alpha\beta\gamma\delta\varepsilon\zeta\theta\iota}\,R^{\mu\nu}{}_{\alpha\beta}\,R^{\rho\sigma}{}_{\gamma\delta}\,R^{\kappa\lambda}{}_{\varepsilon\zeta}\,R^{\xi\phi}{}_{\theta\iota}\,, \tag{5.18}$$

and the last term is proportional to the integrand of the eight-dimensional conformal action with Einstein sector, see (5.16). Then, integrating on the closed manifold M_8 with Euler characteristic $\chi(M_8)$, we find

$$\tilde{S}_8[g_{\mu\nu}] = \tfrac{576\pi^4}{5}\,\chi(M_8) + \tfrac{3}{20}\int_{M_8}\mathrm{d}^8x\;\mathcal{Q}_8\,, \tag{5.19}$$

and the variational derivative $O^{(8)}_{\mu\nu} = \frac{1}{\sqrt{|g|}}\frac{\delta\mathcal{Q}_8}{\delta g^{\mu\nu}}$ is a symmetric rank-two tensor, divergenceless, traceless, of conformal dimension -6, that vanishes on Einstein manifolds.

6 Discussion

The Q-curvature can be defined and explicitly constructed via the ambient method of [33, 56, 57]; see also [38] for a review, as well as the book [39]. That the functional derivative (with respect to the inverse metric) of the integrated Q-curvature on an even-dimensional closed manifold is proportional to the obstruction tensor was already proven in [33], see Theorem 1.1 therein. Moreover, in the Theorem 2.1 of the latter work, the obstruction tensor was shown to obey four defining properties. To paraphrase their Theorem 2.1, if $(M^n,[g])$ is a conformal manifold of even dimension $n=2m\geq 4$, then there exists a natural symmetric 2-tensor $O_{\mu\nu}$, called the *ambient obstruction tensor*, with the following properties:

1. *Naturality:* $O_{\mu\nu}$ is natural, i.e., it can be expressed as a universal polynomial in the metric, its inverse, the curvature, and covariant derivatives of the curvature.

2. *Linearisation:* $O_{\mu\nu}$ is a symmetric, conformally covariant tensor whose expression starts with

$$O_{\mu\nu} = \frac{1}{3-n} \triangle^{m-2} \left(\nabla^\alpha \nabla^\beta W_{\mu\alpha\nu\beta}\right) + \text{lower order terms}, \tag{6.1}$$

where $\triangle = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ (or the D'Alembertian, in Lorentzian signature), and "lower order terms" involve terms with fewer derivatives of the curvature. The tensor $O_{\mu\nu}$ has conformal weight $2-n$, meaning that for $\widehat{g} = e^{2\omega} g$, $\widehat{O}_{\mu\nu} = e^{(2-n)\omega} O_{\mu\nu}$.

3. *Trace and divergence:* traceless and divergence-free: $g^{\mu\nu}O_{\mu\nu} = 0 = \nabla^\mu O_{\mu\nu}$.

4. *Vanishing on conformally Einstein metrics:* if g is conformal to an Einstein metric, then

$$O_{\mu\nu} = 0\,. \tag{6.2}$$

As far as we understand, it is unknown whether these four properties uniquely specify the obstruction tensor; it seems likely that they do.

For possible axiomatic uniqueness results concerning the obstruction tensor, another line of investigation relies on group representation theory and the Bernstein-Gelfand-Gelfand (BGG) machinery as developed for curved geometries in [58] with detour complexes, and in particular its application [59, 60] to the (curved) conformal case, where the obstruction tensor appears as a canonical object whose vanishing allows sequences of differential operators to be conformally invariant and to form a complex. Still, in this curved case set up, we are not aware of the way to translate in group-theoretical terms the condition that the obstruction tensor should vanish on Einstein manifolds, neither are we aware of proofs of uniqueness for the obstruction tensor.

From the perspective of the present work and in the particular dimensions six and eight, the uniqueness of the action, and hence, of the obstruction tensor, appears from the requirement that the latter should vanish on an Einstein metric. For that result, it appeared crucial that the first pointwise invariant, $\mathcal{I}_1$, should appear in the Lagrangian density for conformal gravity, and it is the property (6.1) that encodes the fact that the invariant $\mathcal{I}_1$ should appear in the Lagrangian density. It is tempting to conjecture that, in arbitrary even dimension, the obstruction tensor is uniquely specified by the four conditions given in [33], and therefore that the conformal gravity action that possesses an Einstein sector is unique up to boundary terms, given by the integrated Q-curvature in dimension $2m$.

