The model theory of *m*-ordered differential fields

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Key words m-ordered differential fields, model companion, geometric axiomatization.
MSC (2000) XXXX

In his PhD. thesis ([7]), L. van den Dries studied the model theory of fields (more precisely domains) with finitely many orderings and valuations where all open sets according to the topology defined by an order or a valuation is globally dense according with all other orderings and valuations. Van den Dries proved that the theory of these fields is companionable and that the theory of the companion is decidable (see also [8]).
In this paper we study the case where the fields are expanded with finitely many orderings and an independent derivation. We show that the theory of these fields still admits a model companion in the language ${L_{<,m}}^D = \{+,-,\cdot,D,<_1,\ldots,<_m,1,0\}$. We denote this model companion by $CODF_m$ and give a geometric axiomatization of this theory which uses basic notions of algebraic geometry and some generalized open subsets which appear naturally in this context. This axiomatization allows to recover (just by putting $m=1$) the one given in [4] for the theory *CODF* of closed ordered differential fields. Most of the technics we use here are already present in [2] and [4].
Finally, we prove that it is possible to describe the completions of $CODF_m$ and to obtain quantifier elimination in a slightly enriched (infinite) language. This generalizes van den Dries's results in the "derivation free" case.

1 Preliminaries

1.1 The work of van den Dries

In order to make this paper as self-contained as possible, we briefly recall the results of [7].

Let $m \in \mathbb{N}$, a ***m*-ordered field** M is a field equipped with m orderings $<_1,\ldots,<_m$.
Any such field is a $L_{<,m}$-structure where $L_{<,m} := \{+,\cdot,-,<_1,\ldots,<_m,0,1\}$. We denote by OF_m the $L_{<,m}$-theory of *m*-ordered fields.

Fact 1.1.1 *(*[7, Ch. II, Theorem (1.2)]*): OF_m has a model companion COF_m whose models M satisfy:*

(1) *OF_m;*

(2) *the orderings on M are pairwise independent*[1]*;*

(3) *for any irreducible $f(X,Y) \in M[X,Y]$ and any $a \in M$ such that $f(a,Y)$ changes sign on M w.r.t. to each ordering, there exists $(c,d) \in M^2$ with $f(c,d) = 0$.*

The independence of the orderings leads to some interesting consequences.
Let M be a model of COF_m, a subset U of M^k is **m-open** if $U = U_1 \cap \ldots \cap U_m$ where, for each $i \in \{1,\ldots,m\}$, U_i is a $<_i$-open[2] subset of M^k.

Fact 1.1.2 *(*[7, Ch. III,§1]*): Any m-open subset of M is infinite. Furthermore, the m-open subsets of M form a new topology τ_m on M. The basis of τ_m is given by the subsets $U_1 \cap \ldots \cap U_m$ where each U_i is a basic $<_i$-open subset of M. This topology is Hausdorff and not discrete.*

The author was supported by a FRIA grant during his PhD thesis at the University of Mons-Hainaut and is now supported by a MODNET grant.
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1 This means that they induce different interval topologies on M.
2 A subset of M^k is **$<_i$-open** if it is open for the topology induced by $<_i$.

In [7, Ch. II, Proposition (1.20)] van den Dries showed that, in the case where $m \geq 2$, COF_m is not complete and is not the model completion of OF_m.
Nevertheless he gave a classification of the complete extensions of COF_m that we shortly describe below.
For any field M, let $\mathbf{alg}(\mathbf{M}) := \{a \in M \mid a$ is algebraic over the prime subfield of $M\}$ and denote by $M \models \mathbf{OF_{m,alg}}$ the fact that there exists an extension $N \models COF_m$ of M such that M is algebraically closed in N. The theory $OF_{m,alg}$ is first-order axiomatizable by a list of sentences asserting that each polynomial in $M[X]$ with odd degree has a root in M and that any $a \in M$ which is positive w.r.t. each ordering on M has a square root in M (see [7, Ch. II, Lemma (2.6)]).

