Wild potential good reduction of low dimensional Abelian varieties

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–“I like that too,” said Christopher Robin,
“but what I like *doing* best is Nothing.”
–“How do you do Nothing?” asked Pooh,
after he had wondered for a long time.
–“Well, it’s when people call out at you just
as you’re going off to do it, What are you
going to do, Christopher Robin, and you say,
Oh, nothing, and then you go and do it.”
–“Oh, I see,” said Pooh.
–“This is a nothing sort of thing that we’re
doing now.”
–“Oh, I see,” said Pooh again.
–“It means just going along, listening to all
the things you can’t hear, and not bothering.”

A. A. Milne, The House at Pooh Corner

Abstract

Let p be a prime number. Let F be a finite extension of $\mathbb{Q}_p$, the field of p-adic numbers, and let $\mathcal{A}$ be an Abelian variety over F. To such a geometric object is attached a p-adic representation of the absolute Galois group $\mathcal{G}_F$ of F, namely its p-adic Tate module $\mathbf{V}_p(\mathcal{A})$. These varieties together with their representations have now become classical objects in Arithmetic Geometry, and have been extensively studied since the 1960's. However there still are quite a few open questions in the subject.

The Abelian variety $\mathcal{A}/F$ has good reduction if the special fibre of its Néron model remains an Abelian variety. Let L/F be a finite extension and let $\mathcal{A}_L$ be the extension of scalars of $\mathcal{A}$ to L. We say that $\mathcal{A}$ has potential good reduction over L if $\mathcal{A}_L$ has good reduction. This geometric property is carried over the associated representation $\mathbf{V}_p(\mathcal{A})$, which is then potentially crystalline. Thanks to J.-M. Fontaine's p-adic Hodge Theory, such a representation is described by its associated filtered $(\varphi, \mathcal{G}_F)$-module, a purely semilinear object. Now the extension L/K can either be unramified, tamely ramified, or wildly ramified. As the terminology suggests, the latter is the worst case scenario.

In 2005, M. Volkov (see [Vol05]) gave a characterization of the p-adic representations V of $\mathcal{G}_{\mathbb{Q}_p}$ arising from Abelian varieties over $\mathbb{Q}_p$ with tame potential good reduction. This characterization is given in terms of necessary and sufficient conditions on the filtered $(\varphi, \mathcal{G}_{\mathbb{Q}_p})$-module associated to V. In this thesis we are interested in Abelian varieties defined over F

with wild potential good reduction, that is, Abelian varieties that acquire good reduction over a wildly ramified extension. Not much is known in this situation, and our purpose is to provide the first explicit examples in low dimension.

A well-known result of J.-P. Serre and J. Tate states that when $p > 2\dim(\mathcal{A}) + 1$ the potential good reduction is necessarily tame. For this reason we concentrate on elliptic curves over $\mathbb{Q}_3$ (*i.e.* $\dim(\mathcal{A}) = 1$ and $p = 3$) and Abelian surfaces over $\mathbb{Q}_3$ and $\mathbb{Q}_5$ (*i.e.* $\dim(\mathcal{A}) = 2$ and $p = 3$ or 5).

In the first part of this thesis we provide a full classification of the 3-adic representations of $\mathcal{G}_{\mathbb{Q}_3}$ arising from elliptic curves over $\mathbb{Q}_3$ with potential good reduction. Our classification is obtained as follows. We begin by computing the filtered $(\varphi, \mathcal{G}_{\mathbb{Q}_3})$-modules satisfying the conditions from [Vol05]. Then we show that each representation obtained in this way indeed arises from some elliptic curve over $\mathbb{Q}_3$, which is achieved by constructing minimal Galois pairs introduced in [Vol05]. This classification highlights new phenomena specific to the wild potential good reduction case.

The second part of this thesis is dedicated to Abelian surfaces. Here we do not aim to classify their attached representations, but merely to investigate the inertia subgroups naturally appearing in a situation of potential good reduction, with a focus on the wild ones. In 2005 A. Silverberg and Yu. Zarhin (see [SZ05]) have classified all such possible inertia subgroups over a discretely valued field with perfect residue field. However most of the wild ones are achieved over local fields in equicharacteristic. We show that each such group can actually be realised in mixed characteristic. This result is obtained in three steps. We begin by realising each of these groups as the inertia subgroup of a finite extension of $\mathbb{Q}_p$. We then provide a minimal polarised Galois pair for each such extension. We next use a recent result of S. Philip (see [Phi24]) that guarantees the existence of appropriate filtrations on their $(\varphi, \mathcal{G}_F)$-modules, thus allowing us to lift our Galois pairs to characteristic zero and achieving our goal.

Résumé

Soit p un nombre premier. Soit F une extension finie de $\mathbb{Q}_p$, le corps de nombres p-adiques, et soit $\mathcal{A}$ une variété abélienne définie sur F. On peut associer à un tel objet géométrique son module de Tate p-adique $\mathbf{V}_p(\mathcal{A})$, qui est une représentation p-adique du groupe de Galois absolu $\mathcal{G}_F$ de F. Ces variétés ainsi que les représentations qui leur sont associées sont devenue des objets classiques en Géométrie arithmétique, et ont été étudiées de manière approfondie depuis les années 1960. Cela dit, il reste quand-même un certain nombre de questions ouvertes à leur sujet.

La variété Abélienne $\mathcal{A}/F$ a bonne réduction si la fibre spéciale de son modèle de Néron est également une variété abélienne. Soit L/F une extension finie et soit $\mathcal{A}_L$ l'extension des scalaires de $\mathcal{A}$ à L. On dit que $\mathcal{A}$ a potentielle bonne réduction sur L si $\mathcal{A}_L$ a bonne réduction. Cette propriété géométrique s'interprète sur la représentation associée $\mathbf{V}_p(\mathcal{A})$, qui est alors potentiellement crystalline. Grâce à la théorie de Hodge p-adique développée par J.-M. Fontaine, une telle représentation est décrite par son $(\varphi, \mathcal{G}_F)$-module filtré associé, un objet purement semilinéaire. Une extension finie L/K peut être non ramifiée, modérément ramifiée ou sauvagement ramifiée. Ce dernier est, comme la terminologie le suggère, le pire cas possible.

En 2005, M. Volkov (voir [Vol05]) a fourni une caractérisation des représentations p-adiques V de $\mathcal{G}_{\mathbb{Q}_p}$ provenant des variétés abéliennes sur $\mathbb{Q}_p$ ayant potentielle bonne réduction modérée. Cette caractérisation est donnée sous

forme de conditions nécessaires et suffisantes sur le $(\varphi, \mathcal{G}_{\mathbb{Q}_p})$-module filtré associé à V. Dans cette thèse, nous nous intéressons aux variétés abéliennes définies sur F ayant potentielle bonne réduction sauvage, c'est-à-dire, les variétés abéliennes qui acquièrent bonne réduction sur une extension sauvagement ramifiée. Nous savons peu de chose dans ce cas, et notre objectif est de fournir des exemples concrets en petite dimension.

Un résultat bien connu de J.-P. Serre et J. Tate nous dit que lorsque $p > 2\mathrm{dim}(\mathcal{A}) + 1$, alors la potentielle bonne réduction est nécessairement modérée. Pour cette raison, nous portons notre attention sur les courbes elliptiques définies sur $\mathbb{Q}_3$ (*i.e.* $\mathrm{dim}(\mathcal{A}) = 1$ et $p = 3$) et les surfaces abéliennes définies sur $\mathbb{Q}_3$ et $\mathbb{Q}_5$ (*i.e.* $\mathrm{dim}(\mathcal{A}) = 2$ and $p = 3$ or 5).

Dans la première partie de cette thèse, nous fournissons une classification complète des représentations 3-adiques de $\mathcal{G}_{\mathbb{Q}_3}$ provenant des courbes elliptiques sur $\mathbb{Q}_3$ ayant potentielle bonne réduction. Celle-ci est obtenue de la façon suivante. Nous commençons par déterminer les $(\varphi, \mathcal{G}_{\mathbb{Q}_3})$-modules filtrés qui satisfont les conditions de [Vol05]. Ensuite, nous montrons que chacune des représentations obtenues de cette manière provient bien d'une courbe elliptique définie sur $\mathbb{Q}_3$, ce que nous faisons en construisant des paires galoisiennes minimales (introduites dans [Vol05]). Cette classification met en lumière de nouveaux phénomènes spécifiques à la potentielle bonne réduction sauvage.

La deuxième partie de cette thèse est dédiée aux surfaces abéliennes. Ici le but n'est plus de classifier leur représentations associées, mais de commencer par déterminer les sous-groupes d'inertie qui apparaissent naturellement dans les cas de potentielle bonne réduction, en se concentrant sur les cas sauvages. En 2005, A. Silverberg et Yu. Zarhin (voir [SZ05]) ont classifié tous les sous-groupes d'inertie possibles provenant des surfaces abéliennes définies sur un corps valué à corps résiduel parfait. Cependant, la plupart des cas sauvages sont obtenus sur des corps locaux d'égale caractéristique. Nous montrons que chacun de ces groupes peut être réalisé en caractéristique mixte. Nous obtenons ce résultat en trois étapes. D'abord nous réalisons chaque groupe comme le sous-groupe d'inertie d'une extension finie de $\mathbb{Q}_p$. Ensuite, nous construisons une paire galoisienne minimale polarisée pour chacune de ces extensions. Enfin, un résultat récent de S. Philip (voir [Phi24]) nous garanti l'existence de filtrations appropriées sur leur $(\varphi, \mathcal{G}_F)$-modules associés, nous permettant de relever nos paires galoisiennes en caractéristique zéro et d'ainsi obtenir le résultat désiré.

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Contents

Introduction

The objective of this thesis is to provide explicit examples of wild potential good reduction associated to elliptic curves and Abelian surfaces.

Fix a prime number p and an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, the field of p-adic numbers. We denote by $\mathcal{G}_F$ the absolute Galois group of a finite extension F of $\mathbb{Q}_p$ and by $\mathcal{I}_F$ its inertia subgroup. Let $\mathcal{A}/F$ be a d-dimensional Abelian variety and ℓ a prime number. The set $\mathcal{A}(\overline{\mathbb{Q}_p})$ of points of $\mathcal{A}$ with values in $\overline{\mathbb{Q}_p}$ is an Abelian group. The subgroup of ℓ^n-torsion points of $\mathcal{A}(\overline{\mathbb{Q}_p})$, denoted by $\mathcal{A}[\ell^n]$, is a free $\mathbb{Z}/\ell^n\mathbb{Z}$-module of rank $2d$ over which $\mathcal{G}_F$ acts linearly and continuously. The ℓ-adic representation associated to $\mathcal{A}$, also called Tate module, is

$$\mathbf{V}_\ell(\mathcal{A}) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_{\ell} \mathcal{A}[\ell^n].$$

It is a $2d$-dimensional $\mathbb{Q}_\ell$-vector space equipped with a linear and continuous action of $\mathcal{G}_F$. We denote by

$$\rho_{\ell,\mathcal{A}} : \mathcal{G}_F \longrightarrow \mathrm{Aut}_{\mathbb{Q}_\ell}(\mathbf{V}_\ell(\mathcal{A})) \simeq \mathrm{GL}_{2d}(\mathbb{Q}_\ell)$$

the associated continuous group homomorphism. When $\mathcal{A}/F$ has potentially good reduction and $\ell \neq p$, the image $\rho_{\ell,\mathcal{A}}(\mathcal{I}_F)$ is finite and independent of the prime number ℓ. We say that $\rho_{\ell,\mathcal{A}}$ is potentially unramified and call the above image the inertia subgroup associated to $\mathcal{A}$. This group corresponds to the inertia subgroup $I(K/F)$ of a finite Galois extension K/F of minimal degree over which $\mathcal{A}$ acquires good reduction.

When $\ell = p$ the p-adic representation associated to $\mathcal{A}$ is always infinitely ramified, even when $\mathcal{A}$ has potentially good reduction. The right analogue in this situation is the notion of a potentially crystalline representation. Thanks to Fontaine's p-adic Hodge Theory, such representations can be described by purely semilinear objects. More concretely, there is an equivalence of category between the category of p-adic representations of $\mathcal{G}_F$ crystalline over K, and the category of admissible filtered $(\varphi, \mathrm{Gal}(K/F))$-modules (see [Fon94b, § 5.6] and [CF00, Thm.A]). The notion of a crystalline representation is not restricted to Tate modules of Abelian varieties, but in that case it is easier to interpret it. Let $\mathcal{A}/F$ have good reduction over K and (D, Fil) be the filtered $(\varphi, \mathrm{Gal}(K/F))$-module associated to $\mathbf{V}_p(\mathcal{A})$. The Frobenius φ corresponds to the Frobenius of the special fibre of $\mathcal{A}_K$. The filtration Fil is a lifting datum, carrying the information that we started with an object defined in characteristic 0. The action of the Galois group $\mathrm{Gal}(K/F)$ is a descent datum allowing us to recover $\mathcal{A}$ from $\mathcal{A}_K$.

Now the extension K/F can either be unramified — in which case $\mathcal{A}$ already has good reduction over F — tamely ramified, or wildly ramified. As the terminology suggests, the latter is the worst case scenario. In 2005 M. Volkov (see [Vol05]) gave a characterisation of the p-adic representations V of $\mathcal{G}_{\mathbb{Q}_p}$ arising from Abelian varieties over $\mathbb{Q}_p$ with tame potentially good reduction. The characterisation is given in terms of necessary and sufficient conditions on the filtered $(\varphi, \mathcal{G}_{\mathbb{Q}_p})$-module associated to V. Such a variety acquires good reduction over $\mathbb{Q}_p(\sqrt[e]{p})$ where e is the ramification index of a minimal field of good reduction. As a consequence, the associated inertia subgroup is always cyclic.

In this thesis we are interested in Abelian varieties defined over F with wild potentially good reduction, that is, Abelian varieties that acquire good reduction over a wildly ramified extension. Not much is known in this situation, and our purpose is to provide the first explicit examples in low dimension with control on the field of definition. A well-known result of J.-P. Serre and J. Tate (see [ST68, Cor. 2.2]) states that, when $p > 2\dim(\mathcal{A}) + 1$, the potential good reduction is necessarily tame. For this reason, considering odd primes only, we focus on elliptic curves over $\mathbb{Q}_3$ and Abelian surfaces over $\mathbb{Q}_3$ and $\mathbb{Q}_5$.

Abelian varieties together with their respective Tate modules have been extensively studied since the 1960s. The most classical results in the case of potential good reduction are due to J.-P. Serre and J. Tate in [ST68]. To name but one, the criterion of Néron-Ogg-Shafarevich states that $\mathcal{A}/F$ has potentially good reduction if and only if for any $\ell \neq p$ its ℓ-adic Tate module is potentially unramified. In this case the inertia subgroup associated to $\mathcal{A}$ embeds in the automorphism group of its special fibre, which sheds light on the bridge between Arithmetics and Geometry.

More recently, works of N. Coppola (see [Cop20a], [Cop20b],[Cop25]) give an explicit description of the ℓ-adic representations associated to elliptic curves defined over a p-adic field with wild potentially good reduction. We also mention the complete classification of inertial Weil-Deligne types associated to elliptic curves over $\mathbb{Q}_p$ obtained by L. Dembélé, N. Freitas and J. Voight (see [DFV24]). Although not directly related to representations, the results of Hwang on automorphisms groups of Abelian surfaces over finite fields (see [Hwa21]) have proven useful in the construction of polarised Galois pairs. However, as aforementioned, we are interested in the case $\ell = p$, with a specific focus on wild inertia.

To the best of our knowledge, most authors do not consider the action of the full Galois group $\mathrm{Gal}(K/F)$ on $\mathcal{A}$, but just the one of its inertia subgroup. This amounts to considering $\mathcal{A}$ over the maximal unramified extension of F, or equivalently, its special fibre over an algebraic closure of the residue field. Taking into account the full action is important to us. Not discarding the Galois group of the residue field enables us to be more precise regarding the minimal field of definition of $\mathcal{A}$.

In line with the work of M. Volkov, who classifies all the p-adic representations of $\mathcal{G}_{\mathbb{Q}_p}$ arising from elliptic curves over $\mathbb{Q}_p$ for $p \geq 5$ (see [Vol01]), we deal with the cases of potential good reduction for $p = 3$. Our first main result is the following:

Theorem I (see Section 2.3 and Thm. 2.1). *Let $E/\mathbb{Q}_3$ be an elliptic curve with* *potentially good reduction. The filtered* $(\varphi, \mathcal{G}_{\mathbb{Q}_3})$*-module associated to* E *is isomorphic to one of the objects described in Table 2.1. Conversely, any such object arises from an elliptic curve defined over* $\mathbb{Q}_3$ *with potentially good reduction.*

This classification highlights new phenomena specific to wild potential good reduction. Compared to the tame case, the inertia subgroup is not necessarily cyclic and the possibilities for the minimal fields of good reduction are multiple.

The second part of this thesis is dedicated to Abelian surfaces. Here we do not aim to classify their attached representations, but merely to investigate the inertia subgroups naturally appearing in a situation of potential good reduction, with a focus on the wild ones. In 2005 A. Silverberg and Yu. Zarhin (see [SZ05]) classified all such possible inertia subgroups over a discretely valued field with perfect residue field. However most of the wild ones are achieved over local fields in equicharacteristic. We show that each such group can actually be realised in mixed characteristic:

Theorem II (see Thm. 3.2)**.** *All inertia subgroups associated to potential good reduction of Abelian surfaces appearing in [SZ05, Def. 1.2] (see 3.1.2) for* $p = 3, 5$ *can be realised over a finite unramified extension of* $\mathbb{Q}_p$*.*

In both cases, whether elliptic curves or Abelian surfaces, we observe that the wild ramification subgroups are of order p (for curves over $\mathbb{Q}_3$ and surfaces over $\mathbb{Q}_5$) or p^2 (for surfaces over $\mathbb{Q}_3$). To find non-Abelian examples, one must either consider the even prime number or increase the dimension.

This thesis is organised as follows.

In the first chapter we provide the necessary material to cover the rest of the manuscript. We begin with some brief reminders on potential good reduction of Abelian varieties and their associated representations. We then define the semilinear objects from p-adic Hodge Theory used to explicitely handle the Tate modules of such varieties. Finally, we define the notion of Galois pair following [Vol05], which is crucial in the proofs of our main theorems.

The second chapter is dedicated to the proof of Theorem I, *i.e.*, the full classification of the 3-adic representations of $\mathcal{G}_{\mathbb{Q}_3}$ arising from elliptic curves over $\mathbb{Q}_3$ with potentially good reduction. The first step is to find all possible minimal fields of good reduction for such curves. Thanks to J.-P. Serre (see [Ser71, § 5.6]) we already know the structure of their inertia subgroup. With the help of the [LMFDB] we determine all the possible extensions of $\mathbb{Q}_3$ that occur. Then, for each such extension $K/\mathbb{Q}_3$, we compute the admissible filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules satisfying the conditions from [Vol05, Thm. 5.7, Rmk. 5.11]. To such an object corresponds a potentially crystalline representation of $\mathcal{G}_{\mathbb{Q}_3}$. Finally, we show that every representation obtained in this way indeed arises from some elliptic curve over $\mathbb{Q}_3$, which is achieved by constructing minimal Galois pairs introduced in [Vol05].

In the third chapter we prove Theorem II. We proceed in three steps. First, we begin by realising each of the considered groups as the inertia subgroup of a finite extension of $\mathbb{Q}_p$. We then provide a minimal polarised Galois pair defined over $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$ for each such extension. We next use a recent result of S. Philip (see [Phi24, Thm. 3.10]) allowing us to lift our Galois pairs to characteristic zero and achieving our goal.

CHAPTER 1

Theoretical background

The purpose of this chapter is to describe the different tools that we use in the construction process of Abelian varieties with potentially good reduction. The first two sections are dedicated to the properties satisfied by Tate modules of such varieties and how they translate in terms of filtered $(\varphi, \mathcal{G}_F)$-modules. In the third section, starting from an Abelian variety defined over a finite field, we show how to enrich it with enough additional structure so that it behaves exactly like the special fibre of a variety with potentially good reduction. We discuss the case of tame potential good reduction in the last section.

We adopt the same conventions as J. Milne in [Mil08] regarding Algebraic Geometry. An algebraic variety over a field is a geometrically reduced separated scheme of finite type over that field. An Abelian variety is a complete connected group variety. We assume the reader is familiar with local fields, for which we refer to [Ser68] and [Neu99].

Let p be a prime number and $\overline{\mathbb{Q}_p}$ be a fixed algebraic closure of $\mathbb{Q}_p$, the field of p-adic numbers. We let $\mathcal{G}_F = \mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ be the absolute Galois group of a finite extension F of $\mathbb{Q}_p$. We denote by F^{un} the maximal unramified extension of F inside $\overline{\mathbb{Q}_p}$. We let $\mathcal{I}_F = \mathrm{Gal}(\overline{\mathbb{Q}_p}/F^{\mathrm{un}})$ be the inertia subgroup of $\mathcal{G}_F$. The ring of integers $\mathcal{O}_F$ of F has a unique prime ideal $\mathfrak{p}_F$ and its residue field is $\kappa_F = \mathcal{O}_F/\mathfrak{p}_F$. We let F_0 be the maximal unramified extension of $\mathbb{Q}_p$ inside F and $\sigma \in \mathrm{Gal}(F_0/\mathbb{Q}_p) \simeq \mathrm{Gal}(\kappa_F/\mathbb{F}_p)$ be the Frobenius element. The absolute Galois group of F sits in an exact sequence

$$1 \longrightarrow \mathcal{I}_F \longrightarrow \mathcal{G}_F \longrightarrow \mathcal{G}_{\kappa_F} \longrightarrow 1,$$

where $\mathcal{G}_{\kappa_F}$ is the absolute Galois group of the residue field. While the absolute Galois group of a finite field is well understood, the inertia subgroup $\mathcal{I}_F$ is more mysterious, specifically its maximal pro-p-group $\mathcal{P}_F$, known as the wild ramification subgroup. The *Weil group* $\mathcal{W}_F$ of F is defined by the following exact sequence

$$1 \longrightarrow \mathcal{I}_F \longrightarrow \mathcal{W}_F \longrightarrow \mathbb{Z} \longrightarrow 1,$$

where a lifting of the arithmetic Frobenius is sent to 1. This "decompletion" of $\mathcal{G}_F$ allows us to make sense of the comparison in ℓ of ℓ-adic representations associated to Abelian varieties, as ℓ ranges over all primes.

1.1 Potential good reduction

Let ℓ be a prime number.

Definition 1.1. An *ℓ-adic representation* of $\mathcal{G}_F$ is a finite dimensional $\mathbb{Q}_\ell$-vector space V with a linear and continuous action of $\mathcal{G}_F$, or equivalently, a continuous group homomorphism

$$\rho : \mathcal{G}_F \longrightarrow \mathrm{GL}_n(\mathbb{Q}_\ell) \simeq \mathrm{Aut}_{\mathbb{Q}_\ell}(V).$$

A morphism of ℓ-adic representations is a $\mathbb{Q}_\ell$-linear map that commutes with the action of $\mathcal{G}_F$. We denote by $\mathbf{Rep}_{\mathbb{Q}_\ell}(\mathcal{G}_F)$ the category of ℓ-adic representations of $\mathcal{G}_F$.

Example 1.1. The action of $\mathcal{G}_F$ on ℓth power primitive roots of unity in $\overline{\mathbb{Q}_p}$ induces a continuous morphism

$$\chi_\ell : \mathcal{G}_F \longrightarrow \varprojlim_{m\geq 1}(\mathbb{Z}/\ell^m\mathbb{Z})^\times \simeq \mathbb{Z}_\ell^\times.$$

It is a one-dimensional ℓ-adic representation of $\mathcal{G}_F$ known as the ℓ-adic *cyclotomic character*. We denote by $\mathbb{Q}_\ell(1)$ the one-dimensional $\mathbb{Q}_\ell$-vector space endowed with the action of χ_ℓ.

There are obvious notions of quotients, direct and tensor products of ℓ-adic representations. The identity element is $\mathbb{Q}_\ell$ endowed with the trivial action of $\mathcal{G}_F$. The dual V^* of an ℓ-adic representation V is the $\mathbb{Q}_\ell$-vector space $\mathrm{Hom}_{\mathbb{Q}_\ell}(V, \mathbb{Q}_\ell)$ with the action of $g \in \mathcal{G}_F$ on f defined by $g.f(v) = f(g^{-1}.v)$. The *Tate twist* of a an ℓ-adic representation V, denoted by $V(1)$ is the tensor product $V \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(1)$.

