

Perturbative results for fractional quantum mechanics

Claude Semay,^{*} Clara Tourbez,[†] and Loïc Keszeli[‡]

*Service de Physique Nucléaire et Subnucléaire, Université de Mons,
UMONS Research Institute for Complex Systems, Place du Parc 20, 7000 Mons, Belgium*
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The fractional Schrödinger equation is studied with a kinetic energy that slightly deviates from the usual nonrelativistic form. The harmonic oscillator and the Kepler problem are both treated in the context of small perturbations. The usual perturbation theory is used and compared with the envelope theory. The corrections to the unperturbed Hamiltonians are analytical. The bounds computed with the envelope theory are good for the harmonic oscillator but poorer for the Kepler problem.

Keywords: fractional quantum mechanics, perturbation theory, envelope theory

I. INTRODUCTION

Microscopic physics is very well described by the “usual” quantum mechanics in 3D space relying on the time-dependent Schrödinger equation which is of first order in time derivative and of second order in space derivatives. Fractional quantum theories extend standard quantum mechanics by replacing either the spatial operator, the time derivative, or even the dimensionality of the underlying space with fractional generalizations.

Space-fractional quantum mechanics, introduced by Laskin, replaces the Laplacian in the Schrödinger equation with a fractional Laplacian $(-\hbar^2\Delta)^{\alpha/2}$ defined by the Riesz derivative [1–4]. The resulting dynamics are nonlocal in space and built on the Lévy-path formulation, which is defined for $0 < \alpha \leq 2$, leading to anomalous transport and modified energy spectra. Recent work has further clarified the mathematical properties and physical realizations of fractional Laplacians [5].

Time-fractional evolution modifies the Schrödinger equation by replacing the first-order time derivative with a fractional derivative, usually of Caputo type [6, 7]. The wave function then possesses memory effects, since in this case the evolution depends on the system’s past history. Solutions are typically expressed through Mittag-Leffler functions rather than simple exponentials, and the dynamics may become non-Markovian, making this framework useful for complex and dissipative quantum environments.

Quantum mechanics in a fractional-dimensional space assumes that the effective dimension of space is noninteger ($D \notin \mathbb{N}$) [8, 9]. This approach does not primarily modify derivatives but the geometric measure itself, allowing anisotropic, confined, or fractal-like media to be represented by an effective fractional dimension. Applications include semiconductor heterostructures, critical phenomena, and fractal systems.

The main difference is therefore conceptual: space-fractional quantum mechanics introduces spatial nonlocality through a fractional kinetic operator; time-fractional evolution introduces temporal memory; quantum mechanics in a fractional dimension modifies the geometry of the configuration space. Their connection lies in the common framework of fractional calculus and anomalous dynamics. All three theories seek to describe systems that deviate from ordinary quantum behavior due to long-range correlations, memory effects, or fractal geometry. Some studies even combine these different approaches, which may represent complementary aspects of quantum processes in complex media [10–12].

To our knowledge, there is currently no convincing experimental evidence that nature obeys fractional quantum mechanics (FQM) rather than ordinary quantum mechanics. On the other hand, there are experimental attempts to realize or simulate the dynamics predicted by FQM. A prominent example is an experimental realization of an optical fractional Schrödinger equation (FSE) for femtosecond laser pulses in the temporal domain [13]. In [14], solutions of FSE are computed for a possible future use to explore a wide range of important physical phenomena. The FSE is also an interesting theoretical tool for investigating various phenomena: tunneling and nonlocal transport phenomena in strong laser fields [15, 16]; role of disorder in the vibration spectra of molecules and atoms in solids [17]; equivalent models of quantum gravity [18].

^{*}E-mail: claudesemay@umons.ac.be; ORCID: 0000-0001-6841-9850

[†]E-mail: clara.tourbez@umons.ac.be; ORCID: 0009-0004-0909-6974

[‡]E-mail: Loic.Keszeli@student.umons.ac.be

Although not free from criticism [19, 20], the FSE introduced by Laskin [1–4, 11] is the subject of this paper. In the event that nature had followed the path of FQM, it is expected that α must be very close to 2, otherwise deviations from the usual quantum mechanics should already have been observed. Our first purpose is to compute solutions of the FSE by considering $\alpha = 2 + \epsilon$ with $|\epsilon| \ll 1$ (see Sect. II for the justification for $\epsilon > 0$). This study, which is new up to our knowledge, opens a way to possible comparisons with experiment. Note that perturbations of the FSE have been treated in [21, 22], but the idea was to replace the orbital quantum number l by $l + \alpha$ where α is presumed small, which is quite different from our approach. Our second objective is to compute the same solutions with the envelope theory (ET) [23, 24] which can produce approximate but reliable analytical eigensolutions for a wide variety of problems. The application of the ET for the FQM is also new. The use of both methods, perturbation theory and ET, enables a cross-check of the calculations and challenges the ET as a relevant method for treating fractional quantum equations.

