Frequently hypercyclic operators: recent advances and open problems

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Abstract

We report on recent advances on one of the central notions in linear dynamics, that of a frequently hypercyclic operator. We also include a list of ten open problems.

1 Introduction

Frequent hypercyclicity is one of the most fascinating notions in linear dynamics: it is a natural strengthening of the key concept in linear dynamics, that of hypercyclicity, and it is very close in spirit (though not, as we will see, in a strict sense) to that of linear chaos. Although initiated only ten years ago, the study of frequently hypercyclic operators has seen very deep and important developments, many of them quite recent. In this survey we will discuss some of these advances, and we will highlight several open problems.

Let us first recall the two central concepts in linear dynamics. Throughout this paper, X will denote a separable F-space, that is, a topological vector space whose topology is induced by a complete translation-invariant metric. The reader will lose very little in assuming that X is a separable Banach space. Moreover, $T : X \to X$ will denote a (continuous and linear) operator on X.

Definition 1. (a) An operator T is called *hypercyclic* if there exists some $x \in X$ whose orbit

$$\mathrm{orb}(x, T) = \{x, Tx, T^2x, T^3x, \dots\}$$

is dense in X. In this case, x is called a *hypercyclic vector* for T. The set of hypercyclic vectors is denoted by $HC(T)$.

(b) An operator T is called *chaotic* if it is hypercyclic and if the set of periodic points for T is dense in X.

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For an introduction to linear dynamics we refer to the recent textbooks [8] and [43].

The concept of a frequently hypercyclic operator was introduced in 2004 by F. Bayart and S. Grivaux [5], [6]. The idea is to measure the size of the set

$$N(x,U) = \{n \geq 0 : T^n x \in U\}$$

of powers of T that send a given vector x into a non-empty open set $U \subset X$. For a vector x with a dense orbit, $N(x,U)$ is non-empty for any such set U, and therefore automatically infinite. For a periodic point x, $N(x,U)$ may be as large as a set of the form $\{n_0+kp : k \geq 0\}$ for *some* sets U, but at the price that the orbit misses completely some others. For frequent hypercyclicity, Bayart and Grivaux demand that an orbit meets *every* non-empty open set *often* – in the sense of positive lower density.

Definition 2. An operator T is called *frequently hypercyclic* if there exists some $x \in X$ such that, for any non-empty open set $U \subset X$,

$$\underline{\text{dens}}\{n \geq 0 : T^n x \in U\} > 0.$$

In this case, x is called a *frequently hypercyclic vector* for T. The set of frequently hypercyclic vectors is denoted by $FHC(T)$.

We recall that for a set $A \subset \mathbb{N}_0$,

$$\underline{\text{dens}}\, A = \liminf_{N\to\infty} \frac{1}{N+1} \text{card}\{0 \leq n \leq N : n \in A\}.$$

There are two approaches to frequent hypercyclicity that have been studied in tandem right from the start: a topological approach and a probabilistic, that is ergodic theoretic, approach. These will be discussed in Sections 2 and 3. The subsequent sections follow roughly a chronological order. In Section 4 we ask if any infinite-dimensional separable Banach space admits a frequently hypercyclic operator, in Section 5 we consider the rate of growth of frequently hypercyclic entire functions, Section 6 is concerned with so-called frequently hypercyclic subspaces, and in Section 7 we report on the relationship between frequent hypercyclicity and linear chaos. Throughout the text we will draw attention to some open problems; additional ones will be listed in the final Section 8.

2 Topological approach to frequent hypercyclicity

The first question that comes to mind is if there exist orbits that satisfy the rather strong condition of frequent hypercyclicity. In fact, as we will discuss below, a Baire category approach as in hypercyclicity is not at our disposal. However, in some cases one may obtain frequently hypercyclic orbits by a (countable) constructive procedure. Such a construction is feasible if the operator satisfies the so-called Frequent Hypercyclicity Criterion, see [5], [6], [27].

Theorem 1 (Frequent Hypercyclicity Criterion)**.** *Let T be an operator on a separable F-space. Suppose that there is a dense subset X_0 of X and a map $S : X_0 \to X_0$ such that, for each $x \in X_0$,*

(i) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally,*

(ii) $\sum_{n=0}^{\infty} S^n x$ *converges unconditionally,*

(iii) $TSx = x$.

Then T is frequently hypercyclic and chaotic.