In this work, we have reviewed and re-derived the conformal actions in dimensions 4 and 6 that admit an Einstein sector, and we have compared them with the Q-curvature in the corresponding dimensions. We have then explicitly built the conformal gravity action in eight dimensions that admits an Einstein sector, and have shown that it is unique, up to boundary terms and overall normalisation. We have compared it with the integral of the Q-curvature in eight dimensions and found that they are proportional, up to an additive constant related to the Euler characteristic of the manifold, and up to boundary terms. In arbitrary even dimension, we conjecture that the conformal gravity theory that admits an Einstein sector is unique. It can be defined by the integral of the Q-curvature on the even-dimensional spacetime manifold.

In a forthcoming work in collaboration with Giorgos Anastasiou, Ignacio J. Araya, and Rodrigo Olea, we will further study the eight-conformal gravity action we have built in this work. We plan to discuss its physical properties in relation to holography and the volume-renormalisation of 8D Einstein theory, in particular addressing the problem of the conformal completion of 8D Hilbert-Einstein action, along the lines of [19, 48, 61, 62].[3]

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References

[1] K.S. Stelle, *Classical Gravity with Higher Derivatives*, Gen. Rel. Grav. **9** (1978) 353 [INSPIRE].

[2] R.J. Riegert, *Birkhoff's Theorem in Conformal Gravity*, Phys. Rev. Lett. **53** (1984) 315 [INSPIRE].

[3] P.D. Mannheim and D. Kazanas, *Solutions to the Kerr and Kerr-Newman problems in fourth order conformal Weyl gravity*, Phys. Rev. D **44** (1991) 417 [INSPIRE].

[4] D. Klemm, *Topological black holes in Weyl conformal gravity*, Class. Quant. Grav. **15** (1998) 3195 [gr-qc/9808051] [INSPIRE].

[5] V.D. Dzhunushaliev and H.J. Schmidt, *New vacuum solutions of conformal Weyl gravity*, J. Math. Phys. **41** (2000) 3007 [gr-qc/9908049] [INSPIRE].

[6] R. Emparan and H.S. Reall, *Generalized Weyl solutions*, Phys. Rev. D **65** (2002) 084025 [hep-th/0110258] [INSPIRE].

[7] E.E. Flanagan, *Fourth order Weyl gravity*, Phys. Rev. D **74** (2006) 023002 [astro-ph/0605504] [INSPIRE].

[3] See [63] for a related work.

[8] J. Oliva and S. Ray, *Birkhoff's Theorem in Higher Derivative Theories of Gravity II*, *Phys. Rev. D* **86** (2012) 084014 [arXiv:1201.5601] [INSPIRE].

[9] H. Lü, Y. Pang, C.N. Pope and J.F. Vazquez-Poritz, *AdS and Lifshitz Black Holes in Conformal and Einstein-Weyl Gravities*, *Phys. Rev. D* **86** (2012) 044011 [arXiv:1204.1062] [INSPIRE].

[10] H. Lü, A. Perkins, C.N. Pope and K.S. Stelle, *Black Holes in Higher-Derivative Gravity*, *Phys. Rev. Lett.* **114** (2015) 171601 [arXiv:1502.01028] [INSPIRE].

[11] E. Scholz, *The unexpected resurgence of Weyl geometry in late 20-th century physics*, *Einstein Stud.* **14** (2018) 261 [arXiv:1703.03187] [INSPIRE].

[12] K.S. Stelle, *Renormalization of Higher Derivative Quantum Gravity*, *Phys. Rev. D* **16** (1977) 953 [INSPIRE].

[13] S.L. Adler, *Einstein Gravity as a Symmetry-Breaking Effect in Quantum Field Theory*, *Rev. Mod. Phys.* **54** (1982) 729 [*Erratum ibid.* **55** (1983) 837] [INSPIRE].

[14] A. Zee, *Einstein Gravity Emerging From Quantum Weyl Gravity*, *Annals Phys.* **151** (1983) 431 [INSPIRE].

[15] A. Salvio, *Quadratic Gravity*, *Front. in Phys.* **6** (2018) 77 [arXiv:1804.09944] [INSPIRE].

[16] E.S. Fradkin and A.A. Tseytlin, *Conformal supergravity*, *Phys. Rept.* **119** (1985) 233 [INSPIRE].