The framework of FQM considered and the definition of the perturbation calculations are given in Sect. II. The ET is presented in Sect. III. The case of the harmonic oscillator is treated in Sect. IV. This Hamiltonian is especially important because it can be used as a good approximation to study the lowest eigenvalues of most potential wells. The Kepler problem is treated in Sect. V. It is then possible to propose a connection with the experiment via the hydrogen atom. Concluding remarks and outlook will be given in Sect. VI.

II. FRACTIONAL QUANTUM HAMILTONIAN

The time independent fractional Schrödinger Hamiltonian for a particle in the central 3-dimensional potential considered here is written

$$H = D_\alpha p^\alpha + V(r), \quad (1)$$

where $\mathbf{p}$ is the variable conjugate to the position $\mathbf{r}$, $p = |\mathbf{p}|$, $r = |\mathbf{r}|$, D_α is a constant, and the kinetic part is the fractional Laplacian [25, 26]

$$p^\alpha = (\mathbf{p}^2)^{\alpha/2} = (-\hbar^2 \Delta)^{\alpha/2}, \quad (2)$$

defined by the Riesz derivative [27] in the Laskin formulation. The Hamiltonian (1) is therefore nonlocal.

Several solutions of this Hamiltonian have already been found: free particle, Green function, Kepler problem, 1D power-law potential, infinite potential well, barrier penetration [3, 4, 28–31]. In this exploratory work, our purpose is to investigate small deviations from the ordinary Schrödinger Hamiltonian. To this end, we consider $\alpha = 2 + \epsilon$ in (1) with $|\epsilon| \ll 1$, for two different central potentials. The generic one-body Hamiltonian considered has the following form

$$H_F = \frac{p^{2+\epsilon}}{2m\lambda^\epsilon} + V(r). \quad (3)$$

where λ is a positive constant with the dimension of a momentum. This constant is necessary to maintain the dimensional consistency of the Hamiltonian. Note that we can define a fundamental length L_λ associated with the fundamental momentum λ as

$$L_\lambda = \frac{\hbar}{\lambda}. \quad (4)$$

In the seminal work of Laskin [1], FQM is built on the Lévy-path formulation, which is defined for $0 < \alpha \leq 2$. Nevertheless, $\alpha > 2$ is not mathematically forbidden. In fact, some authors have explored generalized Hamiltonians for arbitrary positive values of α [32–34]. These models can be physically meaningful as effective theories. However, they are usually viewed as higher-order dispersion models rather than as Laskin's FQM. In this paper, we consider both the negative (usual Lévy-path formulation) and the positive (effective models) values for ϵ . Since $|\epsilon|$ is very small, H_F can be approximated by

$$H_F \approx H_\epsilon = H_0 + \epsilon W(p), \quad (5)$$

where

$$H_0 = \frac{\mathbf{p}^2}{2m} + V(r) \quad \text{and} \quad W(p) = \frac{\mathbf{p}^2}{4m} \ln \frac{\mathbf{p}^2}{\lambda^2}. \quad (6)$$

The problem can then be solved by the usual perturbation theory [35–37], but with a perturbation that is momentum dependent. Actually, Hamiltonian (1) can be solved indifferently in a configuration space or in a momentum space. For example, with $V(r) \propto r^2$, this problem treated with the momentum variable is mathematically equivalent to a Schrödinger equation with a power-law potential proportional to r^α treated with the position variable. This is particularly visible in a recent paper [38], in which the Lagrange-mesh method, initially developed in the configuration space, is adapted to direct calculations in momentum space. This is also clear in the formulation of the envelope theory (see Sect. III). The perturbation $W(p)$ grows faster than p^2 . This is not a problem, since the case of a harmonic potential with cubic and quartic perturbations is mathematically analyzed in a correct manner in the configuration space [35–37]. Precautions must be taken for large values of quantum numbers. This will be discussed during the explicit calculations in Sect. IV and V. In the following, two different solvable Hamiltonians H_0 are considered and $\epsilon W(p)$ is treated as a perturbation. The computations will be limited to the determination of the first-order corrections in ϵ to the energies.