Any new notion in linear dynamics is first tested on the weighted backward shifts. Let $w = (w_n)$ be a bounded sequence of non-zero scalars. Then the (unilateral) weighted backward shift B_w is defined on $X = \ell^p$, $1 \leq p < \infty$, or c_0 by

$$B_w(x_n) = (w_2x_2, w_3x_3, w_4x_4, \ldots).$$

Now, if

$$\Big(\frac{1}{|w_2 \cdots w_n|}\Big)_n \in X, \tag{1}$$

then one easily sees that B_w satisfies the Frequent Hypercyclicity Criterion and is therefore frequently hypercyclic. One need only take for X_0 the set of finitely non-zero sequences and for S the forward shift $S(x_n) = (0, x_1/w_2, x_2/w_3, x_3/w_4, \ldots)$. Incidentally, condition (1) characterizes when B_w is chaotic (see [40]).

One naturally wonders which condition on the weights characterizes frequent hypercyclicity of B_w. For $X = c_0$, it was shown by Bayart and Grivaux [7] that there are frequently hypercyclic weighted shifts that are not chaotic and therefore do not satisfy (1). Bayart and Ruzsa [10] recently gave a characterization, which, however, is rather technical.

The counter-example of Bayart and Grivaux also shows that not every frequently hypercyclic operator can satisfy the Frequent Hypercyclicity Criterion (since that criterion implies chaos). Even more, Badea and Grivaux [3] have constructed an operator that is frequently hypercyclic and chaotic while failing to be mixing (the latter says that, for any non-empty open sets $U, V \subset X$, there is some $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$ for all $n \geq N$). Since the conditions of the Frequent Hypercyclicity Criterion also imply that T is mixing (see [43, Theorem 3.4]) we see that not even every frequently hypercyclic and chaotic operator needs to satisfy the Frequent Hypercyclicity Criterion. Thus one is naturally lead to the following.

Problem 1. (a) *Which (strong) dynamical behaviour does the Frequent Hypercyclicity Criterion characterize?*

(b) *Does every chaotic, mixing and frequently hypercyclic operator satisfy the Frequent Hypercyclicity Criterion?*

Let us return to the characterization of frequently hypercyclic weighted shifts. The results for c_0 suggest that a characterizing condition is necessarily also complicated for the spaces ℓ^p. Surprisingly, this is not the case. Bayart and Ruzsa [10] have been able to show that the sufficient condition (1) is also necessary. Their proof is based on an improvement of the well-known result of Erdős and Sarközy on difference sets.

Theorem 2 (Bayart-Ruzsa)**.** *Let B_w be a weighted backward shift on $X = \ell^p$, $1 \leq p < \infty$. Then the following assertions are equivalent:*

(i) *B_w is frequently hypercyclic,*

(ii) *B_w satisfies the Frequent Hypercyclicity Criterion,*

(iii)

$$\sum_{n=2}^{\infty} \frac{1}{|w_2 w_3 \cdots w_n|^p} < \infty$$

($\Longleftrightarrow B_w$ *is chaotic*).

Let us briefly mention that frequently hypercyclic operators are also found among other classes of operators. For example, on the space $H(\mathbb{C})$ of entire functions the differentiation operator $D : H(\mathbb{C}) \to H(\mathbb{C})$, $Df = f'$, and the translation operator $T : H(\mathbb{C}) \to H(\mathbb{C})$, $Tf(z) = f(z+1)$, are frequently hypercyclic [6]. More generally, any non-scalar operator T on $H(\mathbb{C})$ that commutes with D is frequently hypercyclic [25]. For the frequent hypercyclicity of functions of weighted backward shifts, as well as differentiation operators, (weighted) composition operators and adjoint multipliers on Banach (or Fréchet) spaces of analytic or harmonic functions we refer to [6], [7], [15], [22], [23], [18], [50], [19], [45], [52]. The proof of frequent hypercyclicity in these papers often relies on a probabilistic approach, to which we turn in the next section.

In the study of (ordinary) hypercyclicity, the Baire category theorem enters in an unspectacular way. By definition, the set of hypercyclic vectors for an operator is given by

$$HC(T) = \bigcap_{\varnothing \neq U \text{ open}} \bigcup_{n=1}^{\infty} T^{-n}(U).$$

Now, if T is hypercyclic then every vector along a dense orbit is also hypercyclic, so that $HC(T)$ is dense. Moreover, since the space X has to be separable, the above intersection can be reduced to a countable one. This shows that $HC(T)$ is a dense G_δ-set, hence residual, as soon as T is hypercyclic. The importance of this simple observation for the study of hypercyclic operators cannot be overestimated.