[17] I. Antoniadis and E.T. Tomboulis, *Gauge Invariance and Unitarity in Higher Derivative Quantum Gravity*, *Phys. Rev. D* **33** (1986) 2756 [INSPIRE].

[18] J. Maldacena, *Einstein Gravity from Conformal Gravity*, arXiv:1105.5632 [INSPIRE].

[19] G. Anastasiou, I.J. Araya and R. Olea, *Einstein Gravity from Conformal Gravity in 6D*, *JHEP* **01** (2021) 134 [arXiv:2010.15146] [INSPIRE].

[20] C. Fefferman and C.R. Graham, *Conformal invariants*, *Astérisque* **131** (1985) 95.

[21] T. Parker and S. Rosenberg, *Invariants of conformal Laplacians*, *J. Diff. Geom.* **25** (1987) 199.

[22] J. Erdmenger, *Conformally covariant differential operators: Properties and applications*, *Class. Quant. Grav.* **14** (1997) 2061 [hep-th/9704108] [INSPIRE].

[23] H. Lü, Y. Pang and C.N. Pope, *Black Holes in Six-dimensional Conformal Gravity*, *Phys. Rev. D* **87** (2013) 104013 [arXiv:1301.7083] [INSPIRE].

[24] N. Boulanger and J. Erdmenger, *A classification of local Weyl invariants in $D = 8$*, *Class. Quant. Grav.* **21** (2004) 4305 [*Erratum ibid.* **39** (2022) 039501] [hep-th/0405228] [INSPIRE].

[25] N. Boulanger, *A Weyl-covariant tensor calculus*, *J. Math. Phys.* **46** (2005) 053508 [hep-th/0412314] [INSPIRE].

[26] N. Boulanger, J. François and S. Lazzarini, *A classification of global conformal invariants*, *J. Phys. A* **52** (2019) 115201 [arXiv:1809.05445] [INSPIRE].

[27] S. Alexakis, *On the decomposition of global conformal invariants. I*, math/0509571 [INSPIRE].

[28] S. Alexakis, *The decomposition of global conformal invariants*, No. 182, Princeton University Press (2012).

[29] T.N. Bailey, M.G. Eastwood and C.R. Graham, *Invariant Theory for Conformal and CR Geometry*, *Annals Math.* **139** (1994) 491.

[30] T.N. Bailey, M.G. Eastwood and A.R. Gover, *Thomas's Structure Bundle for Conformal, Projective and Related Structures*, *Rocky Mt. J. Math.* **24** (1994) 1191.

JHEP02(2026)101

[31] A.R. Gover, *Invariant Theory and Calculus for Conformal Geometries*, Adv. Math. **163** (2001) 206.

[32] A. Čap and A.R. Gover, *Standard tractors and the conformal ambient metric construction*, Annals Global Anal. Geom. **24** (2003) 231 [math/0207016] [INSPIRE].

[33] C.R. Graham and K. Hirachi, *The ambient obstruction tensor and Q-curvature*, in *AdS/CFT Correspondence: Einstein Metrics and Their Conformal Boundaries*, EMS Press (2005), p. 59–71 [DOI:10.4171/013-1/3].

[34] N. Boulanger, *Algebraic Classification of Weyl Anomalies in Arbitrary Dimensions*, Phys. Rev. Lett. **98** (2007) 261302 [arXiv:0706.0340] [INSPIRE].

[35] N. Boulanger, *General solutions of the Wess-Zumino consistency condition for the Weyl anomalies*, JHEP **07** (2007) 069 [arXiv:0704.2472] [INSPIRE].

[36] T.P. Branson, *The functional determinant*, in *Global Analysis Research Center Lecture Notes Series*, Seul National Univerisity, Vol. 4 (1993) [https://api.semanticscholar.org/CorpusID:117608295].

[37] T. Branson, *Q-curvature and spectral invariants*, in the proceedings of the *24th Winter School "Geometry and Physics"* Palermo, Italy (2005), p. 11–55 [https://eudml.org/doc/220410].

[38] S.-Y.A. Chang, M. Eastwood, B. Ørsted and P.C. Yang, *What is Q-Curvature?*, Acta Appl. Math. **102** (2008) 119.

[39] A. Juhl, *Families of Conformally Covariant Differential Operators, Q-Curvature and Holography*, Birkhäuser Basel (2009) [DOI:10.1007/978-3-7643-9900-9].

[40] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **1** (1960) 21.