III. THE ENVELOPE THEORY

The ET is a method used to compute approximate solutions of N -body Hamiltonians in D dimensions [23, 24]. The basic idea is to replace the initial Hamiltonian H by an auxiliary Hamiltonian $\tilde{H}$, which is solvable. In practice, for $N \geq 3$ and $D \geq 2$, only harmonic oscillator Hamiltonians are usable [39]. For $N \leq 2$ or $D = 1$, other systems are possible [35, 40]. $\tilde{H}$ depends on parameters that are optimized in such a way that one of its particular eigenvalues is as close as possible to the corresponding eigenvalue of H . It is easy to implement for 1 particle, 2 different particles, or N identical particles, but more complicated systems can also be treated. In some favorable cases, it is possible to determine approximate analytical eigenvalues which can provide lower or upper bounds on exact eigenvalues [41]. In other systems, numerical approximation can always be computed. To lighten the paper, only the main properties of the ET are recalled. A description of the method can be found in [42], and a simpler pedagogical presentation is given in [43].

In this paper, we only need to use the method for $N = 1$ and $D = 3$. Let us consider a Hamiltonian of this type

$$H = T(p) + V(r). \quad (7)$$

An approximate eigenvalue E_{ET} is given by solving the following set of equations [24]

$$E_{\text{ET}} = T(p_0) + V(r_0), \quad (8)$$

$$r_0 p_0 = Q\hbar, \quad (9)$$

$$p_0 T'(p_0) = r_0 V'(r_0), \quad (10)$$

where $U'(x) = dU(x)/dx$. If the auxiliary Hamiltonian $\tilde{H}$ is a harmonic oscillator Hamiltonian, the global quantum number Q is given by

$$Q = Q_{\text{HO}} = 2n + l + 3/2. \quad (11)$$

If $\tilde{H}$ is a Hamiltonian for the Kepler problem, then

$$Q = Q_{\text{K}} = n + l + 1. \quad (12)$$

A state is specified by its radial (n) and orbital (l) quantum numbers. Once r_0 and p_0 are computed with (9)–(10), E_{ET} can be found with (8). The parameter r_0 can be interpreted as the approximate mean distance between the particle and the origin, and p_0 as the approximate mean momentum of the particle. Note that (10) is the translation of the master virial theorem [44] for the ET. From the symmetry of (8)–(10), it is clear that it is equivalent to work in position or in momentum variables. Approximations for the eigenvectors can also be computed with the ET, but this is not considered in this work.

The energies computed with the envelope theory are characterized by a strong unnatural degeneracy, as clearly shown by the structure of the global quantum numbers Q_{HO} and Q_{K} . This is inherent to the method because it relies on the solutions of an auxiliary Hamiltonian, which is solvable due to its high degree of symmetry. This drawback can be partly corrected by combining the ET with the dominantly orbital state (DOS) method [41, 45]. Significant improvements can then be obtained for some peculiar Hamiltonians, but the possible variational character of the ET solutions can be lost.

Depending on the choice of $\tilde{H}$ (that is to say of Q) and the structure of H , the ET can provide a lower or upper bound of the exact eigenvalues in some favorable situations [24]. In the following two sections, the only difference

between the studied Hamiltonian H and the chosen auxiliary Hamiltonian $\tilde{H}$ is the kinetic part. In this situation, the possibly variational character of the ET is only controlled by the function $b_T(x)$ such that [24]

$$b_T(x) = T(\sqrt{x}) = \frac{x^{1+\epsilon/2}}{2m\lambda^\epsilon}, \quad (13)$$

with $x \geq 0$. Since

$$b_T(x)'' = \left(1 + \frac{\epsilon}{2}\right) \frac{\epsilon}{2} \frac{x^{\epsilon/2-1}}{2m\lambda^\epsilon} \quad (14)$$

has always the same sign, an upper (lower) bound is obtained if $\epsilon < 0$ ($\epsilon > 0$).