Fact 1.1.3 *(*[7, Ch. II, (2.8)]*): Two models $(M, <_1, \ldots, <_m)$ and $(N, \tilde{<}_1, \ldots, \tilde{<}_m)$ of COF_m are $L_{<,m}$-elementary equivalent iff $(alg(M), <_1, \ldots, <_m)$ and $(alg(N), \tilde{<}_1, \ldots, \tilde{<}_m)$ are $L_{<,m}$-isomorphic*[3].

In order to get a model completion for OF_m (and hence quantifier elimination), van den Dries extended the theory COF_m by adding to the language new predicate symbols $\mathbf{W_{d,k_1,\ldots,k_m}}$ for $d \geq 2, 1 \leq k_i \leq d$ and, as defining axioms, the following list:

$$\forall X_1, \ldots, X_d \ \big(W_{d,k_1,\ldots,k_m}(X_1, \ldots, X_d) \Leftrightarrow \exists Y \bigwedge_i (\varphi_{d,k_i}(<_i, Y, X_1, \ldots, X_d))\big)$$

where $\varphi_{d,k_i}(<_i, Y, X_1, \ldots, X_d)$ is a quantifier free $L_{<,m}$-formula such that, for any model M of OF_m and any $a_1, \ldots, a_d, b \in M$, $\varphi_{d,k_i}(<_i, b, a_1, \ldots, a_d)$ holds iff b is the ${k_i}^{th}$ root of $Y^d + a_1 Y^{d-1} + \cdots + a_d$ in the real closure of M w.r.t. the ordering $<_i$.
Let us denote by $\overline{\mathbf{COF_m}}$ the theory obtained from COF_m by adding this list of axioms and by $\overline{\mathbf{L_{<,m}}}$ the new language with added predicate symbols $W_{d,k_1,\ldots,k_m}$.

Fact 1.1.4 *(*[7, Ch. II, (2.14)]*): The theory $\overline{COF_m}$ has quantifier elimination in $\overline{L_{<,m}}$.*

1.2 Algebraic geometry and semi-generic points

We now give a list of definitions and results from algebraic geometry that we will need in the next section.
We consider here a field M of characteristic zero in a language L extending the usual language of rings and we assume that M is equipped with a derivation D. We will also denote by L^D the language $L \cup \{D\}$ where D is a function symbol interpreted as the derivation. We finally fix a sufficiently saturated elementary L-extension N of M and denote by $\tilde{N}$ the algebraic closure of N.
Let J be a prime ideal of $M[X_1, \ldots, X_k]$, the set

$$V(J) := \{\bar{a} \in \tilde{N}^k \mid \forall p \in J,\ p(\bar{a}) = 0\}$$

is called a **M-variety** of $\tilde{N}^k$.
On the other hand, if V is a M-variety the set

$$I(V) := \{p \in M[X_1, \ldots, X_k] \mid p(\bar{a}) = 0 \text{ for all } \bar{a} \in V(J)\}$$

is a prime ideal of $M[X_1, \ldots, X_k]$.
A tuple $\bar{c} \in V$ is a **M-generic point** of V iff the transcendence degree of $M(\bar{c})$ over M is maximal among the points of V. In this case, we define the **dimension** of V, $\dim(V)$, to be this transcendence degree.
Let now $V \subseteq \tilde{N}^k$ be a M-variety and $\bar{a} \in V$. The **torsor** of V at $\bar{a}$ is the set

$$\tau_{\bar{a}}(V) := \{(v_1, \cdots, v_k) \in \tilde{N}^k \mid p^D(\bar{a}) + \sum_{i=1}^{k} \frac{\partial p}{\partial X_i}(\bar{a}) \cdot v_i = 0 \text{ for all } p \in I(V)\},$$

where p^D is the polynomial of $M[X_1, \ldots, X_k]$ obtained by taking the derivative of the coefficients of p.
The **torsor bundle** of V is the set

$$\tau(V) := \{(\bar{a}, \bar{v}) \in \tilde{N}^{2k} \mid \bar{a} \in V \wedge \bar{v} \in \tau_{\bar{a}}(V)\}.$$

[3] Note that we use the same notation for an ordering on M (resp. N) and for its restriction to $alg(M)$ (resp. $alg(N)$).