A direct consequence is that every vector in X is the sum of two hypercyclic vectors for T:

$$X = HC(T) + HC(T).$$

Indeed, for any $x \in X$, the two residual sets $x - HC(T)$ and $HC(T)$ need to intersect.

It is natural to wonder if Baire continues to have the same impact on frequent hypercyclicity. The answer is a resounding 'no'. It was already observed in [6] and [27] that for many frequently hypercyclic operators the set

$$FHC(T)$$

of frequently hypercyclic vectors for T is of first Baire category. In [27], [41] the authors asked if the set $FHC(T)$ is always of first Baire category. The positive answer was recently given independently by Bayart-Ruzsa [10] (for Banach spaces) and by Moothathu [49] and Grivaux-Matheron [38].

Theorem 3 (Moothathu, Bayart-Ruzsa, Grivaux-Matheron)**.** *For any operator T on a separable F-space the set $FHC(T)$ of frequently hypercyclic vectors is of first Baire category.*

We give here a slight modification of the proof in [49].

Proof. Let $\|\cdot\|$ denote an F-norm that defines the topology of X, see [27]. Note, in particular, that for an F-norm we have that $\|cx\| \leq n\|x\|$ if $c \in \mathbb{K}$ and $n \in \mathbb{N}$ with $|c| \leq n$.

(I) We first show that there exists a non-empty open set V such that the sets

$$V,\, 2V,\, 2^2V,\, 2^3V, \dots$$

are pairwise disjoint. Indeed, for an arbitrary vector $x \in X \setminus \{0\}$ we consider the open ball $V = B(x, \frac{\|x\|}{4})$. It then suffices to show that, for all $n \geq 1$, $V \cap 2^nV = \varnothing$. Now, if $x + w = 2^n(x + \widetilde{w}) \in V \cap 2^nV$ then $(1 - 2^{-n})x = 2^{-n}w - \widetilde{w}$. Since $\frac{1}{1-2^{-n}} \leq 2$ we deduce that

$$\|x\| = \|\tfrac{1}{1-2^{-n}}(1 - 2^{-n})x\| \leq 2\|(1 - 2^{-n})x\| = 2\|2^{-n}w - \widetilde{w}\| < 2 \cdot 2\frac{\|x\|}{4} = \|x\|,$$

which is impossible.

(II) We now show that $FHC(T)$ is of first Baire category. Let U be a non-empty open set whose closure $\overline{U}$ is contained in a set V as given by (I). Then

$$FHC(T) \subset \{x \in X : \underline{\text{dens}}\{k \geq 0 : T^kx \in U\} > 0\} \subset \bigcup_{m=1}^{\infty} \bigcup_{N=0}^{\infty} A_{m,N},$$

where

$$A_{m,N} = \{x \in X : \text{for all } n \geq N,\ \tfrac{1}{n+1}\text{card}\{0 \leq k \leq n : T^kx \in \overline{U}\} \geq \tfrac{1}{m}\}.$$

It is easy to see that each set $A_{m,N}$ is closed.

To finish the proof it suffices to show that each set $A_{m,N}$ has empty interior. Suppose that this is not the case for some $m \geq 1$, $N \geq 0$. We may suppose that the operator T is hypercyclic; let x be a corresponding hypercyclic vector. Since, for any $j \geq 0$, also $2^{-j}x$ is hypercyclic and since $A_{m,N}$ has non-empty interior we can find some $n_j \geq 0$ such that

$$T^{n_j}(2^{-j}x) \in A_{m,N},$$

which implies that there is some $N_0 \geq N$ such that, for any $0 \leq j \leq 2m$,

$$\tfrac{1}{N_0+1}\text{card}\{0 \leq k \leq N_0 : T^kx \in 2^j\overline{U}\} \geq \tfrac{1}{2m}.$$

This is clearly impossible since the sets $\overline{U}$, $2\overline{U}$, $2^2\overline{U}$, $2^3\overline{U}, \dots$ are pairwise disjoint. □