Definition 1.2. An ℓ-adic representation ρ_ℓ of $\mathcal{G}_F$ is *unramified* if $\rho_\ell(\mathcal{I}_F) = 0$. It is *potentially unramified* if $\rho_\ell(\mathcal{I}_F)$ is finite, in which case there exists a finite extension L/F such that $\rho_\ell(\mathcal{I}_L) = 0$.

Recall that $\mathcal{G}_F/\mathcal{I}_F \simeq \mathcal{G}_{\kappa_F}$ is a procyclic group topologically generated by the Frobenius element. An unramified representation of $\mathcal{G}_F$ is equivalent to an ℓ-adic representation of $\mathcal{G}_{\kappa_F}$, which is completely determined by the image of the Frobenius.

Remark 1.1. The image $\chi_\ell(\mathcal{I}_F)$ is infinite if and only if $\ell = p$.

Let $d, n \geq 1$ be integers and $\mathcal{A}/F$ be a d-dimensional Abelian variety. The set $\mathcal{A}(\overline{\mathbb{Q}_p})$ of points of $\mathcal{A}$ with values in $\overline{\mathbb{Q}_p}$ is an Abelian group. The subgroup $\mathcal{A}[\ell^n]$ of ℓ^n-torsion points of $\mathcal{A}(\overline{\mathbb{Q}_p})$ is a free $\mathbb{Z}/\ell^n\mathbb{Z}$-module of rank $2d$. The absolute Galois group $\mathcal{G}_F$ acts linearly on $\mathcal{A}[\ell^n]$ and this action is continuous for the discrete topology. Moreover, the transition maps $\mathcal{A}[\ell^{n+1}] \longrightarrow \mathcal{A}[\ell^n]$ are $\mathcal{G}_F$-equivariant. Taking the inverse limit and tensoring by $\mathbb{Q}_\ell$ leads to the notion of Tate module.

Definition 1.3. The *ℓ-adic representation* of $\mathcal{G}_F$ associated to $\mathcal{A}$, also called its *ℓ-adic Tate module*, is

$$\mathbf{V}_\ell(\mathcal{A}) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_{\ell} \mathcal{A}[\ell^n].$$

It is a $2d$-dimensional representation, we let

$$\rho_{\ell,\mathcal{A}} : \mathcal{G}_F \longrightarrow \mathrm{GL}_{2d}(\mathbb{Q}_\ell) \simeq \mathrm{Aut}_{\mathbb{Q}_\ell}(\mathbf{V}_\ell(\mathcal{A}))$$

be the associated continuous group homomorphism.

For topological reasons, the case $\ell = p$ is the most interesting. A natural question then arises. Given a p-adic representation V of $\mathcal{G}_F$, when does it come from an Abelian variety $\mathcal{A}/F$? That is, under what conditions does there exists $\mathcal{A}/F$ such that $V \simeq \mathbf{V}_p(\mathcal{A})$? In this work, we will focus on Abelian varieties with potentially good reduction. The second chapter of this thesis is dedicated to the full classification of the 3-adic representations of $\mathcal{G}_{\mathbb{Q}_3}$ arising from elliptic curves over $\mathbb{Q}_3$ with potentially good reduction. In the last chapter, extending results of Silverberg and Zarhin (see [SZ05]), we give the possible inertia subgroups associated to Abelian surfaces defined over a finite extension of $\mathbb{Q}_p$ ($p = 3, 5$) with potentially good reduction.

Definition 1.4. An Abelian variety $\mathcal{A}/F$ has *good reduction* if there exists an Abelian scheme $\mathfrak{A}/\mathcal{O}_F$ such that $\mathcal{A} \simeq \mathfrak{A} \times_{\mathcal{O}_F} F$. In which case the special fibre A/κ_F of $\mathfrak{A}$ remains an Abelian variety.

Definition 1.5. An Abelian variety $\mathcal{A}/F$ has *potentially good reduction* if it acquires good reduction over a finite extension L/F *i.e.* the scalar extension $\mathcal{A}_L = \mathcal{A} \times_F L$ of $\mathcal{A}$ to L has good reduction.

When $\dim \mathcal{A} = 1$, *i.e.*, $\mathcal{A} = E$ is an elliptic curve, this property can be easily expressed.

Example 1.2. An elliptic curve E/F has potentially good reduction if and only if its j-invariant $j(E)$ is integral.

The following result, due to J.-P. Serre and J. Tate, known as the Néron-Ogg-Shafarevich criterion, translates the geometric property of good reduction in terms of representations.

Theorem 1.1 ([ST68], Thm. 2.2). *An Abelian variety $\mathcal{A}/F$ has good reduction over L if and only if for any (every) $\ell \neq p$ its associated ℓ-adic representation is* *unramified over L.*

If $\mathcal{A}_0/F$ is an Abelian variety with potentially good reduction, there exists a unique finite extension $M_{\mathcal{A}_0}/F^{\mathrm{un}}$ of minimal degree over which $\mathcal{A}_0$ acquires good reduction. We call that degree the *semi-stability defect* of $\mathcal{A}_0$, denoted by $\mathrm{dst}(\mathcal{A}_0)$. Let $\ell \neq p$ and consider $\mathbf{V}_\ell(\mathcal{A}_0)$ the ℓ-adic representation associated to $\mathcal{A}_0$. Since $\mathcal{A}_0$ has potentially good reduction, there exists L/F finite of minimal ramification index satisfying $\rho_{\ell,\mathcal{A}_0}(\mathcal{I}_L) = 0$. It is then easy to see that

$$M_{\mathcal{A}_0} = \overline{\mathbb{Q}_p}^{\ker(\rho_{\ell,\mathcal{A}_0}|_{\mathcal{I}_L})}.$$

Conversely, if L/F is a finite extension with $L^{\mathrm{un}} = M_{\mathcal{A}_0}$, then $\mathcal{A}_0$ acquires good reduction over L and $\mathrm{dst}(\mathcal{A}_0) = e(L/F)$, it is the minimal ramification index among all fields of good reduction of $\mathcal{A}_0$. It is also worth noticing that if L, L'/F satisfy $L^{\mathrm{un}} = (L')^{\mathrm{un}}$, then they are interchangeable in the sense that $\mathcal{A}_0$ acquires good reduction over L if and only if it acquires it over L'. We can always assume that L/F is totally ramified, we denote by K its Galois closure, K_0 the maximal unramified extension of F inside K and κ the residue field of K. Then, the extension K/L is unramified, $K\mathbb{Q}_p^{\mathrm{un}} = M_{\mathcal{A}_0}$ and

$$\mathrm{Gal}(K/F) = I(K/F) \rtimes \mathrm{Gal}(K/L)$$

with

$$\mathrm{Gal}(K/L) \simeq \mathrm{Gal}(\kappa/\kappa_L) \simeq \mathrm{Gal}(K_0/F).$$

The Galois group $\mathrm{Gal}(K/F)$ acts on $\mathcal{A} = \mathcal{A}_0 \times_F K$. This action extends to the special fibre A/κ which is the scalar extension of the special fibre A_0/κ_F of $\mathcal{A}_0 \times_F L$ (see [ST68]). An element $g \in \mathrm{Gal}(K/F)$ acts on $f \in \mathrm{Aut}_K(\mathcal{A})$ by conjugation by the element $\mathrm{id}_{\mathcal{A}_0} \times_F g \in \mathrm{Aut}_K(\mathcal{A})$. Since the inertia subgroup $I(K/F)$ acts trivially on κ we obtain an antimorphism

$$\nu : I(K/F) \hookrightarrow \mathrm{Aut}_\kappa(A),$$

the injectivity of ν coming from the minimality of K.

Let $\mathcal{A}_0^\vee$ be the *dual Abelian variety* of $\mathcal{A}_0$, it is F-isogenous to $\mathcal{A}_0$, hence also acquires good reduction over K, we then have another antimorphism

$$\nu' : I(K/F) \hookrightarrow \mathrm{Aut}_\kappa(A^\vee).$$

One can show using Weil's descent criterion that for every $g \in \mathrm{Gal}(K/F)$ we have

$$\nu'(g) = (\nu(g)^\vee)^{-1}.$$

Let $\Lambda_0 : \mathcal{A}_0 \longrightarrow \mathcal{A}_0^\vee$ be a *polarisation* and $\Lambda = \Lambda_0 \times_F K$. If $\lambda : A \longrightarrow A^\vee$ is a polarisation coming from Λ, then we have an inclusion (see [Vol05, Lemma. 3.1])

$$\nu(I(K/F)) \subset \mathrm{Aut}_\kappa(A, \lambda).$$

1.2 Filtered $(\varphi, \mathcal{G}_F)$-modules

As we have seen in Section 1.1, when $\mathcal{A}/F$ has potentially good reduction, the kernel of the ℓ-adic representation associated to $\mathcal{A}$ is a subgroup of $\mathcal{I}_F$ of finite index. This cannot be the case anymore when $\ell = p$, indeed, the following diagram

$$\begin{tikzcd}
\mathcal{G}_F \arrow[r, "\rho_{p,\mathcal{A}}"] \arrow[rr, bend right, "\chi_p"'] & \mathrm{GL}_n(\mathbb{Q}_p) \arrow[r, "\det"] & \mathbb{Q}_p^\times
\end{tikzcd}$$

commutes. The determinant of the Tate module of $\mathcal{A}$ is the p-adic cyclotomic character, which is infinitely ramified.

J.-M. Fontaine's p-adic Hodge Theory provides a framework to describe and classify p-adic representations of $\mathcal{G}_F$. The p-adic representations associated to Abelian varieties with potentially good reduction are potentially crystalline. Such representations are described by their associated filtered $(\varphi, \mathcal{G}_F)$-module, a purely semilinear object.

Definition 1.6. A *φ-module* over F is a finite dimensional F_0-vector space D with a σ-semilinear and injective Frobenius $\varphi : D \to D$. A morphism of φ-modules over F is a F_0-linear map that commutes with the Frobenius.

Recall (see [Fon77, Ch. III]) that the *Dieudonné module* $D(A)$ associated to the p-divisible group $A(p)$ of an Abelian variety A over a finite field κ is a φ-module over $\mathrm{Frac}(W(\kappa))$, its Frobenius being induced by the Frobenius endomorphism of A.

Remark 1.2. Let $\mathcal{A}/F$ have good reduction over a minimal Galois extension K/F with special fibre A/κ. Applying the Dieudonné functor to the injection ν from Section 1.1 gives us an embedding

$$I(K/F) \xhookrightarrow{\nu} \mathrm{Aut}_\kappa(A) \hookrightarrow \mathrm{Aut}_\varphi(D(A)).$$

Definition 1.7. A *filtered φ-module* over F is a pair (D, Fil) consisting of a φ-module D with a separated, exhaustive and decreasing filtration Fil on $D_F = F \otimes_{F_0} D$, *i.e.* $\mathrm{Fil} = (\mathrm{Fil}^i D_F)_{i \in \mathbb{Z}}$ such that:

- $\mathrm{Fil}^i D_F$ is a sub-F-vector space of D_F,
- $\bigcap_{i \in \mathbb{Z}} \mathrm{Fil}^i D_F = 0$ (separated),
- $\bigcup_{i \in \mathbb{Z}} \mathrm{Fil}^i D_F = D_F$ (exhaustive).

In the next chapters we will be interested in filtrations Fil on D_F with only one jump. That is, $\mathrm{Fil}^0 D_F = D_F$, $\mathrm{Fil}^1 D_F$ is a non trivial subspace of D_F and $\mathrm{Fil}^2 D_F = \{0\}$. In that case Fil is completely determined by $\mathrm{Fil}^1 D_F$ and we say that (D, Fil) is of *Hodge-Tate type* $(0, 1)$.

Definition 1.8. A morphism of filtered φ-modules over F is a morphism f of φ-modules such that the scalar extension f_F preserves the filtration.

Let K/F be a finite Galois extension.

Definition 1.9. A *filtered* $(\varphi, \mathrm{Gal}(K/F))$*-module* is a filtered φ-module over K with a semilinear action of $\mathrm{Gal}(K/F)$ such that φ is $\mathrm{Gal}(K/F)$-equivariant and the filtration is $\mathrm{Gal}(K/F)$-stable. A morphism of filtered $(\varphi, \mathrm{Gal}(K/F))$-modules is a morphism of filtered φ-modules over K that is also $\mathrm{Gal}(K/F)$-equivariant.

Let (D, Fil) and (D', Fil') be filtered $(\varphi, \mathrm{Gal}(K/F))$-modules with respective Frobenius φ and φ'. Concretely, a morphism $(D, \mathrm{Fil}) \longrightarrow (D', \mathrm{Fil}')$ of filtered $(\varphi, \mathrm{Gal}(K/F))$-modules is a $\mathrm{Gal}(K/F)$-equivariant K_0-linear map $f : D \longrightarrow D'$ such that $f \circ \varphi = \varphi' \circ f$ and $f_K(\mathrm{Fil}^i D_K) \subset \mathrm{Fil}^i D'_K$ for any $i \in \mathbb{Z}$. The filtered $(\varphi, \mathrm{Gal}(K/F))$-modules and their respective morphisms form a category we denote by $\mathbf{MF}_\varphi(\mathrm{Gal}(K/F))$.

We sometimes use the term $(\varphi, \mathcal{G}_F)$-module when the extension K/F is not specified, meaning that the action of $\mathcal{G}_F$ factors through a finite quotient.

Next is the analogue of the Tate twist of a p-adic representation, hence its name.

Definition 1.10. Let $\mathbf{D} = (D, \mathrm{Fil})$ be a filtered $(\varphi, \mathrm{Gal}(K/F))$-module. The *Tate twist* $\mathbf{D}\{-1\}$ of $\mathbf{D}$ is the K_0-vector space D with Frobenius $\varphi\{-1\} = p\varphi$, the same action of $\mathrm{Gal}(K/F)$ and filtration $\mathrm{Fil}^i(D\{-1\})_K = \mathrm{Fil}^{i-1} D_K$.

As in the case of representations, we also have tensor products and duals.

Definition 1.11. The tensor product of two filtered $(\varphi, \mathrm{Gal}(K/F))$-modules (D, Fil) and (D', Fil') is the K_0-vector space $D \otimes_{K_0} D'$ with Frobenius $\varphi \otimes \varphi'$, diagonal action of $\mathrm{Gal}(K/F)$ and ith piece of filtration

$$\mathrm{Fil}^i(D \otimes_{K_0} D')_K = \sum_{j+k=i} \mathrm{Fil}^j D_K \otimes_K \mathrm{Fil}^k D'_K.$$

The rth exterior power $\bigwedge^r_{K_0} D$ of D naturally inherits a filtration by symmetry of the definition of the filtration on the rth tensor power.

Definition 1.12. Let $\mathbf{D} = (D, \mathrm{Fil})$ be a filtered $(\varphi, \mathrm{Gal}(K/F))$-module. The dual $\mathbf{D}^*$ of $\mathbf{D}$ is the K_0-vector space $D^* = \mathrm{Hom}_{K_0}(D, K_0)$ with Frobenius φ^*

defined by

$$\varphi^*(f) = \sigma f \varphi^{-1},$$

an element $g \in \mathrm{Gal}(K/F)$ acts on $f \in D^*$ via $g.f = gfg^{-1}$ and the ith piece of the filtration is

$$\mathrm{Fil}^i D_K^* = (\mathrm{Fil}^{-i+1} D_K)^\perp = \left\{ f_K \in D_K^* \mid \forall x \in \mathrm{Fil}^{-i+1} D_K, f_K(x) = 0 \right\}.$$

Definition 1.13. Let (D, Fil) be a filtered $(\varphi, \mathrm{Gal}(K/F))$-module of Hodge-Tate type $(0, 1)$. A *nondegenerate skew form* on (D, Fil) is a nondegenerate antisymmetric bilinear form β on the underlying K_0-vector space D such that

$$\beta : D \times D \longrightarrow K_0 \{-1\}$$

is a morphism of $(\varphi, \mathrm{Gal}(K/F))$-modules and

$$\forall x \in \mathrm{Fil}^1 D_K,\ \beta_K(x, x) = 0,$$

i.e., $\mathrm{Fil}^1 D_K$ is totally isotropic.

Since $\mathrm{Fil}^1(D^* \{-1\})_K = (\mathrm{Fil}^1 D_K)^\perp$, such a form is equivalent to an antisymmetric isomorphism of filtered $(\varphi, \mathrm{Gal}(K/F))$-modules

$$\delta : D^* \{-1\} \xrightarrow{\sim} D.$$

We will now see how one can attach such a semilinear object to a p-adic representation. Recall that $\mathbf{B}_{\mathrm{cris}}$ is the ring of crystalline periods and $\mathbf{B}_{\mathrm{dR}}$ is the de Rham period ring (see [Fon94a, § 1.5, § 2.3]). For every finite Galois extension K/F, the functor

$$\mathbf{D}_{\mathrm{cris},K} : \mathbf{Rep}_{\mathbb{Q}_p}(\mathcal{G}_F) \longrightarrow \mathbf{MF}_\varphi(\mathrm{Gal}(K/F)) : V \longmapsto (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K}$$

associate to any p-adic representation V of $\mathcal{G}_F$ a filtered $(\varphi, \mathrm{Gal}(K/F))$-module. Indeed, the absolute Galois group $\mathcal{G}_K$ acts on $\mathbf{B}_{\mathrm{cris}}$ and we have $\mathbf{B}_{\mathrm{cris}}^{\mathcal{G}_K} = K_0$ (see [Fon94b, Prop. 5.1.2]). Endowing the tensor product with the diagonal action of $\mathcal{G}_K$ implies that $\mathbf{D}_{\mathrm{cris},K}(V)$ is a finite dimensional K_0-vector space, and the following inequality always holds

$$\dim_{K_0} \mathbf{D}_{\mathrm{cris},K}(V) \leq \dim_{\mathbb{Q}_p} V.$$

The ring $\mathbf{B}_{\mathrm{cris}}$ is endowed with an injective and $\mathcal{G}_K$-equivariant Frobenius φ, hence the map:

$$\varphi \otimes \mathrm{id}_V : \mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V \longrightarrow \mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V,$$

defines a Frobenius (also denoted by φ) on $\mathbf{D}_{\mathrm{cris},K}(V)$ by taking $\mathcal{G}_K$-fixed points. There exists and embedding (see [Fon94a, § 4.1]):

$$\mathbf{B}_{\mathrm{cris}} \otimes_{K_0} K \hookrightarrow \mathbf{B}_{\mathrm{dR}}$$

into the de Rham period ring, which induces:

$$\mathbf{D}_{\mathrm{cris},K}(V) \otimes_{K_0} K \longhookrightarrow \mathbf{D}_{\mathrm{dR}}(V) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K},$$

where $\mathbf{D}_{\mathrm{dR}}$ is the de Rham functor. The ring $\mathbf{B}_{\mathrm{dR}}$ is naturally endowed with a filtration, making of $\mathbf{D}_{\mathrm{dR}}(V)$ and thus $\mathbf{D}_{\mathrm{cris},K}(V) \otimes_{K_0} K$ filtered K-vector spaces, thanks to the above injection. Finally, since $\mathcal{G}_F$ also acts on $\mathbf{D}_{\mathrm{cris},K}(V)$, this action factors through a semilinear action of $\mathrm{Gal}(K/F)$.

Example 1.3. The filtered φ-module over F associated to the p-adic cyclotomic character $\mathbb{Q}_p(1)$ (as a representation of $\mathcal{G}_F$) is $F_0\{-1\}$.

In general, we have $\mathbf{D}_{\mathrm{cris},K}(V(1)) \simeq \mathbf{D}_{\mathrm{cris},K}(V)\{-1\}$, where $V(1)$ is the Tate twist of the p-adic representation V.

Definition 1.14. A p-adic representation V of $\mathcal{G}_F$ is said to be *crystalline* if

$$\dim_{F_0} \mathbf{D}_{\mathrm{cris},F}(V) = \dim_{\mathbb{Q}_p} V.$$

It is *potentially crystalline* if it is crystalline as a representation of $\mathcal{G}_L$ for some finite extension L/F.

We denote by $\mathbf{Rep}^{\mathrm{cris},L}_{\mathbb{Q}_p}(\mathcal{G}_F)$ the category of p-adic representations of $\mathcal{G}_F$ crystalline over L. Abelian varieties with potentially good reduction give rise to such representations. In fact, the following holds.

Theorem 1.2 ([CI99], Thm. 4.7). *An Abelian variety* $\mathcal{A}/F$ *has good reduction (resp. potentially good reduction over L) if and only if* $\mathbf{V}_p(\mathcal{A})$ *is crystalline (resp. potentially crystalline over L).*

It is the p-adic analogue of the Néron-Ogg-Shafarevic criterion (see Theorem 1.1).

Remark 1.3. If $\mathcal{A}/F$ has potentially good reduction over a finite Galois extension K, its associated filtered $(\varphi, \mathrm{Gal}(K/F))$-module (D, Fil) is of Hodge-Tate type $(0, 1)$ and $\dim_K \mathrm{Fil}^1 D_K = \dim \mathcal{A}$.

Theorem 1.3. *Let* K/F *be a finite Galois extension. The functor*

$$\mathbf{D}_{\mathrm{cris},K} : \mathbf{Rep}^{\mathrm{cris},K}_{\mathbb{Q}_p}(\mathcal{G}_F) \longrightarrow \mathbf{MF}_{\varphi}(\mathrm{Gal}(K/F)) : V \longmapsto (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K}$$

is exact, fully faithful, commutes to tensor products and duals.

Proof. We sketch a proof for $K = F = \mathbb{Q}_p$ (see [Fon94b, § 5.6.7] for the general case). In the formalism of B-admissible representations, the ring $\mathbf{B}_{\mathrm{cris}}$ is $(\mathbb{Q}_p, \mathcal{G}_F)$-regular and a crystalline representation is a $\mathbf{B}_{\mathrm{cris}}$-admissible representation. The exactness and the fact it commutes with tensor product and duals is a direct consequence of [Fon94b, Prop. 1.5.2]. A quasi-inverse is given by

$$D \longmapsto \mathrm{Fil}^0((\mathbf{B}_{\mathrm{cris}} \otimes D)^{\varphi=1})$$

where $\varphi = 1$ denotes the φ-fixed points and D is in the essential image of $\mathbf{D}_{\mathrm{cris}}$. Indeed, for V a crystalline representation we have

$$\begin{aligned} \mathrm{Fil}^0((\mathbf{B}_{\mathrm{cris}} \otimes \mathbf{D}_{\mathrm{cris}}(V))^{\varphi=1}) &\simeq \mathrm{Fil}^0((\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\varphi=1}) \\ &= (\mathrm{Fil}^0 \mathbf{B}^{\varphi=1}_{\mathrm{cris}}) \otimes_{\mathbb{Q}_p} V, \end{aligned}$$

and we can conclude with [Fon94a, Thm. 5.3.7(iii)]. □

Remark 1.4. One can choose to use the contravariant version $\mathbf{D}^*_{\mathrm{cris},K}$ of the crystalline functor, defined by

$$V \longmapsto \mathrm{Hom}_{\mathbb{Q}_p[\mathcal{G}_K]}(V, \mathbf{B}_{\mathrm{cris}}).$$

In that case we have an identification $\mathbf{D}^*_{\mathrm{cris},K}(V) \simeq \mathrm{D}_{\mathrm{cris},K}(V^*)$. We will work with this version for the remainder of this thesis.

Note that in the definition of a filtered $(\varphi, \mathrm{Gal}(K/F))$-module (Def. 1.9), we do not require any kind of compatibility between the Frobenius and the filtration while the action of $\mathrm{Gal}(K/F)$ does. It is the missing piece that will allow us to describe the essential image of the functor $\mathbf{D}_{\mathrm{cris},K}$.

Let (D, Fil) be a filtered φ-module. If C is a matrix representing the σ-semilinear endomorphism φ and v is the normalised valuation on K_0, then $v(\det C)$ is independent of C.

Definition 1.15. The *Newton number* associated to a φ-module D is

$$t_{\mathrm{N}}(D) = v(\det C),$$

where C is some matrix representing φ.

Definition 1.16. The *Hodge number* associated to a filtered φ-module (D, Fil) is

$$t_{\mathrm{H}}(D) = \sum_{i \in \mathbb{Z}} i \dim_K(\mathrm{Fil}^i D_K / \mathrm{Fil}^{i+1} D_K).$$

Let (D, Fil) be a fitered φ-module over F, a subobject of (D, Fil) is a φ-stable F_0-subvector space D' of D. It is naturally a filtered φ-module with Frobenius $\varphi' = \varphi|_{D'}$ and filtration Fil' induced by the inclusion $D'_F \subset D_F$.