Remark that, since $M[X_1,\dots,X_k]$ is a Noetherian ring, these two sets are M-definable (i.e. definable with parameters from M) in the usual language of rings.
The following lemma is due to D. Pierce and A. Pillay. It is the key point in all the constructions of geometric axiomatizations for model complete theories of differential fields (see for example [5, 4, 2]).

Lemma 1.2.1
*Let $V \subseteq \tilde{N}^k$ be a M-variety and $\overline{a} \in \tilde{N}^k$ be a M-generic point of V.
For any $\overline{b} \in \tau_{\overline{a}}(V)$, let $W \subseteq \tau(V)$ be the M-variety of which $(\overline{a},\overline{b})$ is a M-generic point.
Then there exist $\overline{c} \in M(\overline{a},\overline{b})^k$ and a derivation $D^\star$ on $M(\overline{a},\overline{b})$ extending D such that $D^\star(\overline{a},\overline{b}) = (\overline{b},\overline{c})$.*

Proof. This lemma follows directly from Theorem (1.1) and Corollary (1.7) in [5]. □

We now recall a definition introduced in [4]:
Let $\overline{a} = (a_1,\dots,a_k) \in \tilde{N}^k$ be a M-generic point of a M-variety V and assume that V has dimension r. Without lost of generality we can assume that $a_1,\dots,a_r$ are algebraically independent over M and $a_{r+1},\dots,a_n$ are algebraic over $K(a_1,\dots,a_r)$.
We then define Q_i to be the polynomial obtained from a minimal polynomial of a_{r+i} over $M(a_1,\dots,a_{r+i-1})$ after replacing $a_1,\dots,a_{r+i-1}$ by the variables $X_1,\dots,X_{r+i-1}$ and cancelling the denominators.
The set of these Q_i are called a system of **canonical semi-generators**[4] of V (associated to $\overline{a}$). Furthermore, we say that a point $\overline{b}$ of V is **semi-generic** if for each $i \in \{1,\dots,k-r\}$,

$$s_{Q_i}(\overline{b}) := \frac{\partial Q_i}{\partial X_{r+i}}(\overline{b}) \neq 0 \text{ and } H(Q_1,\dots,Q_{k-r})(\overline{b}) \neq 0$$

where $H(Q_1,\dots,Q_{k-r})$ is the product of the dominant coefficients of each Q_i when this latter is seen as a polynomial in X_{r+i} with coefficients in $M[X_1,\dots,X_{r+i-1}]$.
Remark that the generic point $\overline{a}$ above is semi-generic.

2 m-ordered fields with a derivation

We now restrict ourselve to the case where M is a m-ordered field equipped with a derivation D which does not interact with the orderings. We denote by ODF_m the theory which consists of the axioms of m-ordered differential fields in the language $L_{<,m}{}^D = \{+,-,\cdot,D,<_1,\dots,<_m,1,0\}$. We will denote this language by L^D for evident typographical convenience $\big(L$ will denote the language $L_{<,m} = \{+,-,\cdot,<_1,\dots,<_m,1,0\}$ of m-ordered rings$\big)$. As previously, N is a sufficiently saturated elementary L-extension of M and $\tilde{N}$ is the algebraic closure of N.

2.1 Geometric axiomatization for the model companion

We are now able to state the main result of this paper.

Theorem 2.1.1
*The theory ODF_m has a model companion, namely the theory $CODF_m$ of closed m-ordered differential fields.
This model companion is axiomatized as follows:
let $(M,D) \models ODF_m$, M is a closed m-ordered differential field if*

(1) $M \models COF_m$;

(2) *for every M-varieties V and $W \subset \tau(V)$ such that W projects generically[5] onto V, if U is a non-empty M-definable m-open subset of W such that $U\mid_M$ contains a semi-generic point of W then $U\mid_M$ contains a point of the form $(\overline{a},D(\overline{a}))$.*

Remark 2.1.2. If $m = 1$ then we recover the geometric axiomatization of $CODF$ given in [4]. In fact, the proof of points (A) and (B) below is a slight adaptation of the proofs of Proposition 1.6 and Theorem 1.7 in [4].