In the light of this result, and unlike for hypercyclic operators, we can no longer be sure that

$$X = FHC(T) + FHC(T).$$

Indeed, Bonilla and the author [27] found that under mild conditions on the frequently hypercyclic operator T,

$$X \neq FHC(T) + FHC(T).$$

This is true, for example, for the multiples λB ($|\lambda| > 1$) of the backward shift on the spaces ℓ^p, $1 \leq p < \infty$, or c_0, and for the differentiation operator D on the space $H(\mathbb{C})$ of entire functions. However, the translation operator $T : H(\mathbb{C}) \to H(\mathbb{C})$, $Tf(z) = f(z+1)$ satisfies

$$H(\mathbb{C}) = FHC(T) + FHC(T).$$

This is, in the final analysis, due to the flexibility of the Runge approximation theorem. It is not clear if the same can happen in a Banach space. The following seems to be still open, see [27], [41].

Problem 2. *Is there an operator T on a Banach space X for which $X = FHC(T) + FHC(T)$ holds?*

3 Probabilistic approach to frequent hypercyclicity

In this section we will only briefly touch upon what is in fact the deepest part of the theory of frequently hypercyclic operators: the use of probabilistic techniques. After pioneering work by Flytzanis [33], a rich edifice of research has been erected by F. Bayart, S. Grivaux, and E. Matheron, see [6], [36], [7], [8], [37], [38] and [9].

We will limit ourselves here to stating what seems to us to be the most pertinent results, without defining the relevant notions of ergodic theory and of measure theory on Fréchet spaces. A starting point for the interested reader might be the earlier survey of the author, [41], and Chapters 5 and 6 of the book of Bayart and Matheron [8]. A recent survey article by Bayart [4] is also highly recommended.

The use of ergodic theoretic techniques in linear dynamics goes back to Flytzanis [33]. He noted that if an operator T on a separable Hilbert space admits an ergodic measure of full support then the operator is hypercyclic. He then showed that, under certain assumptions on the operator T, a sufficiently large supply of unimodular eigenvectors (that is, eigenvectors to eigenvalues of modulus 1) will ensure the existence of a T-ergodic measure of full support. This has set the tone for all the research that followed.

Now, Bayart and Grivaux [5], [6] noted that in view of the Birkhoff ergodic theorem an ergodic measure of full support for T even leads to the operator being frequently hypercyclic. In fact, this observation motivated the notion of frequent hypercyclicity.

With their work, Bayart, Grivaux and Matheron have shed considerable light on the link between the existence of ergodic measures of full support, the set of unimodular eigenvectors, and the frequent hypercyclicity of an operator. Gaussian measures play a prominent (but not exclusive) role in these investigations. In the sequel, let $\mathbb{T}$ denote the unit circle.

The following has evolved over the course of an intensive ten years of research, see [6], [36], [7], [8], [37], [38] and [9].

Theorem 4 (Bayart, Grivaux, Matheron)**.** *Let T be an operator on a separable complex Fréchet space. Consider the following:*

(a) *the unimodular eigenvectors are* perfectly spanning*, that is, for any countable set $D \subset \mathbb{T}$ the linear span of $\bigcup_{\lambda\in\mathbb{T}\setminus D} \ker(T-\lambda)$ is dense in X;*

(b) *there exists an ergodic Gaussian measure of full support for T;*

(c) *T is frequently hypercyclic.*

Then (a)$\Longrightarrow$(b)$\Longrightarrow$(c)*. If X is a Hilbert space or, more generally, a Banach space of cotype 2 then* (a)$\Longleftrightarrow$(b)*.*

There are some Banach spaces where condition (b) does not imply (a), see [7].

For applications it is useful to note that condition (a) is implied by (and in fact equivalent to) the existence of perfect sets $\Lambda_j \subset \mathbb{T}$ ($j \in \mathbb{N}$) and continuous maps $E_j : \Lambda_j \to X$ such that $TE_j(\lambda) = \lambda E_j(\lambda)$ for all $\lambda \in \Lambda_j$ ($j \in \mathbb{N}$) and such that the span of $\bigcup_{j\in\mathbb{N}} E_j(\Lambda_j)$ is dense in X; see [9], [38].

In particular, for weighted backward shifts B_w on $X = \ell^p$, $1 \leq p < \infty$, conditions (a), (b) and (c) are equivalent, and they hold if and only if

$$\sum_{n=2}^{\infty} \frac{1}{|w_2 w_3 \cdots w_n|^p} < \infty;$$

see [9].