Definition 1.17. A filtered φ-module (D, Fil) over F is *admissible* if the two following conditions are satisfied

(1) $t_{\mathrm{H}}(D) = t_{\mathrm{N}}(D)$

(2) For all subobjects D' of D: $t_{\mathrm{H}}(D') \leq t_{\mathrm{N}}(D')$.

The category of admissible filtered $(\varphi, \mathrm{Gal}(K/F))$-modules will be denoted by $\mathbf{MF}^{\mathrm{ad}}_{\varphi}(\mathrm{Gal}(K/F))$. The following strong result shows that it is exactly the essential image of the functor $\mathbf{D}_{\mathrm{cris},K}$.

Theorem 1.4 ([CF00], Thm. A)**.** *Let K/F be a finite Galois extension. The functor*

$$\mathbf{D}_{\mathrm{cris},K} : \mathbf{Rep}_{\mathbb{Q}_p}^{\mathrm{cris},K}(\mathcal{G}_F) \longrightarrow \mathbf{MF}_{\varphi}^{\mathrm{ad}}(\mathrm{Gal}(K/F)) : V \longmapsto (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K}$$

is an equivalence of categories between the category of p-adic representations of $\mathcal{G}_F$ crystalline over K and the category of admissible filtered $(\varphi, \mathrm{Gal}(K/F))$-modules.

As a result, Tate modules of Abelian varieties with potentially good reduction can be described by a purely semilinear object.

An Abelian variety defines a family of compatible ℓ-adic representations for $\ell \neq p$. The Weil representation is its p-adic analogue. Recall that the Weil group $\mathcal{W}_F$ of F is defined by the following exact sequence

$$1 \longrightarrow \mathcal{I}_F \longrightarrow \mathcal{W}_F \xrightarrow{\pi} \mathbb{Z} \longrightarrow 1,$$

where π sends a lifting of the arithmetic Frobenius to 1. We can associate to every $(\varphi, \mathrm{Gal}(K/F))$-module $\mathbf{D}$ a K_0-vector space Δ with a continuous K_0-linear action of $\mathcal{W}_F$ in the following manner:

$$\rho : \mathcal{W}_F \longrightarrow \mathrm{Aut}_{K_0}(\Delta) : w \longmapsto (w \bmod \mathcal{W}_K)\, \varphi^{-\pi(w)},$$

where Δ is the underlying K_0-vector space of $\mathbf{D}$.

Definition 1.18. Let $\mathbf{D}$ be a $(\varphi, \mathrm{Gal}(K/F))$-module, the pair $\mathbf{W}(\mathbf{D}) = (\Delta, \rho)$ is called the *Weil representation* associated to $\mathbf{D}$.

A Weil representation (Δ, ρ) of $\mathcal{W}_F$ is defined over $\mathbb{Q}$ if $\mathrm{Tr}(\rho(w)) \in \mathbb{Q}$ for every $w \in \mathcal{W}_F$.

1.3 Galois pairs

The idea behind the notion of Galois pair is to mirror the situation illustrated in Section 1.1 but starting from an Abelian variety defined over a finite field.

Let $f \geq 1$ be an integer and $K/\mathbb{Q}_{p^f}$ be a finite Galois extension with residue field κ. We denote by K_0 the maximal unramified extension of $\mathbb{Q}_p$ inside K. We let σ be the Frobenius of $\mathrm{Gal}(K_0/\mathbb{Q}_{p^f}) \simeq \mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$. Let $A_0/\mathbb{F}_{p^f}$ be an Abelian variety and $A = A_0 \times_{\mathbb{F}_{p^f}} \kappa$ be the scalar extension of A_0 to κ. The Galois group $\mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$ then acts on $\mathrm{Aut}_\kappa(A)$, and we denote by

$$\mathrm{Ist}_\kappa(A) = \mathrm{Aut}_\kappa(A) \rtimes \mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$$

the semidirect product obtained from this action. We let

$$\mathrm{pr} : \mathrm{Ist}_\kappa(A) \twoheadrightarrow \mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$$

be the natural projection. We recall the definition of a Galois pair introduced in [Vol05, Def. 3.1].

Definition 1.19. Let $K/\mathbb{Q}_{p^f}$ be a finite Galois extension. A *Galois pair* for $K/\mathbb{Q}_{p^f}$ is a triple (A_0, Γ, ν) where $A_0/\mathbb{F}_{p^f}$ is an Abelian variety, Γ is a finite subgroup of $\mathrm{Aut}_\kappa(A)$ and $\nu : \mathrm{Gal}(K/\mathbb{Q}_{p^f}) \to \mathrm{Ist}_\kappa(A)$ is an antimorphism satisfying:

(1) For all $g \in \mathrm{Gal}(K/\mathbb{Q}_{p^f})$, $(\mathrm{pr} \circ \nu)(g) = g \bmod I(K/\mathbb{Q}_{p^f})$

(2) $\mathrm{Im}\nu = \Gamma \rtimes \mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$.

A Galois pair is minimal if ν is injective and κ is minimal with respect to Γ, that is Γ cannot be defined over a smaller extension $\kappa'/\mathbb{F}_{p^f}$.

Lemma 1.1. *Let $K/\mathbb{Q}_{p^f}$ be a finite Galois extension such that*

$$\mathrm{Gal}(K/\mathbb{Q}_{p^f}) \simeq I(K/\mathbb{Q}_{p^f}) \rtimes \mathrm{Gal}(\kappa/\mathbb{F}_{p^f}).$$

In this situation, a Galois pair (A_0, Γ, ν) for $K/\mathbb{Q}_{p^f}$ is equivalent to a $\mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$-equivariant antimorphism

$$I(K/\mathbb{Q}_{p^f}) \longrightarrow \mathrm{Aut}_\kappa(A)$$

with image $\nu(I(K/\mathbb{Q}_{p^f})) = \Gamma$.

Proof. Let (A_0, Γ, ν) be a Galois pair for $K/\mathbb{Q}_{p^f}$. The condition (1) from definition 1.19 is equivalent to the commutativity of the following diagram

$$\begin{tikzcd}
1 \arrow[r] & I(K/\mathbb{Q}_{p^f}) \arrow[r] \arrow[d, "\nu"] & \mathrm{Gal}(K/\mathbb{Q}_{p^f}) \arrow[r] \arrow[d, "\nu"] & \mathrm{Gal}(\kappa/\mathbb{F}_{p^f}) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{Aut}_\kappa(A) \arrow[r] & \mathrm{Ist}_\kappa(A) \arrow[r, "\mathrm{pr}"] & \mathrm{Gal}(\kappa/\mathbb{F}_{p^f}) \arrow[r] & 1
\end{tikzcd}$$

where the two exact sequences split. The condition (2) from definition 1.19 further imposes that the diagram commutes with the sections defining the respective semidirect products, in other words the antimorphism

$$\nu : I(K/\mathbb{Q}_{p^f}) \longrightarrow \mathrm{Aut}_\kappa(A)$$

is $\mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$-equivariant. Conversely, a $\mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$-equivariant antimorphism

$$I(K/\mathbb{Q}_{p^f}) \longrightarrow \mathrm{Aut}_\kappa(A)$$

naturally induces an antimorphism

$$\mathrm{Gal}(K/\mathbb{Q}_{p^f}) \longrightarrow \mathrm{Ist}_\kappa(A)$$

that satisfies conditions (1) and (2) from definition 1.19. □

Definition 1.20. Let $K/\mathbb{Q}_{p^f}$ be a finite Galois extension. A morphism of Galois pairs $(A_0, \Gamma, \nu) \to (A_0', \Gamma', \nu')$ is a $\mathbb{F}_{p^f}$-morphism $\psi_0 : A_0 \to A_0'$ such that for every $g \in \mathrm{Gal}(K/\mathbb{Q}_{p^f})$, the diagram

$$\begin{tikzcd}
A \arrow[r, "\psi^{\overline{g}}"] \arrow[d, "\nu(g)"'] & A' \arrow[d, "\nu'(g)"] \\
A \arrow[r, "\psi^{\overline{g}}"] & A'
\end{tikzcd}$$

commutes, where $\overline{g} = g \bmod I(K/\mathbb{Q}_{p^f})$ and $\psi = \psi_0 \times_{\mathbb{F}_{p^f}} \kappa$.

We say that a morphism of Galois pair is an isogeny if ψ_0 is an isogeny. We refer to [Vol05, § 3] for the categorical properties of Galois pairs.

We will now introduce the notion of a polarised Galois pair, which is motivated by the discussion in the end of Section 1.1.

Definition 1.21. The dual of a Galois pair (A_0, Γ, ν) for $K/\mathbb{Q}_{p^f}$ is $(A_0^\vee, \Gamma^\vee, \nu')$ where

- $\Gamma^\vee = \{\gamma^\vee \mid \gamma \in \Gamma\}$,
- $\nu' : g \mapsto (\nu(g)^\vee)^{-1}$.

A morphism $\lambda_0 : A_0 \to A_0^\vee$ induces a morphism of Galois pairs if and only if for every $\gamma \in \Gamma$, $\gamma^\vee \lambda^\vee \gamma = \lambda$. If λ_0 is a polarisation, it is equivalent to the inclusion $\Gamma \subset \mathrm{Aut}_\kappa(A, \lambda)$.

Definition 1.22. A Galois pair (A_0, Γ, ν) for $K/\mathbb{Q}_{p^f}$ is *polarised* if there exists a polarisation $\lambda_0 : A_0 \to A_0^\vee$ such that

$$\Gamma \subset \mathrm{Aut}_\kappa(A, \lambda),$$

where $\lambda = \lambda_0 \times_{\mathbb{F}_{p^f}} \kappa$.

A Galois pair (A_0, Γ, ν) for $K/\mathbb{Q}_{p^f}$ functorially defines a $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_{p^f}))$-module. Let us denote by D_0 (resp. D) the Dieudonné module of the p-divisible group associated to A_0 (resp. A). If $\dim A_0 = d$, then D_0 is a $2d$-dimensional $\mathbb{Q}_{p^f}$-vector space with linear Frobenius φ_0 and $D = K_0 \otimes_{\mathbb{Q}_p} D_0$ is a $2d$-dimensional K_0-vector space with σ-semilinear Frobenius $\varphi = \sigma \otimes \varphi_0$. The following diagram

$$\begin{tikzcd}
\mathrm{Gal}(K/\mathbb{Q}_{p^f}) \arrow[r, "\nu"] \arrow[dr, "\nu'"'] & \mathrm{Aut}_\kappa(A) \rtimes \mathrm{Gal}(\kappa/\mathbb{F}_{p^f}) \arrow[d, hook, "\text{Dieudonné functor}"] \\
 & \mathrm{Aut}_\varphi(D) \rtimes \mathrm{Gal}(K_0/\mathbb{Q}_{p^f})
\end{tikzcd}$$

defines, thanks to contravariancy of the Dieudonné functor, a group morphism ν' that induces the desired semilinear action.

Remark 1.5. If (A_0, Γ, ν) is a minimal Galois pair for $K/\mathbb{Q}_{p^f}$, the Weil representation $\Delta(A)$ associated to the $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_{p^f}))$-module $D(A)$ is potentially unramified over K, and K is minimal in that regard.

The contravariant functor $(A_0, \Gamma, \nu) \mapsto D(A)$ from the category of Galois pairs for $K/\mathbb{Q}_{p^f}$ to the category of $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_{p^f}))$-modules is faithful. If ψ_0 is an isogeny of Galois pairs, then $D(\psi_0)$ is an isomorphism of $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_{p^f}))$-modules. The Dieudonné module $D(A^\vee)$ of the dual of A is isomorphic to $D(A)^* \{-1\}$, the twist of the dual of $D(A)$.

Let (A_0, Γ, ν) be a Galois pair for $K/\mathbb{Q}_{p^f}$ and let $D = D(A)$ be the $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_{p^f}))$-module associated to $A = A_0 \times_{\mathbb{F}_{p^f}} \kappa$.

Definition 1.23. Let Fil be a filtration on D. A polarisation $\lambda_0 : A_0 \to A_0^\vee$ *lifts* to (D, Fil) if

$$D(\lambda) : D^* \{-1\} \xrightarrow{\sim} D$$

is an antisymmetric isomorphism of filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_{p^f}))$-modules.

Such a morphism induces a non degenerate skewform on (D, Fil) (see Def. 1.13). The following result ensures that when such a form exists, it is induced by a polarisation on A_0 (up to isomorphism).

Proposition 1.1 ([Vol05], Prop. 5.5). *Let* Fil *be a filtration on D such that there is a nondegenerate skew form on* (D, Fil). *Then there exists a filtration* Fil' *on D such that* $(D, \mathrm{Fil}) \simeq (D, \mathrm{Fil}')$ *and a polarisation* λ_0 *that lifts to* (D, Fil').

Let us mention a result due to S. Philip (see [Phi24]) regarding the existence of certain filtrations.

Proposition 1.2. *Let* $A/\overline{\mathbb{F}_p}$ *be an Abelian variety and* λ *be a polarisation on A. Let* $\widehat{D}$ *be the Dieudonné module of A, it is a* φ*-module over* $\widehat{\mathbb{Q}_p^{un}}$. *Let* $L/\widehat{\mathbb{Q}_p^{un}}$ *be a finite Galois extension such that*

$$\mathrm{Gal}(L/\widehat{\mathbb{Q}_p^{un}}) \hookrightarrow \mathrm{Aut}_\varphi(\widehat{D}).$$

There exists an admissible $\mathrm{Gal}(L/\widehat{\mathbb{Q}_p^{un}})$*-stable filtration* Fil *of Hodge-Tate type* $(0,1)$ *on* $\widehat{D}_L$ *such that* λ *lifts to* $(\widehat{D}, \mathrm{Fil})$.

Proof. It is a direct consequence of [Phi24, Thm. 2.2], since Dieudonné modules of Abelian varieties over $\overline{\mathbb{F}_p}$ are *Abelian* in the sense of [Phi24, Def. 2.1]. $\square$

1.4 The tame case

Let $f \geq 1$ and $F = \mathbb{Q}_{p^f}$. Let $\mathcal{A}/F$ be an Abelian variety acquiring good reduction over a totally ramified extension L/F of minimal degree. Denote by K the Galois closure of L and by K_0 be the maximal unramified extension of F inside K. Let $\mathbf{D} = (D, \mathrm{Fil})$ be the filtered $(\varphi, \mathrm{Gal}(K/F))$-module associated to $\mathcal{A}$. We let D_0 be the subspace of elements fixed by $\mathrm{Gal}(K_0/F)$ (seen as a subgroup of $\mathrm{Gal}(K/F)$), and $\varphi_0 = \varphi|_{D_0}$ the F-linear restriction of φ. We denote by $\mathbf{W}(\mathbf{D})$ the Weil representation associated to $\mathbf{D}$. It is known that $\mathbf{D}$ satisfies the following properties:

(1) φ_0 acts semisimply on D_0 and $P_{\mathrm{Char}}(\varphi_0)$ is a p-Weil polynomial

(2) $\mathbf{W}(\mathbf{D})$ is defined over $\mathbb{Q}$

(3) There exists a nondegenerate skew form $\mathbf{D} \times \mathbf{D} \longrightarrow K_0\{-1\}$

(4) $\mathbf{D}$ is of Hodge-Tate type $(0, 1)$.

The Property (1) comes from the Weil conjectures since $P_{\mathrm{Char}}(\varphi_0) = P_{\mathrm{Char}}(\mathrm{Frob}_A)$, where Frob_A is the Frobenius endomorphism of the special fibre A of $\mathcal{A} \times_F L$ (see [Del74, Thm. 1.6]). While (2) is a consequence of [ST68, Thm. 2.2 (ii)]. A polarisation on $\mathcal{A}$ induces a $\mathcal{G}_F$-equivariant symplectic form $\mathbf{V}_p(\mathcal{A}) \times \mathbf{V}_p(\mathcal{A}) \to \mathbb{Q}_p(1)$, hence the Property (3). Property (4) is deduced from the Hodge decomposition of Tate modules of p-divisible groups (see [Tat67, Cor. 2]).

M. Volkov showed that these conditions alone are sufficient to guarantee that a multiple nV of a p-adic representation V of $\mathcal{G}_F$ arises from an Abelian variety over F with tame potentially good reduction.

Theorem 1.5 ([Vol05], Thm. 5.13)**.** *Let V be a p-adic representation of $\mathcal{G}_F$ that becomes crystalline over a finite tame extension K/F. The following are equivalent:*

(i) There exists an integer n and an Abelian variety $\mathcal{A}_0/F$ such that

$$nV \simeq \mathbf{V}_p(\mathcal{A}_0)$$

(ii) $\mathbf{D}_{\mathrm{cris},K}(V)$ *satisfies conditions (1), (2), (3), and (4).*

If in addition, we require $\mathbf{W}(\mathbf{D})$ to be of *Tate type* (see [Vol05, Def. 4.6]), we then have a characterisation of the p-adic representations of $\mathcal{G}_F$ arising from Abelian varieties over F with tame potentially good reduction.

Theorem 1.6 ([Vol05], Thm. 5.7)**.** *Let V be a p-adic representation of $\mathcal{G}_F$ that becomes crystalline over a finite tame extension K/F. The following are equivalent:*

(i) There exists an Abelian variety $\mathcal{A}_0/F$ such that $V \simeq \mathbf{V}_p(\mathcal{A}_0)$

(ii) $\mathbf{D}_{\mathrm{cris},K}(V)$ *satisfies conditions (1), (2), (3), (4) and* $\mathbf{W}(\mathbf{D})$ *is of Tate type.*

The proof partly relies on the existence of Galois pairs, which is guaranteed under the right conditions in the tame case (see [Vol05, Thm. 4.11]). Our results are mainly based on the construction of such pairs in specific wild cases. When a polarised Galois pair can be endowed with a filtration compatible with the polarisation (in the sense of 1.23) it can be lifted to an Abelian variety with potentially good reduction.

Proposition 1.3. *Let (A_0, Γ, ν) be a minimal polarised Galois pair for $K/\mathbb{Q}_{p^f}$ and D be its associated $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_{p^f}))$-module. Assume there exists a Hodge-Tate type $(0,1)$ filtration Fil on D and a nondegenerate skew form on (D, Fil). Then (D, Fil) is attached to the p-adic Tate module of an Abelian variety $\mathcal{A}_0/\mathbb{Q}_{p^f}$ with good reduction over K minimal.*

Proof. Follows from Proposition 1.1 and the proof of [Vol05, Thm. 5.7]. □

In the case of elliptic curves some conditions on the filtered $(\varphi, \mathcal{G}_F)$-module can be weakened.

Remark 1.6. When $\dim_{\mathbb{Q}_p} V = 2$, the Tate type condition on the Weil representation is no longer necessary. Furthermore, condition (3) can be replaced by

$$\bigwedge\nolimits_{K_0}^{2} D = K_0\{-1\} \left(\Leftrightarrow \bigwedge\nolimits_{\mathbb{Q}_p}^{2} V = \mathbb{Q}_p(1)\right),$$

that is, the determinant is the cyclotomic character (see [Vol01, Thm. 5.1]).

CHAPTER 2

The 3-adic representations arising from elliptic curves over $\mathbb{Q}_3$

This chapter is a reproduction of the article [Bos25] by the author.

2.1 Introduction

A p-adic representation V of $\mathcal{G}_{\mathbb{Q}_p}$ arises from an elliptic curve over $\mathbb{Q}_p$ if there exists $E/\mathbb{Q}_p$ such that $V \simeq \mathbf{V}_p(E)$. In this chapter, we classify all 3-adic representations arising from elliptic curves over $\mathbb{Q}_3$ with potentially good reduction, up to isomorphism. Such representations are completely determined by their associated filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_p))$-module, where $K/\mathbb{Q}_p$ is Galois of minimal degree over which V acquires good reduction (see Theorem 1.3). For anything concerning elliptic curves, we refer to [Sil86].

Let $E/\mathbb{Q}_p$ be an elliptic curve acquiring good reduction over a finite Galois extension $K/\mathbb{Q}_p$ with maximal unramified subfield K_0 such that its ramification index $e = e(K/\mathbb{Q}_p)$ is minimal. Let $\mathbf{D} = (D, \mathrm{Fil})$ be its associated filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_p))$-module for the contravariant crystalline functor, D_0 the subspace of elements fixed by $\mathrm{Gal}(K_0/\mathbb{Q}_p)$ and $\varphi_0 = \varphi \restriction_{D_0}$ the $\mathbb{Q}_p$-linear restriction of φ. We have seen (Section 1.3 and Rem. 1.6) that $\mathbf{D}$

satisfies the following properties:

(1) $P_{\mathrm{Char}}(\varphi_0)(X) = X^2 + a_p X + p$, with $|a_p|_\infty \le 2\sqrt{p}$

(2) $\mathbf{W}(\mathbf{D})$ is defined over $\mathbb{Q}$

(3) $\bigwedge_{K_0}^2 \mathbf{D} = K_0\{-1\}$ (*i.e.* $\bigwedge_{\mathbb{Q}_p}^2 \mathbf{V}_p(E) = \mathbb{Q}_p(1)$)

(4) $\mathbf{D}$ is of Hodge-Tate type $(0,1)$

These conditions alone are sufficient to guarantee that a 2-dimensional p-adic representation of $\mathcal{G}_{\mathbb{Q}_p}$ comes from an elliptic curve over $\mathbb{Q}_p$ in the case of tame potential good reduction (see Theorem 1.6 and Remark 1.6). It is not known yet if these are sufficient in the presence of wild potential good reduction as well, however they are still necessary. Starting from these conditions and imposing geometric descent datum and a minimal field of good reduction $K/\mathbb{Q}_3$, we provide a list of isomorphism classes of possible filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules. Then we show that every object in the list arises from an elliptic curve over $\mathbb{Q}_3$ with potentially good reduction.

Some of the classes we describe can directly be deduced from the case $p \ge 5$ when $(e,p) = 1$ (see [Vol01]). Our genuine new results are the cases of wild potential good reduction ($e = 3, 6$ and 12) with $e = 12$ being the first case of non Abelian inertia. We provide proofs in the tame case for the sake of completeness. The classification is synthetised in Table 2.1.

Note that the supersingular traces $a_3 = \pm 3$ occur, which is specific to the $p = 3$ case (compared to $p \ge 5$). One may expect that they should appear every time the reduction is supersingular, and yet this is not the case. The reason behind this absence lies in the structure of the automorphism group of the special fibre, which controls the possible descents. Furthermore, we need to deal with several different fields of good reduction. Indeed, wild finite extensions of $\mathbb{Q}_p^{\mathrm{un}}$ aren't unique as opposed to the tame ones. This leads to interesting new phenomena. The case $e = 12$ is uniform, the five fields are almost indistinguishable. When $e = 3$ the situation is different between the two possible fields. The non Abelian extension occurs for only one possible Frobenius trace and has an infinity of isomorphism classes.

The Abelian extension, on the other hand, occurs for every supersingular traces value but has only two classes for each. Let us finally mention that the ordinary cases have simply disappeared when $e > 2$, again a specific feature of elliptic curves over $\mathbb{F}_3$.

2.2 Strategy

Let $E/\mathbb{Q}_3$ be an elliptic curve with potentially good reduction. We know that $\mathrm{dst}(E) \in \{1, 2, 3, 4, 6, 12\}$ and $\rho_{\ell,E}(\mathcal{I}_{\mathbb{Q}_3})$ is either a cyclic group of order $1, 2, 3, 4, 6$ or the non Abelian semi-direct product of a cyclic group of order 4 by a group of order 3 (see [Ser71, §5.6]). The degree of a minimal field of good reduction is bounded by the image of inertia and the structure of its inertia subgroup is known.

We begin by fixing a semi-stability defect $e \in \{1, 2, 3, 4, 6, 12\}$. The first step is to determine every finite Galois extension with ramification index e that arises as a field of good reduction of some elliptic curve defined over $\mathbb{Q}_3$. Cases $e = 1, 2$ and 4 are tame, hence necessarily given by $\mathbb{Q}_3(\sqrt[e]{3})$. For $e = 3$ we use Local Class Field Theory and the local fields database in [LMFDB], $e = 6$ is then obtained by a ramified quadratic twist. Finally, $e = 12$ is treated using [LMFDB] since the structure of the Galois group and inertia subgroup are well known. We then fix $K/\mathbb{Q}_3$ to be one such extension. The next step is to describe the list of the 2-dimensional filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules $\mathbf{D}$ satisfying properties $(1) - (4)$ for $p = 3$ (see Section 2.3). We then show that given an elliptic curve $E/\mathbb{Q}_3$ with potentially good reduction over K, its associated filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}^*_{\mathrm{cris},K}(V_3(E))$ is necessarily isomorphic to one object $\mathbf{D}$ of our list. Finally, given an object $\mathbf{D}$ in the list, we need to find an elliptic curve $E/\mathbb{Q}_3$ such that

$$\mathbf{D}^*_{\mathrm{cris},K}(V_3(E)) \simeq \mathbf{D},$$

which is done in Section 2.4.