The ET is particularly suitable for perturbative calculations [24]. The first step to solve the Hamiltonian H_ϵ is to find E_{ET} , r_0 and p_0 for the unperturbed Hamiltonian H_0 with the system (8)–(10). The first order ET solution E_ϵ is then given in a second step by the computation of

$$E_\epsilon = E_{\text{ET}} + \Delta E_{\text{ET}} \quad \text{with} \quad \Delta E_{\text{ET}} = \epsilon W(p_0). \quad (15)$$

It is also possible to solve the system (8)–(10) directly for H_ϵ and then perform a first-order expansion in ϵ . However, this approach also leads to the result (15). As an inequality between two functions does not automatically hold for their finite-order limit expansions, it is worth noting that the existence of bounds for the full ET calculation does not guaranty their existence for the associated perturbative calculation. We will check below whether the variational character of the method is preserved for the results obtained.

The effect of a perturbation is often to raise degeneracies. This is not possible with the ET, as previously reported. If the method has a variational character for the Hamiltonian studied, several non-degenerate levels that are characterized by the same global quantum number Q will therefore have the same bound on the energy. It is perhaps possible to partly correct this problem by combining the ET with the DOS method, but this is beyond the scope of this work.

Let us consider the following Hamiltonian, whose characteristics will be useful in the following sections.

$$H = D_\alpha p^\alpha + \text{sgn}(\beta) q^{2r^\beta}, \quad (16)$$

with $\alpha > 0$ and $\beta > -\alpha$. Note that $\alpha = 1$ with $D_\alpha = 1$ corresponds to the kinetic energy of a massless particle, used in semirelativistic models for hadronic physics [46]. The computation of p_0 gives

$$p_0 = \left(\frac{Q^\beta \hbar^\beta |\beta| q^2}{\alpha D_\alpha} \right)^{1/(\alpha+\beta)}. \quad (17)$$

The ET eigenenergies are [41]

$$E_{\text{ET}} = \text{sgn}(\beta)(\alpha + \beta) \left[\left(\frac{D_\alpha}{|\beta|} \right)^\beta \left(\frac{q^2}{\alpha} \right)^\alpha (Q\hbar)^{\alpha\beta} \right]^{1/(\alpha+\beta)}. \quad (18)$$

This formula is similar to the one obtained in [4] by the WKB method for $D = 1$. The ET is such that it gives the exact eigenenergies for a harmonic oscillator ($\alpha = \beta = 2$) with $Q = Q_{\text{HO}}$, and for the Kepler problem ($\alpha = 2, \beta = -1$) with $Q = Q_{\text{K}}$. When $0 < \beta = \alpha < 2$ with $Q = Q_{\text{HO}}$, (18) reduces to upper bounds for the eigenenergies of what is called the fractional generalization of the 3D harmonic oscillator Hamiltonian of standard quantum mechanics in [3]

$$E_{\text{ET}} = 2\sqrt{D_\alpha q^2} ((2n + l + 3/2)\hbar)^{\alpha/2}. \quad (19)$$

The quality of the approximation can be improved by replacing the factor $2n + l + 3/2$ by the one predicted by the DOS method, $\sqrt{2\alpha}n + l + (1 + \sqrt{2\alpha})/2$, but the variational character of the approximation is no longer guaranteed [41, 45].

IV. PERTURBATIVE RESULTS FOR THE HARMONIC OSCILLATOR

Let us first look at the harmonic oscillator with

$$V(r) = \frac{1}{2}m\omega^2 r^2. \quad (20)$$

The eigenenergies are $E_{nl} = Q_{\text{HO}}\hbar\omega$ and the eigenfunctions $\psi_{nl}(\mathbf{p})$ written in the momentum variable $\mathbf{p}$ are given in [47]. The energy perturbations ΔE_{nl} are the expectation value $\langle \psi_{nl}(\mathbf{p}) | \epsilon W(\mathbf{p}) | \psi_{nl}(\mathbf{p}) \rangle$. These integrals can be computed exactly using the Feynman trick [48]. No general formula has been obtained, but the same structure appears for all tested $\{n, l\}$ sets:

$$\Delta E_{nl} = \epsilon \frac{Q_{\text{HO}}}{4} \ln \left(\frac{e^{A_{nl}-\gamma}}{4} \frac{m\hbar\omega}{\lambda^2} \right) \hbar\omega, \quad (21)$$

where γ is the Euler constant and where A_{nl} is a coefficient depending on $\{n, l\}$, some of whose values are given in Table I. Two different $\{n, l\}$ sets corresponding to the same value of Q_{HO} are given to show the removal of degeneracy.