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[4] We use the name semi-generator because the ideal defining V is equal to
$(Q_1,\dots,Q_{k-r}) : H(Q_1,\dots,Q_{k-r})^\infty := \{f \in M[X_1,\dots,X_k] \mid \exists n \in \mathbb{N}\ H(Q_1,\dots,Q_{k-r})^n \cdot f \in (Q_1,\dots,Q_{k-r})\}$.

[5] This means that for any M-generic point $\overline{c}$ of V there exists $\overline{d} \in \tilde{N}^n$ such that $(\overline{c},\overline{d})$ is a M-generic point of W.

In order to prove Theorem 2.1.1 we have to perform two tasks:
(A): to prove that any model of ODF_m extends to a model of $CODF_m$;
(B): to prove that $CODF_m$ is model complete.

To prove (A) we first introduce the following lemma. This result is a direct consequence of the saturation of N and can be seen as a "multi-ordered" analogue of the claim appearing in the proof of [2, Lemma 3.5].

Lemma 2.1.3
For any $l \in \mathbb{N}$, there exist $t_1, \ldots, t_l \in N$ which are infinitesimal over M w.r.t. to each orderings[6] *and algebraically independent over M.*

Proof. Remark first that $t_1, \ldots, t_l$ are infinitesimal over M iff they belong to each M-definable m-open subset of N containing 0.
Furthermore, any such basic m-open subset of N is defined by a L-formula

$$\Theta_M(X) \equiv \bigwedge_{i=1}^{m} \theta_i(X) \ \text{ where each } \theta_i(X) \text{ defines a } <_i\text{-open subset of } M.$$

Hence, we can consider the set $F(X_1, \ldots, X_l)$ of all L-formulas:

$$f(X_1, \ldots, X_l) \neq 0 \wedge \bigwedge_{j=1}^{l} \Theta_M(X_j)$$

where f ranges over $M[X_1, \ldots, X_l]$ and $\Theta_M(X)$ ranges over all the formulas defining a basic m-open subset of M.
By the saturation of N, it is sufficient to prove that each finite conjunction of formulas in $F(X_1, \ldots, X_l)$ is satisfied by a l-tuple of elements of N.
Hence it suffices to prove that there exist $u_1, \ldots, u_l \in N$ such that

$$f(u_1, \ldots, u_l) \neq 0 \wedge \bigwedge_{j=1}^{l} \Theta_M(u_j)$$

where $f(X_1, \ldots, X_l) \in M[X_1, \ldots, X_l]$ is a non-zero polynomial and $\Theta_M(X_j)$ defines a m-open subset of M.
We will proceed by induction on l. Let us remark first that if $l = 1$, the result immediately follows from Fact 1.1.2 since any polynomial $f(X) \in M[X]$ has finitely many roots in N.
Let now $u_1, \ldots, u_{l-1} \in N$ be algebraically independent and infinitesimal over M and consider again a formula

$$f(X_1, \ldots, X_l) \neq 0 \wedge \bigwedge_{j=1}^{l} \Theta_M(X_j).$$

If

$$\forall X_1, \ldots, X_l \in \Theta_M(N) \ \ f(X_1, \ldots, X_l) = 0$$

then, by the inductive hypothesis, $f(v_1, \ldots, v_{l-1}, X_l) \equiv 0$ for any $v_1, \ldots, v_{l-1} \in \Theta_M(M)$. Therefore, since N is an elementary extension of M

$$M \models \forall X_1, \ldots, X_{l-1} \ \Big(\bigwedge_{j=1}^{l-1} \Theta_M(X_j) \to \bigwedge_{i=1}^{d} f_i(X_1, \ldots, X_{l-1}) = 0 \Big)$$

where $f = \sum_{i=1}^{d} f_i {X_l}^i$ for some positive integer d.
Using again the inductive hypothesis, we can deduce that $f_i \equiv 0$ for any $i \in \{1, \ldots, d\}$.
It follows that $f \equiv 0$, a contradiction. □

This result allows us to state the following slight adaptation of [7, Ch. II, Lemma 1.10].

[6] In the sequel we will simply say that an element is infinitesimal over M to denote that it is infinitesimal over M w.r.t. each orderings.