Table 2.1: Filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules arising from elliptic curves over $\mathbb{Q}_3$ with potentially good reduction.

e	Red. type	K	Frob.	$\mathbf{D}$	#Classes
1	Supersing.	$\mathbb{Q}_3$	-3	$\mathbf{D}_{\mathrm{c}}(1;-3;0)$	1
			0	$\mathbf{D}_{\mathrm{c}}(1;0;0)$	1
			3	$\mathbf{D}_{\mathrm{c}}(1;3;0)$	1
	Ord.	$\mathbb{Q}_3$	-2	$\mathbf{D}_{\mathrm{c}}(1;-2;\alpha)$	2
			-1	$\mathbf{D}_{\mathrm{c}}(1;-1;\alpha)$	2
			1	$\mathbf{D}_{\mathrm{c}}(1;1;\alpha)$	2
			2	$\mathbf{D}_{\mathrm{c}}(1;2;\alpha)$	2
2	Supersing.	$\mathbb{Q}_3(\sqrt{3})$	-3	$\mathbf{D}_{\mathrm{c}}(2;-3;0)$	1
			0	$\mathbf{D}_{\mathrm{c}}(2;0;0)$	1
			3	$\mathbf{D}_{\mathrm{c}}(2;3;0)$	1
	Ord.	$\mathbb{Q}_3(\sqrt{3})$	-2	$\mathbf{D}_{\mathrm{c}}(2;-2;\alpha)$	2
			-1	$\mathbf{D}_{\mathrm{c}}(2;-1;\alpha)$	2
			1	$\mathbf{D}_{\mathrm{c}}(2;1;\alpha)$	2
			2	$\mathbf{D}_{\mathrm{c}}(2;2;\alpha)$	2
4	Supersing.	$\mathbb{Q}_3(\sqrt[4]{3})$	0	$\mathbf{D}_{\mathrm{pc}}(4;0;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
3	Supersing.	$L^{\mathrm{na}}(\zeta_4)$	0	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3;0;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
		$L^{\mathrm{ab}} = \mathbb{Q}_3(\pi)$	-3	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(3;-3,\mu;\pi)$	2
			0	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(3;0,\mu;\pi)$	2
			3	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(3;3,\mu;\pi)$	2
6	Supersing.	$L^{\mathrm{na}}(\zeta_4, \sqrt{3})$	0	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(6;0;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
		$L^{\mathrm{ab}}(\sqrt{3})$	-3	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(6;-3,\mu;\pi)$	2
			0	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(6;0,\mu;\pi)$	2
			3	$\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(6;3,\mu;\pi)$	2
12	Supersing.	K_1	0	$\mathbf{D}_{\mathrm{pc}}(12;0;1;\epsilon;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
		K_2	0	$\mathbf{D}_{\mathrm{pc}}(12;0;2;\epsilon;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
		K_3	0	$\mathbf{D}_{\mathrm{pc}}(12;0;3;\epsilon;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
		K_4	0	$\mathbf{D}_{\mathrm{pc}}(12;0;4;\epsilon;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$
		K_5	0	$\mathbf{D}_{\mathrm{pc}}(12;0;5;\epsilon;\alpha)$	$\mathbb{P}^1(\mathbb{Q}_3)$

One last point requires some discussion. Given some unfiltered 2-dimensional $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}$, the set of Hodge-Tate type $(0,1)$ filtrations on $\mathbf{D}$ is in bijection with $\mathbb{P}^1(\mathbb{Q}_3)$. Indeed, by Galois Descent, it is easy to check that the $\mathrm{Gal}(K/\mathbb{Q}_3)$-stable lines in $D_K = K \otimes_{K_0} D$ are in bijection with the lines in

$$D_K^{\mathrm{Gal}(K/\mathbb{Q}_3)} = \{x \in D_K | \forall g \in \mathrm{Gal}(K/\mathbb{Q}_3), g.x = x\}$$

a 2-dimensional $\mathbb{Q}_3$-vector space. This means that if φ has only trivial stable subspaces, there are infinitely many admissible filtrations on $\mathbf{D}$. In the following we will define sets that parameter our filtrations. This fact ensures that these sets will always be non empty, even though it could not be clear at first glance.

2.3 Classification

In this section, We provide the list of admissible filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules satisfying the following geometric conditions:

(1) $P_{\mathrm{Char}}(\varphi_0)(X) = X^2 + a_3X + 3$, with $|a_3|_\infty \leq 2\sqrt{3}$,

(2) $\mathbf{W}(\mathbf{D})$ is defined over $\mathbb{Q}$,

(3) $\bigwedge^2_{K_0} \mathbf{D} = K_0\{-1\}$ (i.e. $\bigwedge^2_{\mathbb{Q}_3} V_3(E) = \mathbb{Q}_3(1)$),

(4) $\mathbf{D}$ is of Hodge–Tate type $(0, 1)$

with $K/\mathbb{Q}_3$ a minimal Galois extension of good reduction associated to an elliptic curve $E/\mathbb{Q}_3$.

2.3.1 The crystalline case

We start our classification with the representations coming from elliptic curves $E/\mathbb{Q}_3$ with good reduction ($K = \mathbb{Q}_3$). There are two distinct cases behaving differently depending on the trace $a_3(\tilde{E})$ of the Frobenius of $\tilde{E}/\mathbb{F}_3$.

The supersingular case

Let $a \in \{-3, 0, 3\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$. We denote by $\mathbf{D}_c(1; a; \alpha)$ the filtered φ-module (of Hodge–Tate type $(0, 1)$) defined by:

- $D = \mathbb{Q}_3 e_1 \oplus \mathbb{Q}_3 e_2$,
- $M_B(\varphi) = \left(\begin{smallmatrix} 0 & -3 \\ 1 & -a \end{smallmatrix}\right)$, where $B = (e_1, e_2)$,
- $\mathrm{Fil}^1 D = (\alpha e_1 + e_2)\mathbb{Q}_3$.

Identifying $\mathbb{P}^1(\mathbb{Q}_3)$ with $\mathbb{Q}_3 \sqcup \{\infty\}$, we let $\alpha e_1 + e_2 = e_1$ when $\alpha = \infty$. For each $a \in \{-3, 0, 3\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$, the filtered φ-module $\mathbf{D}_{\mathrm{c}}(1; a; \alpha)$ satisfies conditions (1)–(4) and is admissible. Condition (1) is obvious and (4) is satisfied by definition. Conditions (2) and (3) as well as admissibility are easily checked by computation.

Proposition 2.1. *Let* $E/\mathbb{Q}_3$ *be an elliptic curve with good reduction such that* $a_3 = a_3(\tilde{E}) \in \{-3, 0, 3\}$ *and* $\mathbf{D} = \mathbf{D}_{\mathrm{cris},K}(V_3(E))$. *There exists an isomorphism of filtered* φ*-modules between* $\mathbf{D}$ *and* $\mathbf{D}_{\mathrm{c}}(1; a_3; 0)$. *Moreover, if* $a, b \in \{-3, 0, 3\}$, *then* $\mathbf{D}_{\mathrm{c}}(1; a; 0)$ *and* $\mathbf{D}_{\mathrm{c}}(1; b; 0)$ *are isomorphic if and only if* $a = b$.

Proof. Let D (resp. D') be the $\mathbb{Q}_3$-vector space associated to $\mathbf{D}$ (resp. $\mathbf{D}_{\mathrm{c}}(1; a_3; 0)$). Let $B = (e_1, e_2)$ and $B' = (e'_1, e'_2)$ be basis for D and D' respectively such that

$$M_B(\varphi) = \begin{pmatrix} 0 & -3 \\ 1 & -a_3 \end{pmatrix} = M_{B'}(\varphi').$$

Such a basis of D always exists since $P_{\mathrm{Char}}(\varphi)(X) = X^2 + a_3 X + 3$ as $\mathbf{D}$ satisfies the condition (1). A $\mathbb{Q}_3$-isomorphism η between D and D' is φ-equivariant if and only if

$$M_{B,B'}(\eta) \in C(M_B(\varphi)),$$

where $C(M_B(\varphi))$ denotes the centralizer of $M_B(\varphi)$ in $\mathrm{M}_2(\mathbb{Q}_3)$. Notice that since

$$C(M_B(\varphi)) = C(\mathbb{Q}_3[M_B(\varphi)])$$

and $P_{\mathrm{Char}}(\varphi)(X)$ is irreducible, the Double Centralizer Theorem implies

$$C(M_B(\varphi)) = \mathbb{Q}_3[M_B(\varphi)] = \mathbb{Q}_3(M_B(\varphi)).$$

In particular, every nonzero element of $\mathbb{Q}_3(M_B(\varphi))$ is an isomorphism of φ-modules between (D, φ) and (D', φ'). Consider $\mathrm{Fil}^1 D = (\alpha e_1 + \beta e_2)\mathbb{Q}_3$, $(\alpha, \beta) \neq (0, 0)$. The matrix

$$\begin{pmatrix} \alpha & -3\beta \\ \beta & \alpha - a_3\beta \end{pmatrix}$$

is invertible because the homogenous polynomial $X^2 - a_3XY + 3Y^2$ only has trivial roots in $(\mathbb{Q}_3)^2$. Let $(\lambda, \mu) \in \mathbb{Q}_3^2$ be the unique solution to the system of equations

$$\begin{pmatrix} \alpha & -3\beta \\ \beta & \alpha - a_3\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It follows that $(\lambda, \mu) \neq (0, 0)$ and

$$\begin{pmatrix} \lambda & -3\mu \\ \mu & \lambda - a_3\mu \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, the map $\lambda \,\mathrm{id} + \mu M_B(\varphi) \in \mathbb{Q}_3[M_B(\varphi)]$ defines an isomorphism of filtered φ-modules between $\mathbf{D}$ and $\mathbf{D}_c(1; a_3; 0)$. One checks the last assertion by a simple computation. □

Remark 2.1. There are 3 isomorphism classes of filtered φ-modules in the supersingular case, one for each value taken by a.

The ordinary case

Let $a \in \{\pm 1, \pm 2\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$. We denote by $\mathbf{D}_c(1; a; \alpha)$ the filtered φ-module defined by:

- $D = \mathbb{Q}_3 e_1 \oplus \mathbb{Q}_3 e_2$,
- $M_B(\varphi) = \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1}3 \end{smallmatrix}\right)$, where $u \in \mathbb{Z}_3^\times$ such that $u + u^{-1}3 = -a$,
- $\mathrm{Fil}^1 D = (\alpha e_1 + e_2)\mathbb{Q}_3$.

For each $a \in \{\pm 1, \pm 2\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$, the filtered φ-module $\mathbf{D}_c(1; a; \alpha)$ satisfies conditions (1)–(4) and is admissible for $\alpha \neq \infty$.

Proposition 2.2. *Let $E/\mathbb{Q}_3$ be an elliptic curve with good reduction such that $a_3 = a_3(\tilde{E}) \in \{\pm 1, \pm 2\}$ and $\mathbf{D} = \mathbf{D}_{\mathrm{cris}}(V_3(E))$. There exists an isomorphism of filtered φ-modules between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(1; a_3; \alpha)$ for some $\alpha \in \{0, 1\}$. Moreover, if $(\alpha, a), (\beta, b) \in \{0,1\} \times \{\pm 1, \pm 2\}$, then $\mathbf{D}_{\mathrm{c}}(1; a; \alpha)$ and $\mathbf{D}_{\mathrm{c}}(1; b; \beta)$ are isomorphic if and only if $(\alpha, a) = (\beta, b)$.*

Proof. Since $\mathbf{D}$ is admissible, the only possible filtrations are defined by a $\mathbb{Q}_3$-line of the form $\mathrm{Fil}^1 D = (\beta e_1 + e_2)\mathbb{Q}_3$ for some $\beta \in \mathbb{Q}_3$. Let $\alpha \in \{0, 1\}$ and D' be the $\mathbb{Q}_3$-vector space associated to $\mathbf{D}_{\mathrm{c}}(1; a_3; \alpha)$. Let $B = (e_1, e_2)$, $B' = (e'_1, e'_2)$ be basis of D and D' respectively, such that

$$M_B(\varphi) = \begin{pmatrix} u & 0 \\ 0 & u^{-1}3 \end{pmatrix} = M_{B'}(\varphi'),\ u \in \mathbb{Z}_3^{\times},\ u + u^{-1}3 = -a_3.$$

Such a basis exists because $\mathbf{D}$ satisfies (1) and $(a_3, 3) = 1$ and thus we have

$$P_{\mathrm{Char}}(\varphi)(X) = X^2 + a_3 X + 3 = (X - u)(X - u^{-1}3)$$

for some $u \in \mathbb{Z}_3^{\times}$. A $\mathbb{Q}_3$-isomorphism η between D and D' is φ-equivariant if and only if

$$M_{B,B'}(\eta) \in C(M_B(\varphi)).$$

This time, since $P_{\mathrm{Char}}(\varphi)(X)$ is a product of distinct linear factors

$$C(\mathbb{Q}_3[M_B(\varphi)]) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} : \lambda, \mu \in \mathbb{Q}_3 \right\}.$$

If $\beta = 0$, then every invertible element of $C(M_B(\varphi))$ defines an isomorphism of filtered φ-modules between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(1; a_3; 0)$. If $\beta \neq 0$, then taking $\lambda = \beta^{-1}$ and $\mu = 1$ gives an isomorphism of filtered φ-modules between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(1; a_3; 1)$. □

Remark 2.2. There are 8 isomorphism classes of filtered φ-modules in the ordinary case, two for each possible value taken by a.

Remark 2.3. The elliptic curves $E/\mathbb{Q}_3$ with ordinary good reduction and $\alpha = 0$ are canonical lifts of their corresponding reduced curve $\tilde{E}/\mathbb{F}_3$.

The supersingular case

Let $a \in \{-3, 0, 3\}$ and $\alpha \in \mathbb{P}^1(K)$. We denote by $\mathbf{D}_{\mathrm{c}}(2; a; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = \mathbb{Q}_3 e_1 \oplus \mathbb{Q}_3 e_2$,
- $M_B(\varphi) = \left(\begin{smallmatrix} 0 & -3 \\ 1 & -a \end{smallmatrix}\right)$,
- $M_B(\tau_2) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$,
- $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$, where $D_K = K \otimes_{K_0} D$.

For each $a \in \{-3, 0, 3\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{c}}(2; a; \alpha)$ satisfies conditions (1)–(4) and is admissible.

Proposition 2.3. *Let $E/\mathbb{Q}_3$ be an elliptic curve with $\mathrm{dst}(E) = 2$ such that $a_3 = a_3(\tilde{E}) \in \{-3, 0, 3\}$ and $\mathbf{D} = \mathbf{D}_{\mathrm{cris},K}(V_3(E))$. There exists an isomorphism of filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(2; a_3; 0)$. Moreover, if a, $b \in \{-3, 0, 3\}$, then $\mathbf{D}_{\mathrm{c}}(2; a; 0)$ and $\mathbf{D}_{\mathrm{c}}(2; b; 0)$ are isomorphic if and only if $a = b$.*

Proof. Let D be the underlying $\mathbb{Q}_3$-vector space associated to $\mathbf{D}$ and $B = (e_1, e_2)$ a basis of D such that

$$M_B(\varphi) = \begin{pmatrix} 0 & -3 \\ 1 & -a_3 \end{pmatrix}.$$

The element τ_2 is seen as a $\mathbb{Q}_3$-automorphism of D and is of order 2. Since $\mathbf{D}$ satisfies conditions (2)–(3), we have $P_{\mathrm{Char}}(\tau_2)(X) \in \mathbb{Q}[X]$ and $\det(\tau_2) = 1$, so that

$$P_{\mathrm{Char}}(\tau_2)(X) = (X + 1)^2$$

thus $\tau_2 = -\mathrm{id}$. The K-line $(1 \otimes e_1)K$ is stable by τ_2 and if $\alpha \in K$, the K-line $(\alpha \otimes e_1 + 1 \otimes e_2)K$ is stable by τ_2 if and only if $\alpha \in \mathbb{Q}_3$. Let $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$ such that $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$ is the K-line defining the filtration of $\mathbf{D}$. The isomorphism between $\mathbf{D}$ and $\mathbf{D}_{\mathrm{c}}(2; a_3; 0)$ is given by $e_1 \mapsto (3/\alpha)e_1 + e_2$ and $e_2 \mapsto -3e_1 + (3/\alpha - a_3)e_2$. □

The ordinary case

Let $a \in \{\pm 1, \pm 2\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$. We denote by $\mathbf{D}_{\mathrm{c}}(2; a; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = \mathbb{Q}_3 e_1 \oplus \mathbb{Q}_3 e_2$,
- $M_B(\varphi) = \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1}3 \end{smallmatrix}\right)$ where $u \in \mathbb{Z}_3^\times$ such that $u + u^{-1}3 = -a$,
- $M_B(\tau_2) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$,
- $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$.

For each $a \in \{\pm 1, \pm 2\}$ and $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{c}}(2; a; \alpha)$ satisfies conditions (1)–(4) and is admissible for $\alpha \neq \infty$.

Proposition 2.4. *Let* $E/\mathbb{Q}_3$ *be an elliptic curve with* $\mathrm{dst}(E) = 2$ *such that* $a_3 = a_3(\tilde{E}) \in \{\pm 1, \pm 2\}$ *and* $\mathbf{D} = \mathbf{D}_{\mathrm{cris},K}(V_3(E))$. *There exists an isomorphism of filtered* $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$*-modules between* $\mathbf{D}$ *and* $\mathbf{D}_{\mathrm{c}}(2; a_3; \alpha)$ *for some* $\alpha \in \{0, 1\}$. *Moreover, if* (α, a), $(\beta, b) \in \{0, 1\} \times \{\pm 1, \pm 2\}$, *then* $\mathbf{D}_{\mathrm{c}}(2; a; \alpha)$ *and* $\mathbf{D}_{\mathrm{c}}(2; b; \beta)$ *are isomorphic if and only if* $(\alpha, a) = (\beta, b)$.

Proof. See the ordinary crystalline case for the description of φ and the supersingular quadratic case for the description of τ_2 and the filtration. □

Remark 2.4. These are exactly the twists by the ramified quadratic character associated to $\mathbb{Q}_3(\sqrt{3})/\mathbb{Q}_3$ of the corresponding crystalline cases.

Remark 2.5. As in the crystalline case, the elliptic curves $E/\mathbb{Q}_3$ with ordinary potentially good reduction and $\alpha = 0$ are canonical lifts of their corresponding reduced curve $\tilde{E}/\mathbb{F}_3$.

2.3.2 The quartic case

Let $E/\mathbb{Q}_3$ with semi-stability defect $\mathrm{dst}(E) = 4$. Again, since 4 and 3 are coprime, the only quartic extension of $\mathbb{Q}_3^{\mathrm{un}}$ is $\mathbb{Q}_3^{\mathrm{un}}(\sqrt[4]{3})$. Let us fix ζ_4 a primitive fourth root of unity and π_4 a root of $X^4 - 3$ in $\overline{\mathbb{Q}_3}$. Consider $L = \mathbb{Q}_3(\pi_4)$, $K = L(\zeta_4)$ its Galois closure and $K_0 = \mathbb{Q}_3(\zeta_4)$ the maximal unramified extension of $K/\mathbb{Q}_3$. Our curve necessarily acquires good reduction over L. Let $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_3)$ be the absolute Frobenius on K_0, $\omega \in \mathrm{Gal}(K/\mathbb{Q}_3)$ a lifting of σ fixing L and τ_4 a generator of $\mathrm{Gal}(K/K_0) = I(K/\mathbb{Q}_3)$. Then $\mathrm{Gal}(K/\mathbb{Q}_3) = \langle \tau_4 \rangle \rtimes \langle \omega \rangle$ with $\tau_4\omega = \omega\tau_4^{-1}$.

Let $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$. We denote by $\mathbf{D}_{\mathrm{pc}}(4;0;\alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = K_0e_1 \oplus K_0e_2$,
- $\varphi(e_1) = e_2,\ \varphi(e_2) = -3e_1$,
- $M_B(\tau_4) = \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix}$,
- $\omega(e_1) = e_1,\ \omega(e_2) = e_2$,
- $\mathrm{Fil}^1 D_K = (\alpha\pi_4^{-1} \otimes e_1 + \pi_4 \otimes e_2)K$.

For each $\alpha \in \mathbb{P}^1(\mathbb{Q}_3)$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{pc}}(4;0;\alpha)$ satisfies conditions (1)–(4) and is admissible.

Proposition 2.5. *Let $E/\mathbb{Q}_3$ be an elliptic curve with* $\mathrm{dst}(E) = 4$ *and* $\mathbf{D} = \mathbf{D}_{\mathrm{cris},K}(V_3(E))$. *There exists an isomorphism of filtered* $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$*-modules between* $\mathbf{D}$ *and* $\mathbf{D}_{\mathrm{pc}}(4;0;\alpha)$. *Moreover, if $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q}_3)$, then* $\mathbf{D}_{\mathrm{pc}}(4;0;\alpha)$ *and* $\mathbf{D}_{\mathrm{pc}}(4;0;\beta)$ *are isomorphic if and only if $\alpha = \beta$.*

Proof. Let D be the underlying K_0-vector space associated to $\mathbf{D}$, the element τ_4 acts K_0-linearly over D and the morphism

$$I(K/\mathbb{Q}_3) \longrightarrow \mathrm{Aut}_{K_0}(D)$$

is injective by minimality of $e(K/\mathbb{Q}_3)$. We identify τ_4 to its image in $\mathrm{Aut}_{K_0}(D)$, it is an element of order 4. Again, because $\mathbf{D}$ satisfies (2)–(3) we have $\det(\tau_4) = 1$ and $P_{\mathrm{Char}}(\tau_4)(X) \in \mathbb{Q}[X]$ so that

$$P_{\mathrm{Char}}(\tau_4)(X) = P_{\min}(\tau_4)(X) = X^2 + 1 = (X - \zeta_4)(X - \zeta_4^{-1}).$$

In particular, τ_4 is diagonalisable in K_0 and has distinct eigenvalues. Let (e_1, e_2) be a diagonalisation basis of τ_4 over K_0. The relation $\tau_4\omega = \omega\tau_4^{-1}$

implies that $\omega(e_i) \in K_0 e_i$, $i = 1, 2$. Denote by $\omega_i = \omega\restriction_{K_0 e_i}$, the group $\langle \omega_i \rangle$ acts semi-linearly over $K_0 e_i$. Descent theory tells us that $(K_0 e_i)^{\langle \omega_i \rangle} \neq \{0\}$. We can then find a basis (e_1, e_2) of D over K_0 which is fixed by ω and such that $\tau_4(e_1) = \zeta_4 e_1$, $\tau_4(e_2) = \zeta_4^{-1} e_2$. Since φ is $\mathrm{Gal}(K/\mathbb{Q}_3)$-equivariant, it commutes to τ_4 and ω. A simple calculation shows that $\varphi(e_1) \in \mathbb{Q}_3 e_2$ and $\varphi(e_2) \in \mathbb{Q}_3 e_1$. Since $\det(\varphi) = 3$, $\varphi(e_1) = a e_2$ and $\varphi(e_2) = -3a^{-1} e_1$, necessarily $a \in \mathbb{Q}_3^\times$. That way we show that there exists a K_0-basis of D such that

$$\begin{cases} \varphi(e_1) = e_2 \wedge \varphi(e_2) = -3e_1 \\ \tau_4(e_1) = \zeta_4 e_1 \wedge \tau_4(e_2) = \zeta_4^{-1} e_2 \\ \omega(e_1) = e_1 \wedge \omega(e_2) = e_2. \end{cases}$$