It is also interesting to note that every operator on a separable complex Fréchet space that satisfies the Frequent Hypercyclicity Criterion admits an ergodic Gaussian measure of full support, which then implies its frequent hypercyclicity, see [9]. If one is not necessarily demanding a Gaussian measure, a simpler proof of this fact was obtained by Murillo and Peris [51], which even works for (real or complex) F-spaces.

The investigations that were started by Bayart and Grivaux in [5], [6] and that culminated in Theorem 4 lead naturally to two questions: can frequent hypercyclicity only arise in the presence of an ergodic (Gaussian) measure of full support? Can it only arise under the existence of sufficiently many unimodular eigenvectors? Both questions have a negative answer: there is a frequently hypercyclic operator on c_0 that has no unimodular eigenvalues and that does not admit any invariant Gaussian measure of full support, see [7]. The latter was substantially improved by Grivaux and Matheron [38] who showed that there is even a frequently hypercyclic operator on $c_0(\mathbb{Z})$ that does not admit any ergodic measure of full support. In fact, both operators are weighted backward shifts.

The best positive result so far is the following, see [38].

Theorem 5 (Grivaux-Matheron)**.** *Let T be a frequently hypercyclic operator on a reflexive Banach space. Then T admits a continuous invariant measure of full support.*

But several natural questions have so far remained open [7], [38], [9].

Problem 3. (a) *Does every frequently hypercyclic operator on an arbitrary Banach space admit an invariant measure of full support?*

(b) *Does every frequently hypercyclic operator on a Hilbert space admit an ergodic measure of full support?*

(c) *Does every frequently hypercyclic operator on a Hilbert space admit an ergodic Gaussian measure of full support?*

(d) *Does every frequently hypercyclic operator on a Hilbert space have unimodular eigenvalues?*

Questions (b)*,* (c) *and* (d) *may also be asked, more generally, for arbitrary reflexive Banach spaces.*

4 Existence of frequently hypercyclic operators, and the spectrum

By an important result of Ansari [1] and Bernal [14], every infinite-dimensional separable Banach space admits a hypercyclic operator. Bonet, Martínez and Peris [24] (see also [8, Theorem 6.36]) have subsequently shown that this result breaks down for chaos: no hereditarily indecomposable complex Banach space supports a chaotic operator. Recall that a Banach space is called hereditarily indecomposable if none of its closed subspaces is decomposable as a direct sum of infinite-dimensional closed subspaces. Such spaces were first constructed by Gowers and Maurey [35].

Shkarin [56] showed that, as for existence, frequent hypercyclicity behaves like chaos. For this, he first studied the spectrum of frequently hypercyclic operators.

Theorem 6 (Shkarin)**.** *Let T be a frequently hypercyclic operator on a complex Banach space. Then its spectrum $\sigma(T)$ has no isolated points.*

The proof uses the theory of entire functions in an ingenious way. It now follows from Shkarin's result and the Riesz theory that no operator of the form

$$T = \lambda I + K, \quad K \text{ compact}$$

can be frequently hypercyclic. But the celebrated Argyros-Haydon theorem [2] shows that there are separable complex Banach spaces on which *every* operator is of this form. Thus, Argyros-Haydon spaces do not admit frequently hypercyclic operators. More generally, Shkarin [56] proved the following analogue of the Bonet-Martínez-Peris result on chaotic operators.

Theorem 7 (Shkarin)**.** *No hereditarily indecomposable complex Banach space supports a frequently hypercyclic operator.*

In a way, the following is the positive counterpart of the results of Bonet, Martínez, Peris and Shkarin, see [30].

Theorem 8 (de la Rosa-Frerick-Grivaux-Peris)**.** *Every infinite-dimensional separable complex Banach space with an unconditional Schauder decomposition admits an operator that is frequently hypercyclic and chaotic.*

Still, the following problem is open, see [30].

Problem 4. *Which Banach spaces admit a frequently hypercyclic operator? Which Banach spaces admit a chaotic operator?*

We will return to this question in the light of a very recent significant advance in frequent hypercyclicity, see Section 7.