[41] E.S. Fradkin and A.A. Tseytlin, *Asymptotic freedom in extended conformal supergravities*, Phys. Lett. B **110** (1982) 117 [INSPIRE].

[42] R.J. Riegert, *A Nonlocal Action for the Trace Anomaly*, Phys. Lett. B **134** (1984) 56 [INSPIRE].

[43] S.M. Paneitz, *A Quartic Conformally Covariant Differential Operator for Arbitrary Pseudo-Riemannian Manifolds (Summary)*, arXiv:0803.4331 [DOI:10.3842/sigma.2008.036] [INSPIRE].

[44] C.R. Graham, R. Jenne, L.J. Mason and G.A.J. Sparling, *Conformally Invariant Powers of the Laplacian, I: Existence*, J. Lond. Math. Soc. **2-46** (1992) 557.

[45] S. Deser and A. Schwimmer, *Geometric classification of conformal anomalies in arbitrary dimensions*, Phys. Lett. B **309** (1993) 279 [hep-th/9302047] [INSPIRE].

[46] C. Fefferman and C.R. Graham, *The ambient metric*, Princeton University Press (2012).

[47] A.R. Gover and L.J. Peterson, *Conformally invariant powers of the Laplacian, Q-curvature, and tractor calculus*, Commun. Math. Phys. **235** (2003) 339 [math-ph/0201030] [INSPIRE].

[48] G. Anastasiou, I.J. Araya, C. Arias and R. Olea, *Einstein-AdS action, renormalized volume/area and holographic Rényi entropies*, JHEP **08** (2018) 136 [arXiv:1806.10708] [INSPIRE].

[49] H. Lü and R. Wen, *Holographic (a,c)-charges and Their Universal Relation in* $d = 6$ *from Massless Higher-order Gravities*, Phys. Rev. D **99** (2019) 126003 [arXiv:1901.11037] [INSPIRE].

[50] R. Aros, F. Bugini and D.E. Diaz, *One-loop divergences in 7D Einstein and 6D conformal gravities*, JHEP **04** (2020) 080 [arXiv:1911.06891] [INSPIRE].

[51] S.A. Fulling, R.C. King, B.G. Wybourne and C.J. Cummins, *Normal forms for tensor polynomials. 1: The Riemann tensor*, Class. Quant. Grav. **9** (1992) 1151 [INSPIRE].

[52] F.-Y. Chen and H. Lü, *Holographic Weyl anomaly in 8D from general higher curvature gravity*, Phys. Rev. D **111** (2025) 086032 [arXiv:2410.16097] [INSPIRE].

[53] J.S. Case et al., *Computing renormalized curvature integrals on Poincaré-Einstein manifolds*, arXiv:2404.11319 [INSPIRE].

[54] J.M. Martin-Garcia et al., *xAct: Efficient tensor computer algebra for the Wolfram language*, https://josmar493.dreamhosters.com (2020).

[55] T. Nutma, *xTras: A field-theory inspired xAct package for mathematica*, Comput. Phys. Commun. **185** (2014) 1719 [arXiv:1308.3493] [INSPIRE].

[56] C. Fefferman and C.R. Graham, *Q-Curvature and Poincaré Metrics*, Math. Res. Lett. **9** (2002) 139.

[57] C.R. Graham and A. Juhl, *Holographic formula for Q-curvature*, Adv. Math. **216** (2007) 841.

[58] A. Čap, J. Slovák and V. Souček, *Bernstein-Gelfand-Gelfand Sequences*, Annals Math. **154** (2001) 97.

[59] R. Gover and L. Peterson, *The ambient obstruction tensor and the conformal deformation complex*, Pacific J. Math. **226** (2006) 309.

[60] T. Branson and A. Rod Gover, *The conformal deformation detour complex for the obstruction tensor*, math/0605192 [INSPIRE].

[61] G. Anastasiou, O. Miskovic, R. Olea and I. Papadimitriou, *Counterterms, Kounterterms, and the variational problem in AdS gravity*, JHEP **08** (2020) 061 [arXiv:2003.06425] [INSPIRE].

[62] G. Anastasiou et al., *Higher-dimensional Willmore energy as holographic entanglement entropy*, JHEP **01** (2025) 081 [arXiv:2409.19485] [INSPIRE].

[63] W. Jia and M. Karydas, *Obstruction tensors in Weyl geometry and holographic Weyl anomaly*, Phys. Rev. D **104** (2021) 126031 [arXiv:2109.14014] [INSPIRE].