We have now described the $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module structure on D. In particular, we see that $a_3 = 0$, i.e. $\tilde{E}_L$ is supersingular, but the two other supersingular values 3 and -3 cannot appear. What is left is to determine the K-line $\mathrm{Fil}^1 D_K$ which defines the filtration; it needs to satisfy the weak admissibility condition and be $\mathrm{Gal}(K/\mathbb{Q}_3)$-stable. Since φ does not have any stable subspaces, it is immediate. The K-line $(1 \otimes e_1)K$ is stable by action of $\mathrm{Gal}(K/\mathbb{Q}_3)$. Let $\beta \in \mathbb{Q}_3$ and $\mathrm{Fil}^1 D_K = (\beta \otimes e_1 + 1 \otimes e_2)K$. One easily shows that $\mathrm{Fil}^1 D_K$ is stable by ω if and only if $\beta \in L$ and by τ_4 if and only if $\pi_4^2 \beta \in K_0$. Then, $\mathrm{Fil}^1 D_K$ is stable by the action of $\mathrm{Gal}(K/\mathbb{Q}_3)$ if and only if $\pi_4^2 \beta \in L \cap K_0 = \mathbb{Q}_3$. Let $\alpha = \pi_4^2 \beta \in \mathbb{Q}_3$, we can rewrite our K-line defining the filtration as

$$\mathrm{Fil}^1 D_K = (\alpha \pi_4^{-1} \otimes e_1 + \pi_4 \otimes e_2)K$$

it is then clear that $\mathbf{D} \simeq \mathbf{D}_{\mathrm{pc}}(4; 0; \alpha)$. Let $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q}_3)$, consider the following filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules: $\mathbf{D} = \mathbf{D}_{\mathrm{pc}}(4; 0; \alpha)$ and $\mathbf{D}' = \mathbf{D}_{\mathrm{pc}}(4; 0; \beta)$. Let $B = (e_1, e_2)$ and $B' = (e'_1, e'_2)$ be K_0-basis of D and D' their respective underlying K_0-vector spaces. Let $\psi \colon \mathbf{D} \longrightarrow \mathbf{D}'$ be a nonzero morphism of filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules. Let $D_0 = D^{\langle \omega \rangle}$ and $D'_0 = (D')^{\langle \omega' \rangle}$. The relation $\psi \circ \omega = \omega' \circ \psi$ implies $\psi(D_0) \subseteq D'_0 = \mathbb{Q}_3 e'_1 \oplus \mathbb{Q}_3 e'_2$. Moreover, $\psi \circ \tau_4 = \tau'_4 \circ \psi$ implies $\psi(e_i) \in K_0 e'_i$, $i = 1, 2$. Then there exists a, $d \in \mathbb{Q}_3$ such that $\psi(e_1) = a e'_1$ and $\psi(e_2) = d e'_2$. Finally, $\psi \circ \varphi = \varphi' \circ \psi$ leads to $a = d$. Denoting by ψ_K the K-linear extension of ψ, we see that $\psi_K(\mathrm{Fil}^1 D_K) \subseteq \mathrm{Fil}^1 D'_K$ if and only if $\alpha = \beta$. □

2.3.3 The cubic case

Let $E/\mathbb{Q}_3$ be an elliptic curve with semi-stability defect $\mathrm{dst}(E) = 3$. There are exactly 9 totally ramified extensions of degree 3 of $\mathbb{Q}_3$ (see [LMFDB]). Since E acquires good reduction over a degree 3 Galois extension of $\mathbb{Q}_3^{\mathrm{un}}$, we are interested in the ones that keep their ramification index after Galois closure. Indeed, if $e(F/\mathbb{Q}_3) = 3$ but $e(F^{\mathrm{Gal}}/\mathbb{Q}_3) > 3$, then $[(F^{\mathrm{Gal}})^{\mathrm{un}} : \mathbb{Q}_3^{\mathrm{un}}] > 3$ is not minimal. There are only 4 such extensions; among these, 3 are Abelian and the last one has a Galois closure of degree 6 with Galois group isomorphic to S_3. One easily shows (using the Kronecker–Weber Theorem) that the three considered Abelian extensions are exactly the degree 3 totally ramified subextensions of $\mathbb{Q}_3(\zeta_{13}, \zeta_9 + \zeta_9^{-1})$, so their compositum with $\mathbb{Q}_3^{\mathrm{un}}$ is $\mathbb{Q}_3^{\mathrm{un}}(\zeta_9 + \zeta_9^{-1})$, and they are therefore interchangeable. This is not the case of the non Abelian extension. We denote by $L^{\mathrm{ab}} = \mathbb{Q}_3(\zeta_9 + \zeta_9^{-1})$ (resp. $L^{\mathrm{na}} = \mathbb{Q}_3(X^3 - 3X^2 + 6)$) a minimal field of good reduction for $E/\mathbb{Q}_3$ in the Abelian (resp. non Abelian) case.

Given an elliptic curve $E/\mathbb{Q}_3$ and $\ell \neq 3$ a prime, we consider:

$$\tau_E \colon I(\overline{\mathbb{Q}_3}/\mathbb{Q}_3) \xrightarrow{\rho_{\ell,E}} \mathrm{GL}_2(\mathbb{Q}_\ell) \hookrightarrow \mathrm{GL}_2(\mathbb{C}).$$

Proposition 2.6. *Let $E, E'/\mathbb{Q}_3$ be elliptic curves with semi-stability defects* $\mathrm{dst}(E) = \mathrm{dst}(E') = 3$. *We have the following equivalence:*

$$\tau_E \simeq_{\mathbb{C}} \tau_{E'} \Longleftrightarrow M_E = M_{E'},$$

where M_E (resp. $M_{E'}$) is the unique extension of $\mathbb{Q}_3^{un}$ of minimal degree over which E (resp. E') acquires good reduction.

Proof. The left to right implication is obvious since $M_E = (\mathbb{Q}_3^{\mathrm{un}})^{\ker(\tau_E)}$, $M_{E'} = (\mathbb{Q}_3^{\mathrm{un}})^{\ker(\tau_{E'})}$ and two isomorphic representations share the same kernel. If $M_E = M_E'$ then $\ker(\tau_E) = \ker(\tau_{E'})$ and both types factors into faithful irreducible representations of $\mathrm{Gal}(M_E/\mathbb{Q}_3^{\mathrm{un}}) \simeq \mathbb{Z}/3\mathbb{Z}$ defined over $\mathbb{Q}$, which are necessarily isomorphic. □

Using [DFV24, Table 1] we see that there are only two isomorphism classes of such objects for $p = e = 3$ so that $L^{\mathrm{ab}}\mathbb{Q}_3^{\mathrm{un}}$ and $L^{\mathrm{na}}\mathbb{Q}_3^{\mathrm{un}}$ are indeed distinct.

The non Abelian case

Let $E/\mathbb{Q}_3$ with $\mathrm{dst}(E) = 3$ acquiring good reduction over L^{na} with $K = L^{\mathrm{na}}(\zeta_4)$ its Galois closure. Denote by K_0 the maximal unramified extension of $K/\mathbb{Q}_3$ and $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_3)$ the absolute Frobenius. Let $\omega \in \mathrm{Gal}(K/\mathbb{Q}_3)$ be a lifting of σ fixing L^{na} and τ_3 a generator of $\mathrm{Gal}(K/K_0) = I(K/\mathbb{Q}_3)$. Then, $\mathrm{Gal}(K/\mathbb{Q}_3) = \langle\tau_3\rangle \rtimes \langle\omega\rangle$ with $\tau_3\omega = \omega\tau_3^{-1}$ (the unique non trivial semi-direct product).

Let $\alpha \in \mathcal{M}_3^{\mathrm{na}} = \{x \in L^{\mathrm{na}} \mid \tau_3(x) = (3\zeta_4 + x)/(1 + \zeta_4 x)\}$. We denote by $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3; 0; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = K_0 e_1 \oplus K_0 e_2$,
- $\varphi(e_1) = e_2,\ \varphi(e_2) = -3e_1$,
- $M_B(\tau_3) = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2}\zeta_4 \\ \frac{1}{2}\zeta_4 & -\frac{1}{2} \end{pmatrix}$,
- $\omega(e_1) = e_1,\ \omega(e_2) = e_2$,
- $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$.

For each $\alpha \in \mathcal{M}_3^{\mathrm{na}}$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3; 0; \alpha)$ satisfies conditions (1)–(4) and is admissible.

Proposition 2.7. *Let $E/\mathbb{Q}_3$ be an elliptic curve with $\mathrm{dst}(E) = 3$ acquiring good reduction over L^{na} and $\mathbf{D} = \mathbf{D}_{\mathrm{cris},K}(V_3(E))$. There exists $\alpha \in \mathcal{M}_3^{\mathrm{na}}$ such that $\mathbf{D}$ and $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3; 0; \alpha)$ are isomorphic as filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules. Moreover, if $\alpha, \beta \in \mathcal{M}_3^{\mathrm{na}}$, then $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3; 0; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(3; 0; \beta)$ are isomorphic if and only if $\alpha = \beta$.*

Proof. Denote by D the underlying K_0-vector space associated to $\mathbf{D}$, the element τ_3 acts K_0-linearly over D and the morphism

$$I(K/\mathbb{Q}_3) \longrightarrow \mathrm{Aut}_{K_0}(D)$$

is injective by minimality of $e(K/\mathbb{Q}_3)$. We identify τ_3 to its image in $\mathrm{Aut}_{K_0}(D)$, it is an element of order 3. Since $\zeta_3 \notin K_0$,

$$P_{\mathrm{Char}}(\tau_3)(X) = P_{\min}(\tau_3)(X) = X^2 + X + 1.$$

Let $B = (e_1, e_2)$ be a K_0-basis of D fixed by ω such that $\varphi(e_1) = e_2$ and $\varphi(e_2) = -3e_1 - a_3e_2$. Such a basis always exists since ω acts semi-linearly over D and $\varphi\omega = \omega\varphi$. Let λ, μ, λ', $\mu' \in K_0$ such that

$$M_B(\tau_3) = \begin{pmatrix} \lambda & \mu' \\ \mu & \lambda' \end{pmatrix}.$$

We already know that $\lambda' = -\lambda - 1$ and $\mu' = P(\lambda)/-\mu$ where $P = P_{\mathrm{Char}}(\tau_3)$. The relations $\tau\omega = \omega\tau^{-1}$ and $\tau\varphi = \varphi\tau$ imply that $a_3 = 0$, $\sigma(\lambda) = -\lambda - 1$, $\sigma(\mu) = -\mu$ and $P(\lambda)/-\mu = 3\mu$. In conclusion:

$$\begin{cases} \varphi(e_1) = e_2 \wedge \varphi(e_2) = -3e_1 \\ \omega(e_1) = e_1 \wedge \omega(e_2) = e_2 \\ M_B(\tau_3) = \left(\begin{smallmatrix} \lambda & 3\mu \\ \mu & -\lambda-1 \end{smallmatrix}\right) \end{cases}$$

with $\lambda \in -\frac{1}{2} + \mathbb{Q}_3\zeta_4$, $\mu \in \mathbb{Q}_3^\times\zeta_4$ and $P(\lambda) + 3\mu^2 = 0$. Let

$$M = \begin{pmatrix} \lambda + \frac{1}{2} & -3\mu - \frac{3}{2}\zeta_4 \\ \mu - \frac{1}{2}\zeta_4 & \lambda + \frac{1}{2} \end{pmatrix}$$

clearly $\det(M) = 0$. Let $(a, b) \in \ker(M)^{\mathrm{Gal}(K/\mathbb{Q}_3)} \subseteq K_0^2$ be a nonzero element. Then

$$B' = (e_1', e_2') = (ae_1 + be_2, -3be_1 + ae_2)$$

is a K_0-basis of D such that

$$\begin{cases} \varphi(e_1') = e_2' \wedge \varphi(e_2') = -3e_1' \\ \omega(e_1') = e_1' \wedge \omega(e_2') = e_2' \\ M_{B'}(\tau_3) = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2}\zeta_4 \\ \frac{1}{2}\zeta_4 & -\frac{1}{2} \end{pmatrix}. \end{cases}$$

Again, we denote by (e_1, e_2) such a basis. One easily checks that $(1 \otimes e_1)K$ and $(1 \otimes e_2)K$ are not stable by τ_3. Let $\alpha \in K^\times$ and $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)$. A simple calculation shows that such a K-line is stable by the action of $\mathrm{Gal}(K/\mathbb{Q}_3)$ if and only if $\alpha \in L^{\mathrm{na}}$ and $\tau_3(\alpha) = (3\zeta_4 + \alpha)/(1 + \zeta_4\alpha)$. Let B, B' be K_0-basis of $\mathbf{D} = \mathbf{D}^{\mathrm{na}}_{\mathrm{pc}}(3; 0; \alpha)$ and $\mathbf{D}' = \mathbf{D}^{\mathrm{na}}_{\mathrm{pc}}(3; 0; \beta)$ respectively. One easily shows that an isomorphism η of $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules between $\mathbf{D}$ and $\mathbf{D}'$ is of the form

$$M_{B,B'}(\eta) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \ a \in \mathbb{Q}_3^\times.$$

Denoting by $\eta_K \colon D_K \longrightarrow D'_K$ the K-linear extension of η, it is then clear that $\eta_K((\alpha \otimes e_1 + 1 \otimes e_2)K) \subseteq (\beta \otimes e_1 + 1 \otimes e_2)K$ if and only if $\alpha = \beta$. ☐

The Abelian case

Let $E/\mathbb{Q}_3$ with $\mathrm{dst}(E)$ acquiring good reduction over $K = L^{\mathrm{ab}}$. Its Galois group $\mathrm{Gal}(K/\mathbb{Q}_3) = I(K/\mathbb{Q}_3) = \langle \tau_3 \rangle$ is cyclic of order 3.

Let $\alpha \in \mathcal{M}^{\mathrm{ab}}_3 = \{x \in L^{\mathrm{ab}} \mid \tau_3(x) = (x-1)/x\}$ and

$$(a, \mu) \in S = (\{-3\} \times \{1, 2\}) \sqcup (\{0\} \times \{-1, 1\}) \sqcup (\{3\} \times \{-2, -1\}).$$

We denote by $\mathbf{D}^{\mathrm{ab}}_{\mathrm{pc}}(3; a, \mu; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = \mathbb{Q}_3 e_1 \oplus \mathbb{Q}_3 e_2$,
- $M_B(\tau_3) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$,
- $M_B(\varphi) = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, $a = -3$, $\mu = 1, 2$;
- $M_B(\varphi) = \left(\begin{smallmatrix} -2 & 1 \\ -1 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -2 \\ 2 & -1 \end{smallmatrix}\right)$, $a = 0$, $\mu = -1, 1$;
- $M_B(\varphi) = \left(\begin{smallmatrix} -1 & -1 \\ 1 & -2 \end{smallmatrix}\right), \left(\begin{smallmatrix} -2 & 1 \\ -1 & -1 \end{smallmatrix}\right)$, $a = 3$, $\mu = -2, -1$;
- $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$.

For each $\alpha \in \mathcal{M}^{\mathrm{ab}}_3$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}^{\mathrm{ab}}_{\mathrm{pc}}(3; a, \mu; \alpha)$ satisfies conditions (1)–(4) and is admissible.

Proposition 2.8. *Let $E/\mathbb{Q}_3$ be an elliptic curve with $\mathrm{dst}(E) = 3$ acquiring good reduction over K and $\mathbf{D} = \mathbf{D}_{\mathrm{cris},K}(V_3(E))$. There exists $\alpha \in \mathcal{M}_3^{\mathrm{ab}}$ and $(a, \mu) \in S$ such that $\mathbf{D}$ and $\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(3; a, \mu; \alpha)$ are isomorphic as filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules. Moreover, if $\alpha, \beta \in \mathcal{M}_3^{\mathrm{ab}}$ and $(a, \mu), (b, \nu) \in S$, then $\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(3; a, \mu; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(3; b, \nu; \beta)$ are isomorphic if and only if $(a, \mu) = (b, \nu)$.*

Proof. Denote by D the underlying $\mathbb{Q}_3$-vector space associated to $\mathbf{D}$, the element τ_3 acts $\mathbb{Q}_3$-linearly over D and the natural morphism

$$I(K/\mathbb{Q}_3) \longrightarrow \mathrm{Aut}_{\mathbb{Q}_3}(D)$$

is injective by minimality of $e(K/\mathbb{Q}_3)$. We identify τ_3 to its image in $\mathrm{Aut}_{\mathbb{Q}_3}(D)$, it is an element of order 3. Since $\zeta_3 \notin \mathbb{Q}_3$,

$$P_{\mathrm{Char}}(\tau_3)(X) = P_{\min}(\tau_3)(X) = X^2 + X + 1.$$

Let $B = (e_1, e_2)$ be a $\mathbb{Q}_3$-basis of D such that:

$$M_B(\tau_3) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Since $P_{\mathrm{Char}}(\varphi)(X) = X^2 + a_3X + 3$ and $\varphi\tau_3 = \tau_3\varphi$, we have

$$M_B(\varphi) = \begin{pmatrix} \lambda & -2\lambda - a_3 \\ 2\lambda + a_3 & -\lambda - a_3 \end{pmatrix}, \ \lambda \in \mathbb{Q}_3$$

with $\det(\varphi) = 3\lambda^2 + 3\lambda a_3 + a_3^2 = 3$, i.e. λ is a root of $3X^2 + 3a_3X + a_3^2 - 3$. But this polynomial has roots in $\mathbb{Q}_3$ if and only if $3 \mid a_3$, so $a_3 \in \{-3, 0, 3\}$. Considering every possible value of a_3 we obtain:

- if $a_3 = -3$, λ is a root of $X^2 - 3X + 2$ i.e. $\lambda \in \{1, 2\}$,
- if $a_3 = 0$, λ is a root of $X^2 - 1$ i.e. $\lambda \in \{-1, 1\}$,
- if $a_3 = 3$, λ is a root of $X^2 + 3X + 2$ i.e. $\lambda \in \{-2, -1\}$.

Observe that $(1\otimes e_1)K$ and $(1\otimes e_2)K$ are not stable by action of $\mathrm{Gal}(K/\mathbb{Q}_3)$. Let $\alpha \in K^\times$, the K-line $(\alpha\otimes e_1 + 1\otimes e_2)K$ is stable by τ_3 if and only if $\tau_3(\alpha) = (\alpha-1)/\alpha$. So that $\mathbf{D} \simeq \mathbf{D}^{\mathrm{ab}}_{\mathrm{pc}}(3; a_3, \lambda; \alpha)$ with α and (a_3, λ) satisfying the desired conditions. Let α, $\beta \in \mathcal{M}^{\mathrm{ab}}_3$ and (a,μ), $(b,\nu) \in S$. Consider $\mathbf{D} = \mathbf{D}^{\mathrm{ab}}_{\mathrm{pc}}(3; a, \mu; \alpha)$ and $\mathbf{D}' = \mathbf{D}^{\mathrm{ab}}_{\mathrm{pc}}(3; b, \nu; \beta)$, we will first show that their underlying $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules are not isomorphic, except in the obvious case. Let B and B' be $\mathbb{Q}_3$-basis of D and D' respectively. A morphism $\eta\colon D \to D'$ commuting to τ_3 and φ must be of the form

$$M_{B,B'}(\eta) = \begin{pmatrix} c & -d \\ d & c-d \end{pmatrix}$$

where $(c,d) \in \mathbb{Q}_3^2$ is in the kernel of the following linear map

$$\begin{pmatrix} \mu-\nu & 2(\nu-\mu)+b-a \\ \nu-\mu+b-a & \nu-\mu \end{pmatrix}.$$

The determinant of this matrix is $-(3(\nu-\mu)^2 + 3(\nu-\mu)(b-a) + (b-a)^2)$. There exists $(c,d) \neq (0,0)$ in the kernel if and only if $(\nu-\mu)$ is a root of

$$3X^2 + 3(b-a)X + (b-a)^2.$$

But such a polynomial has roots in $\mathbb{Q}_3$ if and only if $a = b$, in which case its only root is 0. This shows that if $\mathbf{D}$ and $\mathbf{D}'$ are isomorphic as $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules then $\mu = \nu$ and $a = b$. Now suppose that $(a,\mu) = (b,\nu)$. Let $\mathrm{Fil}^1 D_K = (\alpha\otimes e_1 + 1\otimes e_2)K$ and $\mathrm{Fil}^1 D'_K = (\beta\otimes e_1 + 1\otimes e_2)K$, the K-lines defining the filtrations on $\mathbf{D}$ and $\mathbf{D}'$. If $\alpha = \beta$ taking $c = 1$ and $d = 0$ gives us an obvious isomorphism. In the other case, we see that $(\alpha\beta - \beta + 1)/(\alpha - \beta) \in \mathbb{Q}_3$ and taking some $d \neq 0$ and $c = d(\alpha\beta-\beta+1)/(\alpha-\beta)$ gives us the desired isomorphism. □

Remark 2.6. We observe two differences with the non Abelian case: the supersingular traces 3 and −3 do appear and for each trace value there are two isomorphism classes of $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules (not considering filtration). We will explain the absence of these traces in Section 2.4. These two isomorphism classes are unramified quadratic twists of each other.

2.3.4 The sextic case

This section can be summarised by the following result: if $E/\mathbb{Q}_3$ has a semi-stability defect of 3 then its quadratic twist $E'/\mathbb{Q}_3$ by the character associated to $\sqrt{3}$ has a semi-stability defect of 6, and vice versa. Consequently, if $F/\mathbb{Q}_3$ is a field of good reduction for E, then $F(\sqrt{3})$ is a field of good reduction for E'.

The non Abelian case

Let $E/\mathbb{Q}_3$ with $\mathrm{dst}(E) = 6$ acquiring good reduction over $L^{\mathrm{na}}(\sqrt{3})$ and let $K = L^{\mathrm{na}}(\sqrt{3}, \zeta_4)$ be its Galois closure. We have $\mathrm{Gal}(K/\mathbb{Q}_3) = (\langle\tau_3\rangle \times \langle\tau_2\rangle) \rtimes \langle\omega\rangle$ and $I(K/\mathbb{Q}_3) = \langle\tau_3\rangle \times \langle\tau_2\rangle$ is cyclic of order 6.

Let $\alpha \in \mathcal{M}_6^{\mathrm{na}} = \{x \in L^{\mathrm{na}}(\sqrt{3}) \mid \tau_3(x) = (3\zeta_4 + x)/(1 + \zeta_4 x)\}$. We denote by $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(6; 0; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = K_0 e_1 \oplus K_0 e_2$,
- $\varphi(e_1) = e_2,\ \varphi(e_2) = -3e_1$,
- $M_B(\tau_3) = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2}\zeta_4 \\ \frac{1}{2}\zeta_4 & -\frac{1}{2} \end{pmatrix}$,
- $M_B(\tau_2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,
- $\omega(e_1) = e_1,\ \omega(e_2) = e_2$,
- $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$.

For each $\alpha \in \mathcal{M}_6^{\mathrm{na}}$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(6; 0; \alpha)$ satisfies conditions (1)–(4) and is admissible.

Proposition 2.9. *Let $E/\mathbb{Q}_3$ be an elliptic curve with $\mathrm{dst}(E) = 6$ acquiring good reduction over $L^{\mathrm{na}}(\sqrt{3})$ and $\mathbf{D} = \mathbf{D}_{\mathrm{cris},K}(V_3(E))$. There exists $\alpha \in \mathcal{M}_6^{\mathrm{na}}$ such that $\mathbf{D}$ and $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(6; 0; \alpha)$ are isomorphic as filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules. Moreover, if $\alpha, \beta \in \mathcal{M}_6^{\mathrm{na}}$, then $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(6; 0; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}^{\mathrm{na}}(6; 0; \beta)$ are isomorphic if and only if $\alpha = \beta$.*

Proof. Similar to the cubic non Abelian case using the natural injection

$$I(K/\mathbb{Q}_3) = \langle \tau_2 \rangle \times \langle \tau_3 \rangle \hookrightarrow \mathrm{Aut}_{K_0}(D). \qquad \square$$

The Abelian case

Let $E/\mathbb{Q}_3$ with $\mathrm{dst}(E) = 6$ acquiring good reduction over $K = L^{\mathrm{ab}}(\sqrt{3})$. Then $\mathrm{Gal}(K/\mathbb{Q}_3) = I(K/\mathbb{Q}_3) = \langle \tau_3 \rangle \times \langle \tau_2 \rangle$ is cyclic of order 6.

Let $\alpha \in \mathcal{M}_6^{\mathrm{ab}} = \{x \in L \mid \tau_3(x) = (x-1)/x\}$ and

$$(a, \mu) \in S = (\{-3\} \times \{1, 2\}) \sqcup (\{0\} \times \{-1, 1\}) \sqcup (\{3\} \times \{-2, -1\}).$$

We denote by $\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(6; a, \mu; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module defined by:

- $D = \mathbb{Q}_3 e_1 \oplus \mathbb{Q}_3 e_2$,
- $M_B(\tau_3) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$,
- $M_B(\tau_2) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$,
- $M_B(\varphi) = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, $a = -3$, $\mu = 1, 2$;
- $M_B(\varphi) = \left(\begin{smallmatrix} -2 & 1 \\ -1 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -2 \\ 2 & -1 \end{smallmatrix}\right)$, $a = 0$, $\mu = -1, 1$;
- $M_B(\varphi) = \left(\begin{smallmatrix} -1 & -1 \\ 1 & -2 \end{smallmatrix}\right), \left(\begin{smallmatrix} -2 & 1 \\ -1 & -1 \end{smallmatrix}\right)$, $a = 3$, $\mu = -2, -1$;
- $\mathrm{Fil}^1 D_K = (\alpha \otimes e_1 + 1 \otimes e_2)K$.