Independently of the question of existence one may be interested in identifying the sets $K \subset \mathbb{C}$ that can arise as spectra of frequently hypercyclic operators. Shkarin [56] has shown that a non-empty compact subset K of $\mathbb{C}$ is the spectrum of some hypercyclic operator on a complex Hilbert (or Banach) space if and only if each of its connected components meets the unit circle. By Theorem 6 one needs to add the absence of isolated points in the case of frequent hypercyclicity. Is that enough? This was wrongly claimed in [43, Theorem 9.43], and the authors of that book take full responsibility for the blunder. Beise [12] recently showed that the whole truth is more complicated.

Theorem 9 (Beise)**.** *Let C be a closed and open component of the spectrum of an operator T on a complex Banach space. If $C \subset \{z \in \mathbb{C} : |z| \leq 1\}$ and $C \cap \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\alpha_1}, \ldots, e^{i\alpha_n}\}$, where $\alpha_1, \ldots, \alpha_n$ are linearly independent over the field of rational numbers, then T is not frequently hypercyclic.*

For example, since frequent hypercyclicity is preserved under rotation $T \to \lambda T$, $|\lambda| = 1$, ([8]), no connected compact set $K \subset \{z \in \mathbb{C} : |z| \leq 1\}$ that meets the unit circle in a single point can be the spectrum of a frequently hypercyclic operator. This excludes, in particular, the set $K = [0, 1]$. It is interesting to note that Beise's proof links the dynamics of an arbitrary operator with the dynamics of the translation operator on the space of entire functions.

Thus we are left with the following, see [56].

Problem 5. *Which compact subsets of $\mathbb{C}$ can be the spectrum of a frequently hypercyclic operator on a complex Banach space?*

5 Rate of growth of frequently hypercyclic entire functions

The differentiation operator

$$D : H(\mathbb{C}) \to H(\mathbb{C}), \quad Df = f'$$

is one of the classical hypercyclic operators. The author [39] and Shkarin [55] independently obtained a sharp result on the possible rates of growth of corresponding hypercyclic entire functions: for any function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(r) \to \infty$ as $r \to \infty$ there is some D-hypercyclic entire function f such that

$$|f(z)| = O\Big(\varphi(r)\frac{\exp(r)}{r^{1/2}}\Big) \quad \text{as } |z| = r \to \infty, \tag{2}$$

while there is no D-hypercyclic entire function f that satisfies

$$|f(z)| = O\Big(\frac{\exp(r)}{r^{1/2}}\Big) \quad \text{as } |z| = r \to \infty.$$

Now, D is even frequently hypercyclic [6]. So, how about the corresponding possible rates of growth? Blasco, Bonilla and the author [22] showed that, for any function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(r) \to 0$ as $r \to \infty$ there is no D-frequently hypercyclic entire function f that satisfies

$$|f(z)| = O\Big(\psi(r)\frac{\exp(r)}{r^{1/4}}\Big) \quad \text{as } |z| = r \to \infty.$$

In a positive direction they were only able to obtain a rate of growth as in (2) with $r^{1/2}$ replaced by 1. The question of the optimal rate, see [22], [26], was settled by Drasin and Saksman [32].

Theorem 10 (Drasin-Saksman)**.** *There is a D-frequently hypercyclic entire function f such that*

$$|f(z)| = O\Big(\frac{\exp(r)}{r^{1/4}}\Big) \quad \textit{as } |z| = r \to \infty.$$

They obtained such an entire function by an explicit construction and complex analytic tools without any functional analytic machinery. Interestingly, Nikula [53] found a probabilistic approach to a result that is only slightly weaker than Drasin and Saksman's.

Theorem 11 (Nikula)**.** *Let $(X_n)_{n\geq 0}$ be a sequence of independent identically distributed complex random variables whose law has support $\mathbb{C}$ and such that, for some $a > 0$,*

$$E(e^{t|X_n|}) = O(e^{at^2}) \quad \textit{as } t \to \infty.$$

Then the random power series

$$f(z) = \sum_{n=0}^{\infty} \frac{X_n}{n!} z^n$$

almost surely represents a D-frequently hypercyclic entire function that satisfies

$$|f(z)| = O\Big(\sqrt{\log r}\,\frac{\exp(r)}{r^{1/4}}\Big) \quad \textit{as } |z| = r \to \infty.$$

Mouze and Munnier [50] develop the ideas of Nikula and combine them with the Birkhoff ergodic theorem to obtain random frequently hypercyclic vectors for various operators.

The growth conditions considered so far are radial. Beise and Müller [13] have initiated the study of growth conditions along rays emanating from the origin. In particular they obtain the following.