n	l	Q_{HO}	A_{nl}	$e^{A_{nl}-\gamma}/4$
0	0	3/2	8/3	2.020
1	0	7/2	74/21	4.760
0	2	7/2	352/105	4.010
5	0	23/2	375598/79695	15.63
0	10	23/2	1488711776/334639305	12.00

Table I: A_{nl} values for some $\{n, l\}$ sets with Q_{HO} and numerical values of the characteristic coefficient in the ln-term of (21).

The eigensolutions of the Hamiltonian (3) with potential (20) can be computed with a high accuracy thanks to the Lagrange mesh method [49]: relative errors below 10^{-8} can easily be obtained. So, it is possible to check numerically the precision of formula (21). In the case $\hbar = m = \omega = 1$, the relative error for the ground state energy between the exact solution and the perturbation approximation reach a maximum of 17% for $|\epsilon| = 0.1$ when $\lambda \in [10^{-4}, 10^4]$. This maximal error drops below 0.001% when $|\epsilon| = 0.001$. When $\lambda \in [0.1, 10]$ with $|\epsilon| = 0.1$, the relative errors reach a maximum of 0.8% for the ground state and 1.5% for states with $Q_{\text{HO}} = 23/2$. Hence, formula (21) seems relevant for a wide range of ϵ and $m\hbar\omega/\lambda^2$ values. However, the quality of the perturbation approximation decreases as the excitation energy increases. A similar behavior occurs for quartic perturbations of the harmonic oscillator [36, 37].

The first step of the ET calculation is to compute p_0 , with (9)–(10), for the Hamiltonian H_0 with (20). From (17) and (18), we have

$$p_0 = \sqrt{mQ\hbar\omega}. \quad (22)$$

With $Q = Q_{\text{HO}}$, the ET solution (8) is the exact value $E_{nl}^{\text{ET}} = E_{nl} = Q_{\text{HO}}\hbar\omega$. Using (15), the contribution of the perturbation is

$$\Delta E_{nl}^{\text{ET}} = \epsilon \frac{Q_{\text{HO}}}{4} \ln \left(Q_{\text{HO}} \frac{m\hbar\omega}{\lambda^2} \right) \hbar\omega, \quad (23)$$

Formulas (21) and (22) have a similar structure. The only difference is the replacement of $e^{A_{nl}-\gamma}/4$ by Q_{HO} in the ln-term. Comparisons between these two last quantities for some sets $\{n, l\}$ are given in Table II with the ratio $R_{nl} = (e^{A_{nl}-\gamma}/4)/Q_{\text{HO}}$. All the values of R_{nl} are close to 1, indicating that the ET approximation is quite good for the lowest energy states. Since the ln-function is a monotonically increasing function, $\Delta E_{nl}^{\text{ET}}$ is an upper (lower) bound of ΔE_{nl} if $\epsilon < 0$ ($\epsilon > 0$) for all the cases computed. So, the variational character expected from (14) is preserved. Note that the absolute error $|\Delta E_{nl} - \Delta E_{nl}^{\text{ET}}|$ does not depend on λ due to the ln-term, but this is not the case for the relative error.

Some central potentials $U(r)$ can be developed around the radial position r_m of their minimum,

$$U(r) = U(r_m) + \frac{1}{2}U''(r_m)(r - r_m)^2. \quad (24)$$

For lower-energy states, the wave function is localized near the minimum, and higher-order terms of the potential $U(r)$ are often negligible. So, good approximations of lowest vibrational energies can be obtained as the eigenvalues of the harmonic potential (24) [36, 37]. This is the case for the Morse potential and the Lennard-Jones potential, which are often used to compute the spectra of a diatomic molecule. If $r_m = 0$, as in the case of Gaussian potentials used to model atomic traps, all the lowest states, vibrational and orbital, can be approximated by the eigenvalues of (24). In these situations, an approximation of the perturbation (23) for a potential well can be obtained by replacing ω by $\sqrt{U''(r_m)/m}$.

n	l	Q_{HO}	$e^{A_{nl}-\gamma}/4$	R_{nl}
0	0	1.5	2.020	1.35
1	0	3.5	4.760	1.36
0	2	3.5	4.010	1.15
5	0	11.5	15.63	1.36
0	10	11.5	12.00	1.04

Table II: Comparisons of formulas (21) and (22) for some sets $\{n, l\}$. The last column is the ratio $R_{nl} = (e^{A_{nl}-\gamma}/4)/Q_{\text{HO}}$.