For each $\alpha \in \mathcal{M}_6^{\mathrm{ab}}$, the filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(6; a, \mu; \alpha)$ satisfies conditions (1)–(4) and is admissible.

Proposition 2.10. *Let* $E/\mathbb{Q}_3$ *be an elliptic curve with* $\mathrm{dst}(E) = 6$ *acquiring good reduction over* K *and* $\mathbf{D} = \mathbf{D}_{\mathrm{cris},K}(V_3(E))$*. There exists* $\alpha \in \mathcal{M}_6^{\mathrm{ab}}$ *and* $(a, \mu) \in S$ *such that* $\mathbf{D}$ *and* $\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(6; a, \mu; \alpha)$ *are isomorphic as filtered* $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$*-modules. Moreover, if* $\alpha, \beta \in \mathcal{M}_6^{\mathrm{ab}}$ *and* $(a, \mu), (b, \nu) \in S$*, then* $\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(6; a, \mu; \alpha)$ *and* $\mathbf{D}_{\mathrm{pc}}^{\mathrm{ab}}(6; b, \nu; \beta)$ *are isomorphic if and only if* $(a, \mu) = (b, \nu)$*.*

Proof. Similar to the cubic Abelian case using the following injection

$$I(K/\mathbb{Q}_3) = \langle \tau_2 \rangle \times \langle \tau_3 \rangle \hookrightarrow \mathrm{Aut}_{\mathbb{Q}_3}(D). \qquad \square$$

2.3.5 The dodecic case

If an elliptic curve $E/\mathbb{Q}_3$ has a semi-stability defect $\mathrm{dst}(E) = 12$, then its minimal field of good reduction has Galois closure $K/\mathbb{Q}_3$ satisfying:

$$\begin{cases} \mathrm{Gal}(K/\mathbb{Q}_3) \simeq \mathbb{Z}/3\mathbb{Z} \rtimes D_4 \simeq (\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z} \\ I(K/\mathbb{Q}_3) \simeq \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}. \end{cases}$$

More precisely:

$$\begin{cases} \mathrm{Gal}(K/\mathbb{Q}_3) = (\langle \tau_3 \rangle \rtimes \langle \tau_4 \rangle) \rtimes \langle \omega \rangle \\ I(K/\mathbb{Q}_3) = \mathrm{Gal}(K/K_0) = \langle \tau_3 \rangle \rtimes \langle \tau_4 \rangle \end{cases}$$

with relations:

$$\begin{cases} \mathrm{ord}(\tau_4) = 4 \wedge \mathrm{ord}(\tau_3) = 3 \wedge \mathrm{ord}(\omega) = 2 \\ \tau_4 \tau_3 \tau_4^{-1} = \tau_3^2 = \tau_3^{-1} \\ \omega \tau_4 \omega = \tau_4^{-1} \\ \tau_3 \omega = \omega \tau_3. \end{cases}$$

This follows from the structure of $\mathrm{Aut}_{\mathbb{F}_9}(\tilde{E})$, where $\tilde{E}$ is the special fibre of $E_K = E \times_{\mathbb{Q}_3} K$. Looking at [LMFDB] we see that there are exactly 10 such fields, namely:

- $K_1 = \mathbb{Q}_3(X^{12} + 3X^4 + 3)$,
- $K_2 = \mathbb{Q}_3(X^{12} - 3X^{11} - 3X^9 + 3X^7 - 3X^4 - 3)$,
- $K_3 = \mathbb{Q}_3(X^{12} + 3)$,
- $K_4 = \mathbb{Q}_3(X^{12} + 9X^{10} + 9X^9 - 9X^8 + 6X^6 + 9X^5 - 9X^4 - 3X^3 + 9X^2 - 9X - 12)$,
- $K_5 = \mathbb{Q}_3(X^{12} + 9X^{11} + 9X^{10} + 9X^9 + 9X^8 - 9X^7 - 12X^6 - 9X^2 - 3)$,
- $K_6 = \mathbb{Q}_3(X^{12} + 3X^{10} - 3X^9 - 3X^7 + 3X^6 + 3X^5 + 3X^4 + 3X^3 - 3)$,
- $K_7 = \mathbb{Q}_3(X^{12} - 3X^{11} - 3X^{10} + 3X^9 + 3X^5 - 3X^4 + 3X^3 + 3)$,

- $K_8 = \mathbb{Q}_3(X^{12} - 9X^{11} + 9X^9 - 9X^8 + 9X^7 - 12X^6 + 3X^3 + 9X^2 + 9X - 12)$,
- $K_9 = \mathbb{Q}_3(X^{12} + 9X^{11} + 9X^{10} - 3X^9 - 9X^8 - 9X^7 + 3X^6 + 9X^5 - 9X^4 + 6X^3 - 9X^2 - 9X + 12)$,
- $K_{10} = \mathbb{Q}_3(X^{12} - 9X^{11} + 6X^9 + 9X^8 + 3X^6 + 9X^5 + 9X^4 + 3X^3 - 9X^2 + 9X + 3)$.

Looking at the lattice of each such field extension over $\mathbb{Q}_3$, we observe that $K_i^{\mathrm{un}} = K_j^{\mathrm{un}}$ if and only if $i \equiv j \bmod 5\mathbb{Z}$, so that there are in fact 5 fields of good reduction. Furthermore, every one of these 5 fields appears as the reduction field of some elliptic curve (see [FK22, Thm. 17(7)]). For $i \in \{1, \dots, 5\}$, we let L_i be the maximal totally ramified sub-extension of K_i, so that $K_i = L_i(\zeta_4)$.

For $i \in \{1, \dots, 5\}$ and $\epsilon \in \{0, 1\}$ we consider

$$\mathcal{M}_{12}^{i,\epsilon} = \{x \in L_i \mid \tau_3(x) = (x + 3(-1)^{\epsilon+1})/(1 + x(-1)^{\epsilon}) \wedge \tau_4(x) = -x\}.$$

We let $K_0 = \mathbb{Q}_3(\zeta_4)$ be the maximal unramified extension of $\mathbb{Q}_3$ in K_i which is independent of i. For $\alpha \in \mathcal{M}_{12}^{i,\epsilon}$, we denote by $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \alpha)$ the filtered $(\varphi, \mathrm{Gal}(K_i/\mathbb{Q}_3))$-module defined by:

- $D = K_0 e_1 \oplus K_0 e_2$,
- $M_B(\tau_4) = \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix}$,
- $M_B(\tau_3) = \begin{pmatrix} -\frac{1}{2} & \frac{(-1)^{\epsilon+1}3}{2} \\ \frac{(-1)^{\epsilon}}{2} & -\frac{1}{2} \end{pmatrix}$,
- $\varphi(e_1) = e_2;\ \varphi(e_2) = -3e_1$,
- $\omega(e_1) = e_1;\ \omega(e_2) = e_2$,
- $\mathrm{Fil}^1 D_{K_i} = (\alpha \otimes e_1 + 1 \otimes e_2)K_i$.

The filtered $(\varphi, \mathrm{Gal}(K_i/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \alpha)$ satisfies conditions (1)–(4) and is admissible for each $i \in \{1, \dots, 5\}$, $\epsilon \in \{0, 1\}$ and $\alpha \in \mathcal{M}_{12}^{i,\epsilon}$.

Proposition 2.11. *Let $E/\mathbb{Q}_3$ be an elliptic curve with* $\mathrm{dst}(E) = 12$ *acquiring good reduction over K_i for some $i \in \{1, \ldots, 5\}$ and $\mathbf{D} = \mathbf{D}_{\mathrm{cris},K_i}(V_3(E))$. There exists $\epsilon \in \{0, 1\}$ and $\alpha \in \mathcal{M}_{12}^{i,\epsilon}$ such that $\mathbf{D}$ and $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \alpha)$ are isomorphic as filtered $(\varphi, \mathrm{Gal}(K_i/\mathbb{Q}_3))$-modules. Moreover, if ϵ, $\epsilon' \in \{0, 1\}$ and $\alpha \in \mathcal{M}_{12}^{i,\epsilon}$, $\beta \in \mathcal{M}_{12}^{i,\epsilon'}$, then $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}(12; i; \epsilon'; \beta)$ are isomorphic if and only if $(\alpha, \epsilon) = (\beta, \epsilon')$.*

Proof. Let D be the underlying K_0-vector space associated to $\mathbf{D}$. As usual, the inertia subgroup of $K_i/\mathbb{Q}_3$ injects in $\mathrm{Aut}_{K_0}(D)$ and we identify τ_4 and τ_3 to their respective image. As in the quartic case we show that there is a K_0-basis $B = (e_1, e_2)$ of D such that:

$$\begin{cases} M_B(\tau_4) = \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4^{-1} \end{pmatrix} \\ \omega(e_1) = e_1 \wedge \omega(e_2) = e_2 \\ \varphi(e_1) = e_2 \wedge \varphi(e_2) = -3e_1. \end{cases}$$

The relations between τ_3 and τ_4 as well as ω and φ implies that there is some $\epsilon' \in \{0, 1\}$ such that

$$M_B(\tau_3) = \begin{pmatrix} -\frac{1}{2} & \frac{3(-1)^{\epsilon'+1}}{2} \\ \frac{(-1)^{\epsilon'}}{2} & -\frac{1}{2} \end{pmatrix}.$$

A simple calculation shows that the K_i-lines of $D_{K_i} = K_i \otimes_{K_0} D$ stable by the action of $\mathrm{Gal}(K_i/\mathbb{Q}_3)$ are of the form

$$(\alpha \otimes e_1 + 1 \otimes e_2)K_i$$

with $\alpha \in L_i$ satisfying the desired conditions. Let ϵ, $\epsilon' \in \{0, 1\}$ and $\alpha \in \mathcal{M}_{12}^{i,\epsilon}, \beta \in \mathcal{M}_{12}^{i,\epsilon'}$. Looking only at their underlying $(\varphi, \mathrm{Gal}(K_i/\mathbb{Q}_3))$-modules, we see that $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon'; \beta)$ are isomorphic if and only if $\epsilon = \epsilon'$. Now supposing $\epsilon = \epsilon'$ and adding the filtration, we check that a morphism between $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon; \beta)$ must be of the form $\lambda\,\mathrm{id}$ with $\lambda \in \mathbb{Q}_3^\times$, so that necessarily $\alpha = \beta$. □

Remark 2.7. As in the cubic Abelian case, observe that $\mathbf{D}_{\mathrm{pc}}(12; 0; i; 0; \alpha)$ and $\mathbf{D}_{\mathrm{pc}}(12; 0; i; 1; \alpha)$ are unramified quadratic twists of each other as unfiltered $(\varphi, \mathrm{Gal}(K_i/\mathbb{Q}_3))$-modules.

2.4 Elliptic curves with given Tate module

In this section, we show that every elliptic curve E defined over $\mathbb{Q}_3$ with good reduction over K minimal has its associated filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module isomorphic to an object of the list 2.1.

Proposition 2.12. *Every (unfiltered)* $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$*-module appearing in Table 2.1 comes from a minimal Galois pair for* $K/\mathbb{Q}_3$*.*

Proof. We only treat the wild cases that are not quadratic twists, i.e. the cubic and dodecic ones. Let us denote $\tilde{E} = \tilde{E}_0 \times_{\mathbb{F}_3} \mathbb{F}_9$. The minimal Galois pairs are given in Table 2.2. It is not hard to see that ν is injective and that the field of definition of Γ is minimal. Each of these objects gives rise to a $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module which is necessarily in our list by construction (they have the right Frobenius and Galois action). Except for the non abelian cubic case, there are always two isomorphisms classes of $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-modules in our list (see Section 2.3). We have only checked that one of them comes from a Galois pair, but in fact both do since they are unramified quadratic twists of each other. As an illustration, we give an explicit construction of the obtained $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module for $e = 12$ and $K = K_i$ for some $i \in \{1, \ldots, 5\}$. Denote by D_0 (resp. D) the Dieudonné module of the 3-divisible group associated to $\tilde{E}_0$ (resp. $\tilde{E}$). Since $P_{\mathrm{Char}}(\varphi_0) = P_{\mathrm{Char}}(\mathrm{Frob}_{A_0})$, our φ-module D has the right trace $a_3 = 0$. The action of $\mathrm{Gal}(K/\mathbb{Q}_3)$ on D is obtained via the antimorphism ν:

$$\begin{tikzcd}
\mathrm{Gal}(K/\mathbb{Q}_3) \arrow[r, hook, "\nu"] \arrow[dr, hook, "\nu'"'] & \mathrm{Aut}_{\mathbb{F}_9}(\tilde{E}) \rtimes \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3) \arrow[d, hook, "\text{Dieudonné functor}"] \\
& \mathrm{Aut}_{\varphi}(D) \rtimes \mathrm{Gal}(K_0/\mathbb{Q}_3)
\end{tikzcd}$$

We have $\nu'(I(K/\mathbb{Q}_3)) \subseteq \mathrm{Aut}_\varphi(D)$, so that the action of $I(K/\mathbb{Q}_3) = \langle\tau_3\rangle \rtimes \langle\tau_4\rangle$ is K_0-linear, while the action of ω is σ-semilinear. It is then obvious to see that D is isomorphic to $\mathbf{D}_{\mathrm{pc}}(12; 0; i; \epsilon)$ as a $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module for some $\epsilon \in \{0, 1\}$. $\square$

Remark 2.8. When $a_3(\tilde{E}_0) = \pm 3$, a Galois pair for $K = L^{\mathrm{na}}(\zeta_4)/\mathbb{Q}_3$ is never minimal because $\mathrm{Aut}_{\mathbb{F}_9}(\tilde{E})$ is too small compared to $\mathrm{Gal}(K/\mathbb{Q}_3)$. It is another way to see why those traces are absent from our list in that case.

Table 2.2: Minimal Galois pairs for $e = 3, 12$.

e	K	Tr.	Min. Gal. pair	$(\varphi, \mathcal{G}_{\mathbb{Q}_3})$-mod.
3	$L^{\mathrm{na}}(\zeta_4)$	0	$\tilde{E}_0 : y^2 = x^3 + x$ $\Gamma = \langle\tau\rangle$, 3-Syl. of $\mathrm{Aut}_{\mathbb{F}_9}(\tilde{E})$ $\nu : \tau_3 \mapsto \tau, \omega \mapsto f_\sigma$	$\mathbf{D}^{\mathrm{na}}_{\mathrm{pc}}(3; 0)$
	L^{ab}	-3	$\tilde{E}_0 : y^2 = x^3 - x + 1$ $\Gamma = \langle\tau\rangle$, 3-Syl. of $\mathrm{Aut}_{\mathbb{F}_3}(\tilde{E}_0)$ $\nu : \tau_3 \mapsto \tau$	$\mathbf{D}^{\mathrm{ab}}_{\mathrm{pc}}(3; -3)$
		0	$\tilde{E}_0 : y^2 = x^3 - x$ $\Gamma = \langle\tau\rangle$, 3-Syl. of $\mathrm{Aut}_{\mathbb{F}_3}(\tilde{E}_0)$ $\nu : \tau_3 \mapsto \tau$	$\mathbf{D}^{\mathrm{ab}}_{\mathrm{pc}}(3; 0)$
		3	$\tilde{E}_0 : y^2 = x^3 - x - 1$ $\Gamma = \langle\tau\rangle$, 3-Syl. of $\mathrm{Aut}_{\mathbb{F}_3}(\tilde{E}_0)$ $\nu : \tau_3 \mapsto \tau$	$\mathbf{D}^{\mathrm{ab}}_{\mathrm{pc}}(3; 3)$
12	K_i	0	$\tilde{E}_0 : y^2 = x^3 + x$ $\Gamma = \mathrm{Aut}_{\mathbb{F}_9}(\tilde{E})$ $\nu : \omega \mapsto f_\sigma$	$\mathbf{D}_{\mathrm{pc}}(12; 0; i; 1)$

2.4.1 A complete classification

To every 3-adic potentially crystalline representation V of $\mathcal{G}_{\mathbb{Q}_3}$ corresponds an admissible filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$-module $\mathbf{D}_{\mathrm{cris},K}(V)$. This association is functorial in a fully faithful way (see Theorem 1.4). In this section, we will show that every object described in Table 2.1 comes from an elliptic curve over $\mathbb{Q}_3$ with potentially good reduction. It turns out that we can use the same tools and ingredients as M. Volkov in her treatment of the tame case (see [Vol05]).

Theorem 2.1. *Let* $\mathbf{D}$ *be one of the filtered* $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_3))$*-module in Table 2.1 There exists an elliptic curve* $E/\mathbb{Q}_3$ *such that* $\mathbf{D} \simeq \mathbf{D}_{\mathrm{cris},K}(V_3(E))$.

Proof. We sketch the proof, following the arguments of [Vol05, Thm. 5.7]. The φ_0-module D_0 comes from an elliptic curve $\tilde{E}_0/\mathbb{F}_3$ with the right Frobenius (via the Dieudonné module of its 3-divisible group). Let $\tilde{E} = \tilde{E}_0 \times_{\mathbb{F}_3} k$. Since $\mathbf{D} = \mathbf{D}_{\mathrm{cris},K}(V)$ for some crystalline representation V of $\mathrm{Gal}(\overline{\mathbb{Q}_3}/K)$ with Hodge–Tate weights $(0,1)$, there exists a 3-divisible group $G/\mathcal{O}_K$ lifting $\tilde{E}(p)/k$ with Tate module isomorphic to V (see [Bre00, Thm. 5.3.2]). By the Serre–Tate Theorem, the triple $(G, \tilde{E}, \tilde{G} \xrightarrow{\sim} \tilde{E}(p))$ determines an elliptic curve E/K with good reduction (i.e. an elliptic scheme over $\mathcal{O}_K$) such that $V_3(E) \simeq V$. Finally, a minimal Galois pair $(\tilde{E}_0, \Gamma, \nu)$ for $K/\mathbb{Q}_3$ (which always exists in the tame case by [Vol05, Thm. 4.11] and in the wild case by Proposition 2.12) furnishes the necessary descent datum to obtain an elliptic curve $E_0/\mathbb{Q}_3$ such that $E = E_0 \times_{\mathbb{Q}_3} K$ and $V \simeq V_3(E_0)$. □

CHAPTER 3

Inertia subgroups associated to Abelian surfaces over $\mathbb{Q}_p$

3.1 Introduction

3.1.1 Context

Let $F/\mathbb{Q}_p$ be a finite extension and $\mathcal{A}/F$ a d-dimensional Abelian variety with potentially good reduction. Recall that for ℓ a prime

$$\rho_{\ell,\mathcal{A}} : \mathcal{G}_F \longrightarrow \mathrm{GL}_{2d}(\mathbb{Q}_\ell)$$

is the ℓ-adic representation associated to $\mathcal{A}$. When $\ell \neq p$, the image $\rho_{\ell,\mathcal{A}}(\mathcal{I}_F)$ is a finite group, independent of the prime ℓ. We call this group the *inertia subgroup* associated to $\mathcal{A}$. It corresponds to the inertia subgroup of a finite Galois extension of F of minimal degree over which $\mathcal{A}$ acquires good reduction. We have seen in Chapter 2 that in the wild cases, the possibilities for such fields are multiple. For instance, when $E/\mathbb{Q}_3$ is an elliptic curve with semi-stability defect $\mathrm{dst}(E) = 3$, we have two fields with distinct Galois groups (but same inertia). When $\mathrm{dst}(E) = 12$, their are five possible fields with the same Galois structure (see Table 2.1).

Our motivation is to determine the minimal fields of good reduction for Abelian surfaces over $\mathbb{Q}_p$ ($p = 3, 5$). We consider this to be the first step towards a classification of the p-adic representations of the absolute Galois group $\mathcal{G}$ of $\mathbb{Q}_p$, arising from such surfaces. Let us assume that we know which extensions can occur, as well as the structure of their respective Galois groups. Then, we could compute all the admissible filtered $(\varphi, \mathcal{G})$-modules satisfying the geometric conditions introduced in Chapter 1. That is, the conditions that are known to be satisfied by Tate modules of Abelian varieties with potentially good reduction. Although this is not necessarily sufficient to obtain a complete classification, it will nevertheless reduce the possibilities to a selected list of potential candidates. With such a list, we could then verify which representation actually comes from a surface by using the tools developed in [Vol05]. In the case of elliptic curves over $\mathbb{Q}_3$, this verification was done by constructing minimal Galois pairs explicitely. Before considering such a classification, we first want to determine the Galois structure of such extensions.

In their paper *"Inertia groups and Abelian surfaces"*, A. Silverberg and Yu. G. Zarhin (see [SZ05]) classify all the possible finite groups that occur as inertia subgroups associated to Abelian surfaces. These groups are defined over a discretely valued field with perfect residue field. These results can be seen as a continuation of the results of J.-P. Serre on elliptic curves (see [Ser71]). Though the classification is complete, most of the groups associated to potential good reduction are realised over the maximal unramified extension of a local field of characteristic $p > 0$.

The goal of this chapter is to extend the results from [SZ05] to the case of potential good reduction of Abelian surfaces defined over a local field of mixed characteristic. We do not merely wish to realise each group over the maximal unramified extension F^{un} of F. We would like to find the smallest finite extension of $\mathbb{Q}_p$ over which each group can be achieved. Our focus will be directed towards $p = 3$ and 5, the only cases (with $p = 2$) where wild ramification occurs.

In the second section, we show that each group associated to potential good reduction in [SZ05] can be realised as the inertia subgroup of a finite extension of $\mathbb{Q}_p$. In a sense, an answer to the totally ramified inverse Galois problem over an unramified extension of $\mathbb{Q}_p$, for some specific finite groups. In the third section, we construct minimal *polarised* Galois pairs for each such extension. When $p = 3$ the Abelian surface over $\mathbb{F}_3$ is the twofold product of a supersingular elliptic curve over $\mathbb{F}_3$. When $p = 5$ it is the Jacobian of a genus 2 hyperelliptic curve over $\mathbb{F}_5$. In the last section, we combine the results of Section 3.3 with a recent result of S. Philip to ensure the existence of a Abelian surfaces $\mathcal{A}$ over a p-adic field with prescribed inertia subgroup. As a consequence, every inertia subgroup associated to potential good reduction in [SZ05] can be realised in mixed characteristic.

3.1.2 Notations

Keeping the notations from [SZ05], we let T_{12} be the unique nontrivial semidirect product of $\mathbb{Z}/3\mathbb{Z}$ by $\mathbb{Z}/4\mathbb{Z}$ and H_{24} be the unique nontrivial semidirect product of $\mathbb{Z}/3\mathbb{Z}$ by $\mathbb{Z}/8\mathbb{Z}$. Let $\rho_{36} : \mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}((\mathbb{Z}/3\mathbb{Z})^2)$, the group homomorphism defined by sending a generator of $\mathbb{Z}/4\mathbb{Z}$ to the inverse morphism. Let $\rho_{72} : \mathbb{Z}/8\mathbb{Z} \to \mathrm{Aut}((\mathbb{Z}/3\mathbb{Z})^2)$ defined by sending a generator of $\mathbb{Z}/8\mathbb{Z}$ to an element of order 4. We let

$$H_{36} = (\mathbb{Z}/3\mathbb{Z})^2 \rtimes_{\rho_{36}} \mathbb{Z}/4\mathbb{Z},$$

and

$$H_{72} = (\mathbb{Z}/3\mathbb{Z})^2 \rtimes_{\rho_{72}} \mathbb{Z}/8\mathbb{Z}.$$

Let $\rho_{20} : \mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$ be the group homomorphism defined by sending a generator of $\mathbb{Z}/4\mathbb{Z}$ to an element of order 2. Let $\rho_{40} : \mathbb{Z}/8\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$ be the group homomorphism defined by sending a generator of $\mathbb{Z}/8\mathbb{Z}$ to an element of order 4. We let

$$H_{20} = \mathbb{Z}/5\mathbb{Z} \rtimes_{\rho_{20}} \mathbb{Z}/4\mathbb{Z},$$

and

$$H_{40} = \mathbb{Z}/5\mathbb{Z} \rtimes_{\rho_{40}} \mathbb{Z}/8\mathbb{Z}.$$

Note that H_{36} is not a subgroup of H_{72} but H_{20} is a normal subgroup of H_{40}. We consider the following lists ([SZ05, Def. 1.2]):

$$\begin{aligned}
\Sigma(0,2) &= \{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/10\mathbb{Z}, \mathbb{Z}/12\mathbb{Z}\}\,, \\
\Sigma_3(0,2) &= \Sigma(0,2) \cup \{\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}, S_3, \mathbb{Z}/3\mathbb{Z}\times S_3, T_{12}, \\
&\qquad \mathbb{Z}/3\mathbb{Z}\times T_{12}, H_{24}, H_{36}, H_{72}\}\,, \\
\Sigma_5(0,2) &= \Sigma(0,2) \cup \{H_{20}, H_{40}\}\,.
\end{aligned}$$

The first one corresponds to all the possible inertia subgroups arising from Abelian surfaces with potentially good reduction defined over a valued field of residue characteristic $p > 5$, *i.e.* the tame cases. The second and third ones correspond to the cases of residue characteristic 3 and 5 respectively, which are the ones we are interested in.