Theorem 12 (Beise-Müller)**.** *Let $K \subset \mathbb{C}$ be a convex compact set whose intersection with the unit circle contains a continuum. Then there is an entire function of exponential type that is D-frequently hypercyclic and such that, for any $\varepsilon > 0$,*

$$|f(z)| = O\big(\exp(H_K(z) + \varepsilon|z|)\big)$$

for all $z \in \mathbb{C}$; here, $H(K) = \sup\{\mathrm{Re}(zu) : u \in K\}$ is the support function of K.

For example, a D-frequently hypercyclic function can be of exponential type 1 and tend to zero exponentially in the sector $\{z : |\arg(z)| \geq \alpha\}$, with $\alpha > \frac{\pi}{2}$ fixed.

We mention that growth rates in terms of L^p-averages and growth rates for entire functions that are frequently hypercyclic for the translation operator $Tf(z) = f(z+1)$ can be found in [22], [23], [32] and [11]. In [13] the authors consider arbitrary operators that commute with D and obtain corresponding non-radial growth rates.

6 Frequently hypercyclic subspaces

We have seen in Section 2 that the set $FHC(T)$ of frequently hypercyclic vectors for an operator T is always of first Baire category. But can this set nonetheless be *large* in an algebraic sense? One of the fundamental results in linear dynamics, due to Herrero [44] and Bourdon [29], states that the set $HC(T)$ contains a dense linear subspace of X (except, of course, the zero vector) as soon as T is hypercyclic. Their proof leads immediately to the same result for $FHC(T)$ if T is frequently hypercyclic, see Bayart and Grivaux [6]. Indeed, if x is a frequently hypercyclic vector for T then span $\mathrm{orb}(x, T)$ is a dense linear subspace contained in $FHC(T) \cup \{0\}$.

The following, however, seems to be open.

Problem 6. *Can the set $FHC(T) \cup \{0\}$ be a linear subspace of X if T is frequently hypercyclic?*

For hypercyclicity, the corresponding problem has a positive answer: by a deep result of Read [54] there exists an operator T on ℓ^1 for which every non-zero vector is hypercyclic, so that $HC(T) \cup \{0\} = X$. But by Section 2 the set $FHC(T) \cup \{0\}$ is always a proper subset of X.

Bernal and Montes [16], [48] studied a different notion of largeness: for a given operator T, does the set $HC(T)$ contain an infinite-dimensional closed subspace (except 0)? They showed that for some hypercyclic operators the answer is positive, while for others it is negative.

Such a subspace is nowadays called a *hypercyclic subspace*, and the set $HC(T)$ is called *spaceable*. We refer to an excellent recent survey by Bernal, Pellegrino and Seoane [17] on spaceability, lineability and related topics.

González, León and Montes [34] obtained the following characterization: if an operator T on a separable complex Banach spaces satisfies the Hypercyclicity Criterion (see [43, Theorem 3.12]) then it possesses a hypercyclic subspace if and only if there exists an infinite-dimensional closed subspace M_0 of X and an increasing sequence (n_k) of positive integers such that

$$T^{n_k}x \to 0 \quad \text{for all } x \in M_0.$$

Now, Bonilla and the author [28] introduced the corresponding notion of a frequently hypercyclic subspace, that is, an infinite-dimensional closed subspace consisting (except 0) of frequently hypercyclic vectors.

Theorem 13 ([28]). *Let X be a separable F-space with a continuous norm and T an operator on X. If*

(i) (a) *T satisfies the Frequent Hypercyclicity Criterion, or*
(b) *X is a complex Banach space, and the unimodular eigenvectors for T are perfectly spanning (see Section 3),*

(ii) *there exists an infinite-dimensional closed subspace M_0 of X such that*

$$T^n x \to 0 \quad \textit{for all } x \in M_0,$$

then T possesses a frequently hypercyclic subspace, that is, $FHC(T)$ is spaceable.

The assumption of the existence of the subspace M_0 is rather strong. To give a concrete example we note that the operator T on $C_0(\mathbb{R}_+)$, $Tf(x) = \lambda f(x+a)$ ($\lambda > 1$, $a > 0$) is easily seen to satisfy the assumptions of the theorem and thus possess a frequently hypercyclic subspace.