V. PERTURBATIVE RESULTS FOR THE KEPLER PROBLEM

The Kepler problem corresponding to the $2D$ hydrogen atom is treated in [30]. The WKB solutions of the same $3D$ system are computed in [3, 4]. In this section, we repeat the procedure of the previous one for the hydrogen atom with

$$V(r) = -\frac{1}{r}, \quad (25)$$

written in atomic units. Within the same units, the eigenenergies¹ are $E_{nl} = -1/(2Q_K^2)$ and the eigenfunctions $\psi_{nl}(\mathbf{p})$ written in the momentum variable $\mathbf{p}$ are given in [47]. The energy perturbations ΔE_{nl} are the expectation value $\langle \psi_{nl}(\mathbf{p}) | \epsilon W(p) | \psi_{nl}(\mathbf{p}) \rangle$. Again, these integrals can be computed exactly using the Feynman trick [48]. No general formula has been obtained, but the same structure appears for all tested $\{n, l\}$ sets:

$$\Delta E_{nl} = \epsilon \frac{1}{2Q_K^2} \ln \left(\frac{Q_K e^{B_{nl}}}{\lambda} \right), \quad (26)$$

where B_{nl} is a coefficient depending on $\{n, l\}$, some of whose values are given in Table III. Two different $\{n, l\}$ sets corresponding to the same value of Q_K are given to show the removal of degeneracy, which is relatively larger than for the harmonic oscillator case.

n	l	Q_K	B_{nl}	$Q_K e^{B_{nl}}$
0	0	1	1/3	1.396
1	0	2	11/15	4.164
0	1	2	1/5	2.443
5	0	6	131/143	15.00
0	5	6	1/13	6.480

Table III: B_{nl} values for some $\{n, l\}$ sets with Q_K and numerical values of the characteristic coefficient in the ln-term of (26).

As in the case of the harmonic potential, the eigensolutions of the Hamiltonian (3) with the potential (25) can be calculated with high precision thanks to the Lagrange mesh method, and the precision of formula (26) can be checked numerically. When $\lambda \in [0.1, 10]$ with $|\epsilon| = 0.1$, the relative errors reach a maximum of 6% for the ground state and 33% for states with $Q_K = 6$. This error is divided by around 100 for $|\epsilon| = 0.001$ within the same range of λ . So, formula (26) seems relevant for a significative range of ϵ and λ values. As in the case of the harmonic oscillator, the quality of the perturbation approximation decreases as the excitation energy increases.

Evaluating p_0 for the Hamiltonian H_0 in atomic units with (25) using (9)–(10), we have, from (17) and (18),

$$p_0 = \frac{1}{Q}. \quad (27)$$

¹ To avoid overloading the text with symbols, the notations used in the previous section (E_{nl} , p_0 , ...) will also be used unchanged in this one.

If $Q = Q_K$, the ET solution (8) is the exact value $E_{nl}^{\text{ET}} = E_{nl} = -1/(2Q_K^2)$. Using (15), the contribution of the perturbation is

$$\Delta E_{nl}^{\text{ET}} = \epsilon \frac{1}{2Q_K^2} \ln \left(\frac{1}{Q_K \lambda} \right). \quad (28)$$

Formulas (26) and (27) have a similar structure. The only difference is the replacement of $Q_K e^{B_{nl}}$ by Q_K^{-1} in the ln-term. Comparisons between these two last quantities for some $\{n, l\}$ sets are given in Table IV with the values of the ratio $R_{nl} = Q_K e^{B_{nl}} / Q_K^{-1}$. In contrast to the harmonic oscillator system, they increase dramatically with the excitation levels. This deterioration is tempered by the ln-function, but the quality of the ET approximation is clearly poorer than for the harmonic oscillator. This is not really surprising, since a correlation between the presence of a divergence and a decrease in the accuracy of the ET was demonstrated in [50], albeit in a different context. As in the previous section, $\Delta E_{nl}^{\text{ET}}$ is an upper (lower) bound of ΔE_{nl} if $\epsilon < 0$ ($\epsilon > 0$) for all the cases computed. So, the variational character expected from (14) is preserved, but the quality of the bound diminishes rapidly with increasing quantum numbers. Again, the absolute error $|\Delta E_{nl} - \Delta E_{nl}^{\text{ET}}|$ does not depend on λ due to the ln-term, but this is not the case for the relative error.