We denote by F_5 the unique semidirect product of $\mathbb{Z}/5\mathbb{Z}$ by $\mathbb{Z}/4\mathbb{Z}$ where $\mathbb{Z}/4\mathbb{Z}$ acts faithfully on $\mathbb{Z}/5\mathbb{Z}$. We let $\mathrm{M}_4(2)$ be the unique semidirect product of $\mathbb{Z}/8\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$ that contains two cyclic subgroups of order 8. While not being associated to surfaces themselves, these two groups will appear in our constructions.

3.2 Totally ramified inverse Galois problem

There are five groups that particularly stand out in the lists defined above, namely the groups H_{24}, H_{36} and H_{72} in $\Sigma_3(0,2)$; H_{20} and H_{40} in $\Sigma_5(0,2)$. It is not immediatly obvious that they all can be realised as the inertia subgroup of a finite extension of $\mathbb{Q}_3$ and $\mathbb{Q}_5$ respectively. The main result of this section is a positive answer to the inverse Galois problem for each such group in mixed characteristic.

In the following, we will often refer to *The L-functions and modular forms database* [LMFDB]. Their database of p-adic fields has been very useful to us, mostly to shorten the proofs and limit unnecessary calculations. We will also refer to *GroupNames* [GN] when basic facts about finite groups are needed.

3.2.1 Inertia groups over $\mathbb{Q}_3$

This section is dedicated to the construction of finite extensions of $\mathbb{Q}_3$ with respective inertia subgroups H_{24}, H_{36} and H_{72}. We begin with H_{24}.

Lemma 3.1. *There exists a finite Galois extension $K_{24}/\mathbb{Q}_3$ such that*

$$H_{24} \simeq I(K_{24}/\mathbb{Q}_3).$$

Proof. We see in the [LMFDB] that there is no totally ramified Galois extension of $\mathbb{Q}_3$ with Galois group isomorphic to $\mathbb{Z}/8\mathbb{Z}$. Fortunately, the cyclic group of order 8 can be realised as the inertia subgroup of an extension of $\mathbb{Q}_3$. Let $f_1(X) = X^8 + 3$ and K_1 be the field of decomposition of f_1 over $\mathbb{Q}_3$. The extension $K_1/\mathbb{Q}_3$ is of degree 16 with inertia subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z}$. Let $f_2(X) = X^3 + 3$ and K_2 be the field of decomposition of f_2 over $\mathbb{Q}_3$. The extension $K_2/\mathbb{Q}_3$ is totally ramified with Galois group isomorphic to S_3. Furthermore, we have $[K_1 \cap K_2 : \mathbb{Q}_3] = 2$. Let $K_{24} = K_1K_2$ be the compositum of K_1 and K_2. It is a degree 24 extension of $\mathbb{Q}_9$ with inertia subgroup having among its quotients S_3 and $\mathbb{Z}/8\mathbb{Z}$. This property characterises H_{24} among the groups of order 24 (see [GN]). We illustrate this construction in Figure 3.1.

Note that H_{24} can also be realised by taking $f_1(X) = X^8 + 6$ and $f_2(X) = X^3 + 6$. While this choice does not have an impact on the inertia subgroup, it gives rise to another Galois group. This construction shows that H_{24} cannot be the Galois group of a totally ramified extension of $\mathbb{Q}_3$. Otherwise, the cyclic group of order 8 would be the Galois group of a totally ramified extension of $\mathbb{Q}_3$. □

Before we start our calculations for H_{36}, consider the following commutative diagram

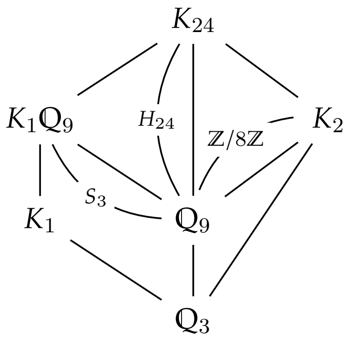

Figure 3.1: Partial lattice of K_{24} over $\mathbb{Q}_3$.

$$\begin{tikzcd}
& & \mathbb{Z}/4\mathbb{Z} \arrow[r, "\rho_{36}"] \arrow[d, two heads] \arrow[dll, hook'] & \mathrm{Aut}((\mathbb{Z}/3\mathbb{Z})^2) \\
D_4 \arrow[r, two heads] & D_4/(\mathbb{Z}/2\mathbb{Z})^2 \arrow[r, "\simeq"] & \mathbb{Z}/2\mathbb{Z} \arrow[ur, hook] &
\end{tikzcd}$$

It induces an action of D_4 on $(\mathbb{Z}/3\mathbb{Z})^2$ that satisfies

$$H_{36} \lhd (\mathbb{Z}/3\mathbb{Z})^2 \rtimes D_4.$$

We can now proceed.

Lemma 3.2. *There exists a finite Galois extension* $K_{36}/\mathbb{Q}_3$ *such that*

$$H_{36} \simeq I(K_{36}/\mathbb{Q}_3).$$

Furthermore, we have an isomorphism

$$\mathrm{Gal}(K_{36}/\mathbb{Q}_3) \simeq (\mathbb{Z}/3\mathbb{Z})^2 \rtimes D_4.$$

Proof. Let $f_1(X) = X^9 + 3X^3 + 9X^2 + 9X + 3$ and K_1 be the field of decomposition of f_1 over $\mathbb{Q}_3$. The extension $K_1/\mathbb{Q}_3$ is totally ramified of degree 18 with Galois group isomorphic $\mathbb{Z}/3\mathbb{Z} \rtimes S_3$ (the unique nontrivial semidirect product of $\mathbb{Z}/3\mathbb{Z}$ by S_3). Let $f_2(X) = X^4 + 3$ and K_2 be the field of decomposition of f_2 over $\mathbb{Q}_3$. The extension $K_2/\mathbb{Q}_3$ is of degree 8 with Galois group D_4 and inertia subgroup $\mathbb{Z}/4\mathbb{Z}$. Since there are only 3 extensions of $\mathbb{Q}_3$ of degree 2 (see [LMFDB]), we necessarily have $[K_1 \cap K_2 : \mathbb{Q}_3] = 2$. Let $K_{36} = K_1K_2$ be the compositum of K_1 and K_2. The inertia subgroup $I(K_{36}/\mathbb{Q}_3)$ is of order 36 having among its quotients $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z} \rtimes S_3$, which characterises H_{36} (see [GN]). Since the cyclic group of order 4 cannot be realised as the Galois group of a totally ramified extension of $\mathbb{Q}_3$, neither can H_{36}. We can describe the Galois group $\mathrm{Gal}(K_{36}/\mathbb{Q}_3)$. It is the only group of order 72 having among its quotients D_4 and $\mathbb{Z}/3\mathbb{Z} \rtimes S_3$, and H_{36} as a subgroup, namely:

$$(\mathbb{Z}/3\mathbb{Z})^2 \rtimes D_4.$$

Let L_{36} denotes one of the ramified degree 2 extension of K_1. It corresponds to one of the two non normal subgroups of order 2 of $\mathrm{Gal}(K_{36}/\mathbb{Q}_3)$ inside $\mathrm{Gal}(K_{36}/K_1) \simeq (\mathbb{Z}/2\mathbb{Z})^2$. We illustrate our constructions in Figure 3.2.

Furthermore, the following exact sequence

$$\begin{tikzcd}
 & & & \mathrm{Gal}(K_{36}/L_{36}) \arrow[dl, dashed, bend right] \arrow[d, equal] & \\
1 \arrow[r] & I(K_{36}/\mathbb{Q}_3) \arrow[r] & \mathrm{Gal}(K_{36}/\mathbb{Q}_3) \arrow[r] & \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3) \arrow[r] & 1,
\end{tikzcd}$$

splits, *i.e.*, we have $\mathrm{Gal}(K_{36}/\mathbb{Q}_3) \simeq I(K_{36}/\mathbb{Q}_3) \rtimes \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$.

□

The last group of this section is H_{72}.

Lemma 3.3. *There exists a finite Galois extension* $K_{72}/\mathbb{Q}_3$ *such that*

$$H_{72} \simeq I(K_{72}/\mathbb{Q}_3).$$

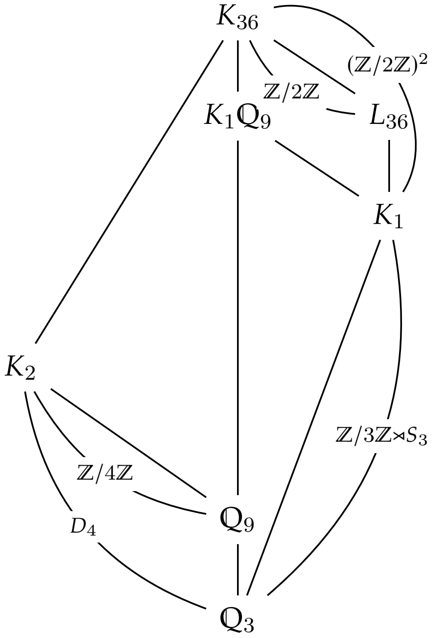

Figure 3.2: Partial lattice of K_{36} over $\mathbb{Q}_3$.

Proof. Consider the situation illustrated by the Figure 3.3. The existence of K_1 and K_2 is provided by the [LMFDB]. Indeed, let $f_1(X) = X^6 + 3X + 3$ and $f_2(X) = X^8 + 3$. We can then take K_1 (resp. K_2) to be the field of decomposition of f_1 (resp. f_2) over $\mathbb{Q}_3$. Both K_1 and K_2 contain the three extensions of degree 2 of $\mathbb{Q}_3$. Indeed, their respective Galois group have exactly 3 normal subgroups of index 2. More precisely, these three extensions sit in the unique degree 8 extension of $\mathbb{Q}_3$ with Galois group D_4, which is a quotient of both $\mathrm{Gal}(K_1/\mathbb{Q}_3)$ and $\mathrm{Gal}(K_2/\mathbb{Q}_3)$. This means that $[K_1 \cap K_2 : \mathbb{Q}_3] = 8$. As a result, $I(K_{72}/\mathbb{Q}_3)$ is of order 72 with quotients $(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$. This characterises H_{72} among the other order 72 groups and gives us the desired isomorphism. $\square$

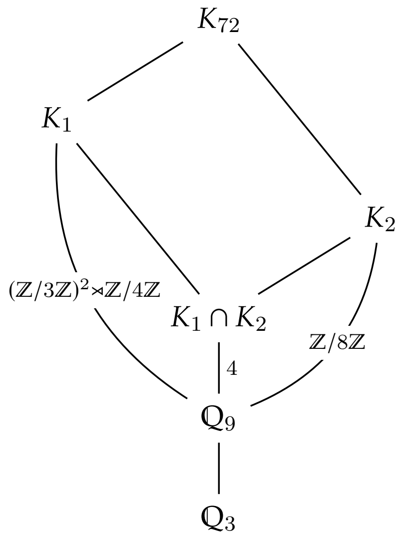

Figure 3.3: Partial lattice of K_{72} over $\mathbb{Q}_3$.

Remark 3.1. Note that we have described the Galois group of K_{36} over $\mathbb{Q}_3$, but not the ones of K_{24} and K_{72}. Even though we could, it will not be needed for the remainder of this chapter.

3.2.2 Inertia groups over $\mathbb{Q}_5$

This section is dedicated to the construction of extensions of $\mathbb{Q}_5$ with respective inertia subgroups H_{20} and H_{40}.

Before we begin, consider the following commutative diagram

$$\begin{tikzcd}
 & & \mathbb{Z}/8\mathbb{Z} \arrow[r, two heads] \arrow[lld] \arrow[d, two heads] & \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z}) \\
\mathrm{M}_4(2) \arrow[r, two heads] & \mathrm{M}_4(2)/(\mathbb{Z}/2\mathbb{Z})^2 \arrow[r, "\simeq"] & \mathbb{Z}/4\mathbb{Z} \arrow[ru] &
\end{tikzcd}$$

It induces an action of $\mathrm{M}_4(2)$ on $\mathbb{Z}/5\mathbb{Z}$ that satisfies

$$H_{40} \lhd \mathbb{Z}/5\mathbb{Z} \rtimes \mathrm{M}_4(2).$$

We can now proceed with our calculations.

Lemma 3.4. *Let* $e \in \{20, 40\}$. *There exists a finite Galois extension* $K_e/\mathbb{Q}_5$ *such that*

$$H_e \simeq I(K_e/\mathbb{Q}_5).$$

Furthermore, we can choose K_{40} *in such a way that*

$$\mathrm{Gal}(K_{40}/\mathbb{Q}_5) \simeq \mathbb{Z}/5\mathbb{Z} \rtimes \mathrm{M}_4(2).$$

Proof. Let K_{20} be the field of decompostion of $X^{20} + 15X^8 + 5$ over $\mathbb{Q}_5$. We have isomorphisms

$$\mathrm{Gal}(K_{20}/\mathbb{Q}_5) \simeq I(K_{20}/\mathbb{Q}_5) \simeq H_{20}.$$

The group H_{20} is the only one (among the other four) realised as the Galois group of a totally ramified extension. It is also the only one that can be found explicitely in the [LMFDB], under the label 5.1.20.27a1.1. Just like in residue characteristic 3, the cyclic group of order 8 is never the Galois group of a totally ramified extension of $\mathbb{Q}_5$. Consider the situation illustrated by the figure 3.4. The existence of K_1 and K_2 is once again provided by the [LMFDB]. For instance, let $f_1(X) = X^8 + 5$ and $f_2(X) = X^5 + 5$. Denote by K_1 (resp. K_2) the field of decomposition of f_1 (resp. f_2) over $\mathbb{Q}_5$. The extension $K_1/\mathbb{Q}_5$ is of order 16, with inertia subgroup $\mathbb{Z}/8\mathbb{Z}$. The extension $K_2/\mathbb{Q}_5$ is totally ramified with Galois group isomorphic to F_5. As H_{40} is characterised by the fact that it is the only group of order 40 having $\mathbb{Z}/8\mathbb{Z}$ and F_5 among its quotients, we necessarily have

$$H_{40} \simeq I(K_{40}/\mathbb{Q}_5).$$

The structure of the Galois group will be needed for Section 3.3. It turns out that, of the two possible groups of order 80 that could occur in this

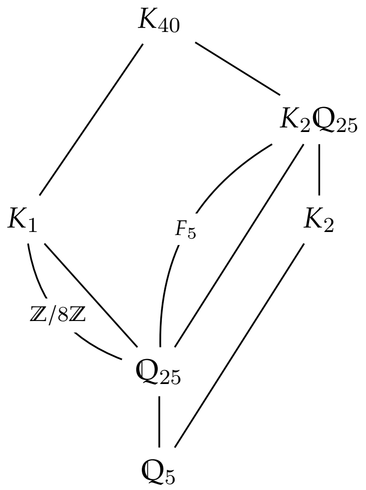

Figure 3.4: Partial lattice of K_{40} over $\mathbb{Q}_5$

situation, both do. We can chose K_1 and K_2 such that the unique degree 4 extension of K_2 is not contained in the unique degree 8 Galois subextension of K_1, and in that case: $\mathrm{Gal}(K_{40}/K_2) \simeq (\mathbb{Z}/2\mathbb{Z})^2$. In the other case, we have $\mathrm{Gal}(K_{40}/K_2) \simeq \mathbb{Z}/4\mathbb{Z}$. From now on we only consider the first case. Since $\mathbb{Z}/5\mathbb{Z} \rtimes \mathrm{M}_4(2)$ is the only group of order 80 having H_{40} and $(\mathbb{Z}/2\mathbb{Z})^2$ as normal subgroups, we have our desired isomorphism. We give another description of the Galois group of K_{40} over $\mathbb{Q}_5$. Let L_{40} be a totally ramified extension of degree 40 of $\mathbb{Q}_5$, corresponding to one of the two non normal subgroups of order 2 of $\mathrm{Gal}(K_{40}/\mathbb{Q}_5)$ inside $\mathrm{Gal}(K_{40}/K_2)$. We have the following split exact sequence

$$\begin{tikzcd}
 & & & \mathrm{Gal}(K_{40}/L_{40}) \arrow[dl, dashed, bend right] \arrow[d, equal] & \\
1 \arrow[r] & I(K_{40}/\mathbb{Q}_5) \arrow[r] & \mathrm{Gal}(K_{40}/\mathbb{Q}_5) \arrow[r] & \mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5) \arrow[r] & 1.
\end{tikzcd}$$

□

We are now able to prove the main result of this section.

Proposition 3.1. *Let* $p \in \{3,5\}$*. For every* $H \in \Sigma_p(0,2)$*, there exists* $K_H/\mathbb{Q}_p$ *such that*

$$I(K_H/\mathbb{Q}_p) \simeq H.$$

Proof. The cases of H_e for $e \in \{24,36,72,20,40\}$ have all been treated individually. To be completely exhaustive, we briefly discuss the other wild cases. The field of decomposition of X^5+20X^4+5 (resp. $X^{10}+20X^8+5$) over $\mathbb{Q}_5$ is totally ramified with Galois group $\mathbb{Z}/5\mathbb{Z}$ (resp. $\mathbb{Z}/10\mathbb{Z}$). The permutation group S_3 is realised as the Galois group of the field of decomposition of X^3+3 over $\mathbb{Q}_3$, which is totally ramified. The remaining cases are cartesian products of previously achieved inertia subgroups associated to elliptic curves over $\mathbb{Q}_3$ (see Sections 2.3.3, 2.3.4 and 2.3.5). For example, the group $\mathbb{Z}/3\mathbb{Z}\times T_{12}$ is realised as the inertia subgroup of the compositum of $\mathbb{Q}_3(X^3-3X^2+6)$ and $\mathbb{Q}_3(X^{12}+3)$. □

3.3 Minimal polarised Galois pairs

Let $f \geq 1$ be an integer. Let $K/\mathbb{Q}_{p^f}$ be a finite Galois extension with residue field $\kappa/\mathbb{F}_{p^f}$. Further assume that $\mathrm{Gal}(K/\mathbb{Q}_{p^f}) \simeq I(K/\mathbb{Q}_{p^f}) \rtimes \mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$. Recall that a polarised Galois pair for $K/\mathbb{Q}_{p^f}$ consists of a polarised Abelian variety (A_0,Λ_0) over $\mathbb{F}_{p^f}$ and a $\mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$-equivariant morphism

$$\nu : I(K/\mathbb{Q}_p) \longrightarrow \mathrm{Aut}_\kappa(A,\Lambda)$$

where $A = A_0 \times_{\mathbb{F}_p} \kappa$ and $\Lambda = \Lambda_0 \times_{\mathbb{F}_p} \kappa$ are the scalar extensions of A_0 and Λ_0 to κ. Let Γ be the image of the morphism ν. We say that a Galois pair is minimal if ν is injective and $\kappa/\mathbb{F}_{p^f}$ is minimal for Γ, *i.e.*, there is no proper subextension $\kappa'/\mathbb{F}_{p^f}$ such that $\Gamma \subset \mathrm{Aut}_{\kappa'}(A,\Lambda)$.

Let $p \in \{3,5\}$, $H \in \Sigma_p(0,2)$ and $K_H/\mathbb{Q}_p$ with inertia subgroup H (which we know exists thanks to Proposition 3.1). The goal of this section is to construct a minimal polarised Galois pair for each extension $K_H/\mathbb{Q}_{p^f}$ (with $f \geq 1$ depending on H). We begin by proving a result on polarisations, due to Hwang in [Hwa21], which will frequently be used in what follows.

Lemma 3.5. *Let Γ be a finite stable subgroup of $\mathrm{Aut}_\kappa(A)$ for the action of $\mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$. There exists a polarisation Λ_0 on A_0 such that*

$$\Gamma \subset \mathrm{Aut}_\kappa(A, \Lambda_0 \times_{\mathbb{F}_{p^f}} \kappa).$$

Proof. Let λ_0 be any polarisation on A_0. The scalar extension $\lambda = \lambda_0 \times_{\mathbb{F}_{p^f}} \kappa$ is a polarisation on A. Define

$$\Lambda = \sum_{\gamma \in \Gamma} \gamma^\vee \lambda \gamma,$$

it is also a polarisation on A since Γ is finite. The inclusion $\Gamma \subset \mathrm{Aut}_\kappa(A, \Lambda)$ then holds by construction. Let σ be a generator $\mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$. Since Γ is stable, the element σ only swaps the terms of the sum that defines Λ, and we have $\Lambda^\sigma = \Lambda$. This means that Λ is defined over $\mathbb{F}_{p^f}$, *i.e.*, there exists a polarisation Λ_0 on A_0 such that $\Lambda = \Lambda_0 \times_{\mathbb{F}_{p^f}} \kappa$. □

3.3.1 Automorphisms over $\mathbb{F}_3$

In this section we provide minimal polarised Galois pairs for $p = 3$. We begin with $K_{24}/\mathbb{Q}_3$.

Lemma 3.6. *There exists an Abelian surface $A_0/\mathbb{F}_3$ and a polarisation Λ on $A = A_0 \times_{\mathbb{F}_3} \mathbb{F}_9$, defined over $\mathbb{F}_9$ such that*

$$H_{24} \hookrightarrow \mathrm{Aut}_{\mathbb{F}_9}(A, \Lambda).$$

Proof. Consider the following elliptic curve over $\mathbb{F}_3$

$$E_0 : y^2 = x^3 - x.$$

Let $A_0 = E_0^2$, $E = E_0 \times_{\mathbb{F}_3} \mathbb{F}_9$ and $A = E^2 = A_0 \times_{\mathbb{F}_3} \mathbb{F}_9$. We have $\mathrm{Aut}_{\mathbb{F}_9}(E) \simeq \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$, let $u \in \mathrm{Aut}_{\mathbb{F}_9}(E)$ of order 4. Then, we can see $\mathrm{GL}_2(\mathbb{Z}[u])$ as a subgroup of $\mathrm{Aut}_{\mathbb{F}_9}(A)$. The group

$$\Gamma_{24} = \left\langle \begin{pmatrix} -1 & u \\ u+1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \right\rangle$$

is a subgroup of $\mathrm{Aut}_{\mathbb{F}_9}(A)$ isomorphic to H_{24}. By Lemma 3.5, there exists a polarisation Λ on A satisfying $\Gamma_{24} \subset \mathrm{Aut}_{\mathbb{F}_9}(A, \Lambda)$. Recall that the Frobenius σ of $\mathbb{F}_9$ acts on u via $u^\sigma = u^{-1}$. Thus, the group Γ_{24} is not stable by the action of $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$, in particular, the polarisation Λ is not necessarily defined over $\mathbb{F}_3$. Note that $\mathbb{F}_9$ is minimal, since $\mathrm{Aut}_{\mathbb{F}_3}(A_0) \simeq \mathrm{GL}_2(\mathbb{Z}[v])$, where $v \in \mathrm{Aut}_{\mathbb{F}_3}(E_0)$ is of order 3. □

At the moment of writing, I haven't been able to construct a stable copy of H_{24} inside the automorphism group of an Abelian surface over $\mathbb{F}_3$. The problem comes from the element of order 8. The one considered in Lemma 3.6 generates with its Galois conjugate an infinite subgroup. While, for example, the "archetypal" one:

$$\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix},$$

generates a subgroup of order 16.

In that setting, these are in fact the only two phenomena that can occur. An element of order 8 in $\mathrm{GL}_2(\mathbb{Z}[u])$ either generates an infinite group or a finite group of order 16 with its conjugate for the action of $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$. This means that it is not possible to find a stable copy of H_{24} in $\mathrm{GL}_2(\mathbb{Z}[u])$. We prove this affirmation.