Menet [46] recently noted that if T satisfies the Frequent Hypercyclicity Criterion then condition (ii) may be weakened to the following: there exists an infinite-dimensional closed subspace M_0 of X and a set $A \subset \mathbb{N}$ of positive lower density such that

$$T^n x \underset{n\in A}{\longrightarrow} 0 \quad \text{for all } x \in M_0.$$

This allowed him to show that certain weighted backward shifts B_w on the space ℓ^p, $1 \le p < \infty$, possess a frequently hypercyclic subspace. But we are still very far from an analogue of the characterization of González, León and Montes, even for weighted shifts. Thus we have the following, see [46].

Problem 7 (Menet). (a) *Characterize the weighted backward shifts B_w on ℓ^p, $1 \le p < \infty$, that possess a frequently hypercyclic subspace.*

(b) *More generally, characterize the operators on a separable (complex) Banach space that possess a frequently hypercyclic subspace.*

Menet [46] also found a necessary condition for an operator to possess a frequently hypercyclic subspace. This enabled him to solve affirmatively a problem posed in [28].

Theorem 14 (Menet). *There exists a frequently hypercyclic operator that possesses a hypercyclic subspace but not a frequently hypercyclic subspace.*

Such an operator can be taken to be a weighted backward shift B_w on ℓ^p, $1 \le p < \infty$. Some related results can be found in Bès and Menet [20].

7 Frequent hypercyclicity versus linear chaos

Since the introduction of frequent hypercyclicity by Bayart and Grivaux in 2004 the relationship of this notion with linear chaos has intrigued the researchers. It became quickly clear that the two concepts are not equivalent. By an intricate construction, Bayart and Grivaux [7] were able to show that there is a weighted backward shift on c_0 that is frequently hypercyclic but not chaotic. In fact, the shift does not even possess a single non-trivial periodic point and, worse, not even a unimodular eigenvalue. Bayart and Grivaux also found a non-chaotic frequently hypercyclic operator on a Hilbert space. But the following remained open, see [7].

Problem (Bayart-Grivaux)**.** *Is every chaotic operator frequently hypercyclic?*

This problem has since been iterated in various articles, and it has come to be considered as one of the major questions in linear dynamics; see, for example, [8, Chapter 6] and [43, Chapter 9]. Menet [47] very recently obtained a counter-example.

Theorem 15 (Menet)**.** *There exists a chaotic operator that is not frequently hypercyclic.*

More precisely, he constructed such an operator on the spaces ℓ^p, $1 \leq p < \infty$, and c_0 as a perturbation of (essentially) a weighted forward shift by an upper-triangular matrix with very few entries.

On the positive side, Menet [47] showed that every chaotic operator satisfies a weak form of frequent hypercyclicity, called reiterative hypercyclicity, where the lower density is replaced by upper Banach density, see [21].

In view of the non-existence of chaotic or of frequently hypercyclic operators in certain Banach spaces one may be interested in the following, see [47]:

Problem 8 (Menet)**.** *Are there Banach spaces that support a chaotic operator but no frequently hypercyclic operator, or vice versa?*

8 Further problems

We end this paper by recalling two major open problems in the theory of frequent hypercyclicity; they have been posed by Bayart and Grivaux [6].

First, it is well known, and easily proven by a Baire category argument, that if T is an invertible hypercyclic operator then T^{-1} is also hypercyclic. But since Baire loses its power in frequent hypercyclicity the following needs to be attacked by different means.

Problem 9. *If T is an invertible frequently hypercyclic operator, is then also T^{-1} frequently hypercyclic?*

Bayart and Ruzsa [10] have shown that T^{-1} enjoys a weak form of frequent hypercyclicity, called $\mathcal{U}$-frequent hypercyclicity.

For a long time Herrero's question if the direct sum $T \oplus T$ of a hypercyclic operator T is also hypercyclic was a driving force behind much of the research in linear dynamics. As Bayart and Grivaux [6, p. 5085] say themselves, this question motivated their introduction of the notion of frequent hypercyclicity. Herrero's question was answered in the negative by De la Rosa and Read [31]. The corresponding problem for frequent hypercyclicity, however, remains open.

Problem 10. *If T is a frequently hypercyclic operator, is then also $T \oplus T$ frequently hypercyclic?*

The intermediate problem if $T \oplus T$ is hypercyclic whenever T is frequently hypercyclic ([6]) was answered positively by Peris and the author [42].

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