n	l	Q_K	$Q_K e^{B_{nl}}$	R_{nl}
0	0	1	1.396	1.4
1	0	2	4.164	8.3
0	1	2	2.443	4.9
5	0	6	15.00	90.
0	5	6	6.480	39.

Table IV: Comparisons of formulas (26) and (27) for some sets $\{n, l\}$. The last column is the ratio $R_{nl} = Q_K e^{B_{nl}} / Q_K^{-1}$.

As the above results are related to the hydrogen atom, it seems feasible to obtain information about the possible values of the parameters ϵ and λ , in the event that nature deviates slightly from what is generally expected. Equation (26) can be rewritten

$$\frac{\Delta E_{nl}}{E_{nl}} = -\epsilon \ln \left(\frac{Q_K e^{B_{nl}}}{\lambda} \right). \quad (29)$$

We can expect this dimensionless quantity to be much less than 1, otherwise deviations from the usual quantum mechanics should already have been observed. To obtain reliable phenomenological estimations, it is necessary to compute the effect of the fractional corrections on several observables, such as accurate transition energies, and compare them with fine experimental measurements, such as QED corrections. This huge work is left for further studies. Nevertheless, we can try to propose a very crude bound on the parameters ϵ and λ . A good estimation of the ground state energy of the hydrogen atom is given by the Rydberg constant. The relative uncertainty about this quantity is around 10^{-12} [51]. This suggests that

$$\left| \frac{\Delta E_{00}}{E_{00}} \right| = \left| \epsilon \ln \left(e^{1/3} L_\lambda \right) \right| \lesssim 10^{-12}, \quad (30)$$

with $L_\lambda = 1/\lambda$ in units of Bohr radius.

VI. CONCLUDING REMARKS

The fractional Schrödinger equation has been investigated as a small perturbation of the standard Schrödinger equation by introducing a kinetic-energy operator that deviates slightly from the usual nonrelativistic form. To ensure dimensional consistency, a positive constant with the dimensions of momentum (or inverse length) has been introduced. In this exploratory study, two illustrative systems, the harmonic oscillator and the Kepler problem, have been examined. For the latter, a potential connection with experimental observations has been suggested.

For both Hamiltonians, analytical expressions for the lowest-energy states have been derived within the framework of perturbation theory. Their accuracy has been validated through numerical calculations for various parameter values.

Analytical approximations for the complete spectra have also been obtained using envelope theory. For all states investigated, the envelope theory eigenenergies provide either upper or lower bounds. These bounds are generally of good quality for the harmonic oscillator across the spectrum. In the Kepler case, the agreement remains satisfactory for the ground state but deteriorates rapidly as the excitation energy increases.

Envelope theory can readily be extended to many-body systems with nonstandard kinetic-energy operators, particularly when all the particles are identical. It therefore offers a practical framework for obtaining approximate eigensolutions of many-body Hamiltonians arising in fractional quantum mechanics. A reasonable level of accuracy may be expected for harmonic-oscillator systems. Although its performance for more general interactions remains to be assessed, the method deserves further investigation given the intrinsic difficulty of solving many-body quantum systems.

The spectra obtained with envelope theory exhibit strong spurious degeneracy. This feature is intrinsic to the method, which relies on an auxiliary Hamiltonian whose solvability stems from its high degree of symmetry. This drawback can be partially corrected by combining the envelope theory with the dominant orbital state method, leading in some cases to significant improvements. Exploring such hybrid approaches in the context of fractional quantum mechanics appears to be a promising direction for future work.

Since conventional perturbation theory yields accurate predictions for the eigenenergies of the fractional Schrödinger equation, it would be worthwhile to extend the analysis to wavefunctions and derived observables. In the case of the hydrogen atom, such calculations could provide a more direct assessment of the physical relevance of fractional quantum mechanics and, ultimately, offer a means of confronting the theory with high-precision experimental data.

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