Lemma 3.7. *Let $E/\mathbb{F}_9$ be as in Lemma 3.6. Let $u \in \mathrm{Aut}_{\mathbb{F}_9}(E)$ be any element of order 4. There exists no stable copy of H_{24} in $\mathrm{GL}_2(\mathbb{Z}[u])$ for the action of $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$.*

Proof. Let $U \in \mathrm{GL}_2(\mathbb{Z}[u])$ of order 8 and assume $\det(U) = u$. Let $\sigma \in \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ denote the Frobenius. Then, the subgroup generated by U and

U^σ is either of order 16 or infinite. Indeed, an simple computation shows that $\mathrm{Tr}(UU^\sigma) \in 4\mathbb{Z}$ and $\det(UU^\sigma) = 1$. So that UU^σ is either of infinite order or finite of order 4. Assume the later, then $\mathrm{Tr}(UU^\sigma) = \mathrm{Tr}(U(U^\sigma)^{-1})$. Since $\det(U(U^\sigma)^{-1}) = -1$, the element $U(U^\sigma)^-1$ is of order 2. The same is true for $U^{-1}U^\sigma$. Since $U^2 \neq (U^\sigma)^2$, we showed that $\langle U, U^\sigma\rangle$ contains at least two elements of order 2 while H_{24} only has one. Furthermore, one can show, looking at their relations, that U and U^σ generate a group isomorphic to $\mathrm{M}_4(2)$. □

Remark 3.2. Note that $\mathrm{M}_4(2)$ does not appear in $\Sigma_3(0, 2)$. While it can be realised as an inertia subgroup over $\mathbb{Q}_5$, it is not possible over $\mathbb{Q}_3$.

Of course, there exists automorphisms of order 8 outside $\mathrm{GL}_2(\mathbb{Z}[u])$, like:

$$\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$$

for instance. Again,

$$\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}\begin{pmatrix} 0 & v^{-1} \\ u & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}^4$$

are two distincts elements of order 2. If we want to show that H_{24} is not realisable over $\mathbb{Q}_3$, we need to consider the full automorphism group $\mathrm{GL}_2(\mathrm{End}_{\mathbb{F}_9}(E))$.

We proceed with $K_{36}/\mathbb{Q}_3$. Recall that $L_{36}/\mathbb{Q}_3$ is a totally ramified subextension of degree 36 inside K_{36}.

Lemma 3.8. *There exists an Abelian surface $A_0/\mathbb{F}_3$ and a polarisation Λ_0 on A_0, defined over $\mathbb{F}_3$ such that*

$$H_{36} \hookrightarrow \mathrm{Aut}_{\mathbb{F}_9}(A_0 \times_{\mathbb{F}_3} \mathbb{F}_9, \Lambda_0 \times_{\mathbb{F}_3} \mathbb{F}_9).$$

Furthermore, this injection is $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$*-equivariant, the action on the left-hand side coming from* $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3) \simeq \mathrm{Gal}(K_{36}/L_{36})$.

Proof. We consider the same elliptic curve E_0 as in Lemma 3.6, let $A_0 = E_0^2$, $E = E_0 \times_{\mathbb{F}_3} \mathbb{F}_9$ and $A = E^2$. Let $u, v \in \mathrm{Aut}_{\mathbb{F}_3}(E)$, with u of order 4 and v of order 3. The subgroup

$$\Gamma_{36} = \left\langle \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -v & 0 \\ 0 & -v \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \right\rangle$$

of $\mathrm{Aut}_{\mathbb{F}_9}(A)$ is isomorphic to H_{36}. Since Γ_{36} is stable by the action of $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$, by Lemma 3.5, there exists a polarisation Λ_0 on A_0 satisfying

$$\Gamma_{36} \subset \mathrm{Aut}_{\mathbb{F}_9}(A, \Lambda_0 \times_{\mathbb{F}_3} \mathbb{F}_9).$$

Thus, it makes sense to consider the semidirect product, and as expected, we have an isomorphism

$$\Gamma_{36} \rtimes \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3) \simeq I(K_{36}/\mathbb{Q}_3) \rtimes \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3).$$

This isomorphism doesn't guarantee that the action of $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ on Γ_{36} and the inertia subgroup are the same. One can show that

$$\langle (\mathrm{id}, \sigma) \rangle < \langle (-\mathrm{id}, 1), (\mathrm{id}, \sigma) \rangle \lhd \Gamma_{36} \rtimes \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3).$$

The action of $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ on Γ_{36} comes from an element of order 2 inside the unique normal copy of $(\mathbb{Z}/2\mathbb{Z})^2$ in $\Gamma_{36} \rtimes \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$. We have the exact same situation in terms of local fields

$$\mathrm{Gal}(K_{36}/L_{36}) < \mathrm{Gal}(K_{36}/K_1\mathbb{Q}_9) \lhd \mathrm{Gal}(K_{36}/\mathbb{Q}_3).$$

This proves the existence of a $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$-equivariant morphism

$$I(K_{36}/\mathbb{Q}_3) \xrightarrow{\sim} \Gamma_{36} \subset \mathrm{Aut}_{\mathbb{F}_9}(A_0 \times_{\mathbb{F}_3} \mathbb{F}_9, \Lambda_0 \times_{\mathbb{F}_3} \mathbb{F}_9),$$

and, as for Γ_{24} (see Lemma 3.6), the extension $\mathbb{F}_9$ is minimal in that regard. □

Remark 3.3. Note that $\Gamma_{36} \subset \mathrm{Aut}_{\mathbb{F}_9}(E) \times \mathrm{Aut}_{\mathbb{F}_9}(E)$, the projection on $\mathrm{Aut}_{\mathbb{F}_9}(E)$ defines a Galois pair (E_0, Γ_{36}, ν'). This Galois pair is not minimal (otherwise H_{36} would be associated to an elliptic curve) but $K' = K_{36}^{\mathrm{Ker}(\nu')}$ corresponds to one of the fields in Section 2.3.5. Letting $\Gamma'_{36} = \nu'(I(K'/\mathbb{Q}_3))$ gives us one of the minimal Galois pairs $(E_0, \Gamma'_{36}, \nu')$ for $K'/\mathbb{Q}_3$ that appear in Table 2.2.

We will now construct a polarised Galois pair for $K_{72}/\mathbb{Q}_3$.

Lemma 3.9. *There exists an Abelian surface $A_0/\mathbb{F}_3$ and a polarisation Λ on $A = A_0 \times_{\mathbb{F}_3} \mathbb{F}_9$, defined over $\mathbb{F}_9$ such that*

$$H_{72} \hookrightarrow \mathrm{Aut}_{\mathbb{F}_9}(A, \Lambda).$$

Proof. We keep the exact same notations as in Lemma 3.6. Consider the group

$$\Gamma_{72} = \left\langle \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix}, \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \right\rangle,$$

which is a subgroup of $\mathrm{Aut}_{\mathbb{F}_9}(A)$, and is isomorphic to H_{72}. As usual, we know there exists a polarisation Λ on A such that $\Gamma_{72} \subset \mathrm{Aut}_{\mathbb{F}_9}(A, \Lambda)$ using Lemma 3.5. As we have already seen with Γ_{24}, since Γ_{72} is not stable by $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$, this polarisation is not necessarily defined over $\mathbb{F}_3$. □

Remark 3.4. We encounter a similar issue as with H_{24}, since H_{72} also contains a unique element of order 2 (see Lemma 3.7).

3.3.2 Automorphisms over $\mathbb{F}_5$

In this section we provide polarised Galois pairs for $p = 5$. We begin with $K_{20}/\mathbb{Q}_5$.

Lemma 3.10. *There exists an Abelian surface $A_0/\mathbb{F}_5$ and a polarisation Λ_0 on A_0, defined over $\mathbb{F}_5$ such that*

$$H_{20} \hookrightarrow \mathrm{Aut}_{\mathbb{F}_5}(A_0, \Lambda_0).$$

Proof. Consider the following hyperelliptic curve over $\mathbb{F}_5$

$$C_0 : y^2 = x^5 - x.$$

Let $A_0 = \mathrm{Jac}(C_0)$ be its Jacobian. Since C_0 is of genus 2, A_0 is an Abelian surface, and it is also defined over $\mathbb{F}_5$. The set of maps defined on points by

$$(x, y) \longmapsto (a^2 x + b, ay); \; a \in \mathbb{F}_5^\times, b \in \mathbb{F}_5$$

is a subgroup Γ_{20} of $\mathrm{Aut}_{\mathbb{F}_5}(C_0) \subset \mathrm{Aut}_{\mathbb{F}_5}(A_0)$, isomorphic to H_{20}. We obtain the desired polarisation on A_0 using Lemma 3.5. □

There is only one polarised Galois pair left, the one associated to $K_{40}/\mathbb{Q}_5$. Recall that L_{40} is a totally ramified degree 40 subextension of K_{40}.

Lemma 3.11. *There exists an Abelian surface* $A_0/\mathbb{F}_5$ *and a polarisation* Λ_0 *on* A_0*, defined over* $\mathbb{F}_5$ *such that*

$$H_{40} \hookrightarrow \mathrm{Aut}_{\mathbb{F}_{25}}(A_0 \times_{\mathbb{F}_5} \mathbb{F}_{25}, \Lambda_0 \times_{\mathbb{F}_5} \mathbb{F}_{25}).$$

Furthermore, this injection is $\mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$*-equivariant, the action on the left-hand side coming from* $\mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5) \simeq \mathrm{Gal}(K_{40}/L_{40})$.

Proof. Let A_0 be as in Lemma 3.10 and let $A = A_0 \times_{\mathbb{F}_5} \mathbb{F}_{25}$, which is isomorphic to $\mathrm{Jac}(C_0 \times_{\mathbb{F}_5} \mathbb{F}_{25})$. The set of maps defined on points by

$$(x, y) \longmapsto (a^2x + b, ay); \ a \in \mathbb{F}_{25} \text{ such that } a^8 = 1, b \in \mathbb{F}_5,$$

is a subgroup Γ_{40} of $\mathrm{Aut}_{\mathbb{F}_{25}}(C) \subset \mathrm{Aut}_{\mathbb{F}_{25}}(A)$, isomorphic to H_{40}. Furthermore, it is stable by the action of $\mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$. By Lemma 3.5 there exists a polarisation Λ_0 on A_0 such that $\Gamma_{40} \subset \mathrm{Aut}_{\mathbb{F}_{25}}(A, \Lambda_0 \times_{\mathbb{F}_5} \mathbb{F}_{25})$. The semidirect product $\Gamma_{40} \rtimes \mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$ is the unique one that contains a normal subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. When we realised H_{40} as an inertia subgroup over $\mathbb{Q}_5$ we discussed (recall Lemma 3.4) the two possible associated Galois groups that could arise in that setting. We consider the one satisfying

$$\mathrm{Gal}(K_{40}/\mathbb{Q}_5) \rhd \mathrm{Gal}(K_{40}/K_2) \simeq (\mathbb{Z}/2\mathbb{Z})^2.$$

In that setting, we necessarily have

$$\Gamma_{40} \rtimes \mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5) \simeq I(K_{40}/\mathbb{Q}_5) \rtimes \mathrm{Gal}(K_{40}/L_{40}).$$

Here L_{40} is a totally ramified subextension of $\mathbb{Q}_5$ inside K_{40} corresponding to one of the two non normal subgroups of order 2 of $\mathrm{Gal}(K_{40}/\mathbb{Q}_5)$. It is then clear that, in this situation, there exists a $\mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$-equivariant morphism

$$I(K_{40}/\mathbb{Q}_5)\xrightarrow{\sim}\Gamma_{40} \subset \mathrm{Aut}_{\mathbb{F}_{25}}(A, \Lambda).$$

It is clear that $\mathbb{F}_{25}$ is minimal for Γ_{40}. Indeed, the Honda-Tate algebra of A_0 over $\mathbb{F}_5$ is the quaternion algebra over $\mathbb{Q}(\sqrt{5})$ ramified at the two infinite places. □

Remark 3.5. Though they used different techniques, working directly over the local field $\overline{\mathbb{F}}_p((t))$, we used the same subgroups of automorphisms that have been defined in [SZ05] to obtain our results.

We are now able to prove the main result of this section.

Proposition 3.2. *Let* $p \in \{3, 5\}$. *For every* $H \in \Sigma_p(0, 2)$, *there exists an integer* $f \geq 1$ *and a minimal polarised Galois pair for* $K_H/\mathbb{Q}_{p^f}$.

Proof. As in Proposition 3.1, each "hard" case has been treated individually. The cyclic groups of order 5 and 10 are subgroups of H_{20}. Let $E_0/\mathbb{F}_3$ be as in Lemma 3.6. The permutation group S_3 is realised over $\mathbb{F}_3$ as a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \subset \mathrm{Aut}_{\mathbb{F}_3}(E_0^2)$. The cartesian products for $p = 3$ are realised using the following injection

$$\mathrm{Aut}_\kappa(E)\times \mathrm{Aut}_\kappa(E) \hookrightarrow \mathrm{Aut}_\kappa(E^2).$$

Except for $\Gamma = \mathbb{Z}/3\mathbb{Z}\times T_{12}$ and $\mathbb{Z}/12\mathbb{Z}$, which have to be realised over $\mathbb{F}_9$, in every other case we have $\kappa = \mathbb{F}_3$. □

Remark 3.6. Note that the minimal f for each Galois pair attached to a group $\Gamma \subset \mathrm{Aut}_\kappa(A)$ is exactly the minimal residue degree of a Galois extension with inertia subgroup isomorphic to Γ.

Remark 3.7. With the notations of Section 2.3.5, the group H_{36} can also be realised as the inertia subgroup of the compositum $K_1K_3/\mathbb{Q}_3$.

3.4 The main result

We have been able to construct minimal polarised Galois pairs for each group in $\Sigma_p(0,2)$. The following result of S. Philip, allow us to conclude. That is, we can show that each such group can be achieved in mixed characteristic.

Theorem 3.1 ([Phi24], Thm. 3.10)**.** *Let H be the inertia subgroup of a finite Galois extension of p-adic fields. Let A_0 be an Abelian variety over a finite field of characteristic p and Λ_0 a polarisation on A_0 such that*

$$H \hookrightarrow \mathrm{Aut}_{\overline{\mathbb{F}_p}}(A_0, \Lambda_0).$$

Then there exists an Abelian variety $\mathcal{A}$ of dimension $\dim A_0$ defined over a finite extension of $\mathbb{Q}_p$ with associated inertia subgroup isomorphic to H.

We can now prove the main result of this chapter.

Theorem 3.2. *Let $p \in \{3,5\}$. For every $H \in \Sigma_p(0,2)$, there exists a finite extension $F/\mathbb{Q}_p$ and an Abelian surface $\mathcal{A}/F$ with potentially good reduction over K such that*

$$I(K/F) \simeq H.$$

Proof. The Proposition 3.2 gives us the needed embedding to apply Theorem 3.1. □

As satisfaying as this answer may seem, we believe we could be more specific regarding the field of definition of $\mathcal{A}$. Indeed, the $\mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$-equivariance together with the minimality of κ are not used in the proof of Theorem 3.2 after all. We are fairly confident that these additional assumptions would allow us to descend $\mathcal{A}$ on an Abelian variety $\mathcal{A}_0$ defined over $\mathbb{Q}_{p^f}$. The proof of Theorem 3.1 partly relies on a technical result regarding the existence of certain filtrations on $(\varphi, \mathcal{G}_F)$-modules (see Proposition 1.2). Unfortunately, the proof relies on the Dieudonné-Manin decomposition of isocrystals. In particular, the considered φ-modules are defined $\overline{\mathbb{F}_p}$, *i.e.*, are $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$-vector spaces. Since our φ-modules are defined over $\mathbb{F}_{p^f}$, the Theorem cannot be applied directly and needs to be adapted. We still believe that a similar result should hold in our situation. If not, we should be able to compute those filtrations explicitely in each individual case, like we did for elliptic curves over $\mathbb{Q}_3$. Those filtrations are the necessary data that allow us to lift our Abelian varieties defined over finite fields in characteristic zero.

We consider the following assumption.

Assumption 3.1. Let D be the $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_{p^f}))$-module associated to a minimal polarised Galois pair for $K/\mathbb{Q}_{p^f}$. There exists an admissible Hodge-Tate type $(0, 1)$ filtration Fil on D and an antisymmetric isomorphism of filtered $(\varphi, \mathrm{Gal}(K/\mathbb{Q}_{p^f}))$-modules

$$\delta : D^*\{-1\} \xrightarrow{\sim} D.$$

Under this additional hypothesis, we prove the following.

Proposition 3.3. *Let $p \in \{3, 5\}$ and assume 3.1 holds. For every $H \in \Sigma_p(0, 2)$, there exists an Abelian surface $\mathcal{A}_0/\mathbb{Q}_{p^f}$ with potentially good reduction over K such that*

$$I(K/\mathbb{Q}_p) \simeq H,$$

where $f \geq 1$ is an integer depending only on H, and K is minimal.

Proof. We know by Proposition 3.1, that there exists $K/\mathbb{Q}_p$ with inertia subgroup H. The Proposition 3.2 gives us a minimal polarised Galois pair for K/F, where $F = \mathbb{Q}_{p^f}$ for some f depending on H. We will denote by (A_0, Λ_0) the polarised Abelian surface associated to this Galois pair. Let κ be the residue field of K. Denote by $A = A_0 \times_{\mathbb{F}_{p^f}} \kappa$ and $\Lambda = \Lambda_0 \times_{\mathbb{F}_{p^f}} \kappa$ the scalar extensions of A_0 and Λ_0 to κ. Let D be the Dieudonné module of the p-divisible group $A(p)$ associated to A. It is a φ-module, its Frobenius coming from the Frobenius of A. We use the $\mathrm{Gal}(\kappa/\mathbb{F}_{p^f})$-equivariant injection

$$I(K/F) \hookrightarrow \mathrm{Aut}_\kappa(A, \Lambda),$$

to construct an anti-morphism

$$\nu : \mathrm{Gal}(K/F) \hookrightarrow \mathrm{Aut}_\kappa(A,\Lambda) \rtimes \mathrm{Gal}(\kappa/\mathbb{F}_{p^f}).$$

By contravariancy of the Dieudonné functor, this anti-morphism induces a morphism ν'

$$\begin{tikzcd}
\mathrm{Gal}(K/F) \arrow[r, hook, "\nu"] \arrow[dr, hook, "\nu'" swap] & \mathrm{Aut}_\kappa(A,\Lambda) \rtimes \mathrm{Gal}(\kappa/\mathbb{F}_{p^f}) \arrow[d, hook, "\text{Dieudonné functor}"] \\
 & \mathrm{Aut}_\varphi(D) \rtimes \mathrm{Gal}(K_0/F)
\end{tikzcd}$$

where $K_0 = \mathrm{Frac}(W(\kappa))$. It endows D with the structure of a $(\varphi, \mathrm{Gal}(K/F))$-module. By the Assumption 3.1, there exists an admissible filtration Fil on $D_K = D \otimes_{K_0} K$ of Hodge-Tate type $(0,1)$, and an antisymmetric isomorphism of filtered $(\varphi, \mathrm{Gal}(K/F))$-modules

$$\delta : D^*\{-1\} \xrightarrow{\sim} D.$$

The filtered $(\varphi, \mathrm{Gal}(K/F))$-module (D, Fil) then corresponds to a representation V of $\mathcal{G}_F$, crystalline over K. By Proposition1.1, there exists a filtration Fil' and a polarisation λ_0 on A_0 such that λ_0 lifts to (D, Fil') and $(D, \mathrm{Fil}) \simeq (D, \mathrm{Fil}')$. From here, we have all the ingredients to follow [Vol05, Thm. 5.7], as in Theorem 2.1. We thus obtain an Abelian surface $\mathcal{A}_0/F$ with potentially good reduction over K minimal. □

Remark 3.8. Actually, we can show that the groups in $\Sigma_3(0,2)$ that are subgroup of $T_{12} \times T_{12}$ can be realised over $\mathbb{Q}_3$. Every such group is the inertia subgroup of the compositum of two well-chosen extensions K_1 and K_2 found in Sections 2.3.3, 2.3.4 and 2.3.5. It is then enough to consider the product $E_1 \times E_2$ of elliptic curves over $\mathbb{Q}_3$ such that K_i is a minimal field of good reduction for E_i.

Bibliography

[Bos25] Giovanni Bosco. "The 3-adic representations arising from elliptic curves over $\mathbb{Q}_3$ with potential good reduction". *Journal de Théorie des Nombres de Bordeaux* 37.1 (2025), pp. 79–103.

[Bre00] Christophe Breuil. "Groupes p-divisibles, groupes finis et modules filtrés". *Annals of Mathematics* 152.2 (2000), pp. 489–549.

[CI99] Robert Coleman and Adrian Iovita. "The Frobenius and monodromy operators for curves and abelian varieties". *Duke Mathematical Journal* 97.1 (1999), pp. 171–215.

[CF00] Pierre Colmez and Jean-Marc Fontaine. "Construction des représentations p-adiques semi-stables". *Inventiones Mathematicae* 140.1 (2000), pp. 1–43.

[Cop20a] Nirvana Coppola. "Wild Galois representations: Elliptic Curves over a 3-adic field". *Acta Arithmetica* 195.3 (2020), pp. 289–303.

[Cop20b] Nirvana Coppola. "Wild Galois representations: Elliptic curves over a 2-adic field with non-abelian inertia action". *International Journal of Number Theory* 16.06 (2020), pp. 1199–1208.

[Cop25] Nirvana Coppola. *Wild Galois representations: elliptic curves with wild cyclic reduction*. 2025. arXiv: `2506.20562`.

[Del74] Pierre Deligne. "La conjecture de Weil : I". *Publications Mathématiques de l'IHES* 43 (1974), pp. 273–307.

[DFV24] Lassina Dembélé, Nuno Freitas, and John Voight. *On Galois inertial types of elliptic curves over* $\mathbb{Q}_\ell$. 2024. arXiv: `2203.07787`.

[GN] Tim Dokchitser. *GroupNames*. `https://people.maths.bris.ac.uk/~matyd/GroupNames/index.html`. 2026.

[Fon77] Jean-Marc Fontaine. *Groupes p-divisibles sur les corps locaux*. Astérisque 47-48. Société mathématique de France, 1977.

[Fon94a] Jean-Marc Fontaine. "Le corps des périodes p-adiques". In: *Périodes p-adiques. Séminaire du Bures-sur-Yvette, France, 1988*. Vol. 223. Astérisque. Société Mathématique de France, 1994, pp. 59–111.

[Fon94b] Jean-Marc Fontaine. "Représentations p-adiques semi-stables". In: *Périodes p-adiques. Séminaire du Bures-sur-Yvette, France, 1988*. Vol. 223. Astérisque. Société Mathématique de France, 1994, pp. 113–184.

[FK22] Nuno Freitas and Alain Kraus. *On the symplectic type of isomorphisms of the p-torsion of elliptic curves*. Vol. 1361. Memoirs of the American Mathematical Society. American Mathematical Society, 2022, pp. v+105.

[Hwa21] WonTae Hwang. "On a classification of the automorphism groups of polarized abelian surfaces over finite fields". *Finite Fields and Their Applications* 72 (2021). 101809.

[LMFDB] LMFDB. *The L-functions and Modular Forms Database*. `http://www.lmfdb.org`. 2026.

[Mil08] James S. Milne. *Abelian Varieties (v2.00)*. 2008.

[Neu99] Jürgen Neukirch. *Algebraic Number Theory*. Vol. 322. Grundlehren der mathematischen Wissenschaften. Springer, 1999.

[Phi24] Séverin Philip. *Groupes de monodromie finie des variétés abéliennes*. 2024. arXiv: `2412.11900`.

[Ser68] Jean-Pierre Serre. *Corps locaux*. Vol. 8. Publications de l'institut de Mathématiques de l'Université du Nancago. Hermann, 1968.

[Ser71] Jean-Pierre Serre. "Propriétés galoisiennes des points d'ordre fini des courbes elliptiques". *Inventiones Mathematicae* 15 (1971), pp. 259–331.

[ST68] Jean-Pierre Serre and John Tate. "Good Reduction of Abelian Varieties". *Annals of Mathematics* 88.3 (1968), pp. 492–517.

[SZ05] Alice Silverberg and Yuri Zarhin. “Inertia groups and abelian surfaces”. *Journal of Number Theory* 110.1 (2005), pp. 178–198.

[Sil86] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*. Vol. 106. Graduate Texts in Mathematics. Springer, 1986, pp. xii+400.

[Tat67] John Tate. “p-Divisible Groups”. In: *Proceedings of a Conference on Local Fields*. Springer, 1967, pp. 158–183.

[Vol01] Maja Volkov. “Les représentations ℓ-adiques associées aux courbes elliptiques définies sur $\mathbb{Q}_p$”. *Journal für die Reine und Angewandte Mathematik* 535 (2001), pp. 65–101.

[Vol05] Maja Volkov. “A class of p-adic Galois representations arising from Abelian varieties over $\mathbb{Q}_p$”. *Mathematische Annalen* 331.4 (2005), pp. 889–923.