Corollary 2.1.4
Let $M \models OF_m$, $f(\overline{X},Y) \in M[\overline{X},Y]$ be irreducible and $\overline{a} \in M^l$ be such that $f(\overline{a},Y)$ changes signs on M w.r.t. each ordering of M.
Then there exist a m-ordered field $\overline{M}$ extending M and a root $(\overline{c},d) \in \overline{M}$ of f such that:

- *$\overline{c}$ is infinitesimally close (component by component) to $\overline{a}$;*
- *the components of $\overline{c}$ are algebraically independent over M.*

Proof. Let $t_1,\dots,t_l \in N$ be given by Lemma 2.1.3 and put, for any $j \in \{1,\dots,l\}$, $c_j = a_j + t_j$. Note that the orderings on M extend to the field $M(c_1,\dots,c_l)$.
Since $t_1,\dots,t_l$ are infinitesimal over M, $f(\overline{c},Y)$ changes sign on $M(\overline{c})$ w.r.t. each ordering of $M(\overline{c})$. Hence, by the intermediate value property for real closed fields, it has a root in the m different (one for each ordering) real closures of $M(\overline{c})$.
Since f is irreducible in $M(\overline{c})[Y]$, the ideal $(f(\overline{c},Y))$ is real and then that we can extend any ordering of $M(\overline{c})$ to an ordering on the field $M(\overline{c})[Y]/(f(\overline{c},Y)) = \overline{M}$.
Just put $d \equiv Y mod(f(\overline{c},Y))$ to get the desired conclusion. □

Proof. of (A):
Let $(M,D) \models ODF_m$. By [7, Ch. I, Lemmas 1.8 and 1.10] and the extension theorem for derivation [3, Ch. X, section 7, Theorem 7], we can assume that M is a model of COF_m.
Let V,W,U be as in the hypothesis of Theorem 2.1.1 and assume that V (resp. W) has dimension r (resp. $r+s$).
Let $Q_1,\dots,Q_{n-r},Q_{n+1},\dots,Q_{2n-s}$ be a system of canonical generators of W such that $Q_1,\cdots,Q_{n-r}$ is a system of canonical generators of V.
By hypothesis, $U\mid_M$ contains a semi-generic point $(\overline{a},\overline{b})$ of W.
Consider now the following Taylorian development:

$$Q_1(a_1,\dots,a_r,a_{r+1}+\varepsilon) = \underbrace{Q_1(a_1,\dots,a_{r+1})}_{=0} + \underbrace{s_{Q_1}(a_1,\dots,a_{r+1})}_{\neq 0} \cdot \varepsilon + o(\varepsilon).$$

Let $(\eta_1,\dots,\eta_m)$ be the sign of the element $s_{Q_1}(a_1,\dots,a_{r+1})$ of M, i.e. each η_i is the sign of $s_{Q_1}(a_1,\dots,a_{r+1})$ w.r.t. $<_i$.
Remark that $Q_1(a_1,\dots,a_r,a_{r+1}+\varepsilon)$ takes the sign of $s_{Q_1}(a_0,\dots,a_{r+1})\cdot\varepsilon$ and hence $Q_1(a_0,\dots,a_r,a_{r+1}+\varepsilon)$ changes sign on M for each orderings, following the sign of ε.
By Lemma 2.1.4, we can find a root $(c_1,\dots,c_r,c_{r+1})$ of Q_1 in a m-ordered extension of M such that $c_1 - a_1,\dots,c_r - a_r$ are infinitesimal over M and $c_1,\dots,c_r$ are algebraically independent over M. Remark that, since polynomials are continuous w.r.t. each order topology, c_{r+1} is infinitesimally close to a_{r+1}.
Repeating the same argument recursively first on each a_{r+i} and then on each b_{s+j} (for any $i \in \{1,\dots,n-r\}$ and any $j \in \{1,\dots,n-s\}$), we find for each i (resp. each j) an element c_i (resp. d_j) in a m-ordered extension of M such that

$$Q_i(c_1,\cdots,c_{r+i-1},c_{r+i}) = 0 \ \big(\text{resp. } Q_{n+j}(\overline{c},d_1,\cdots,d_{s+j-1},d_{s+j}) = 0\big)$$

and c_{r+i} (resp. d_{s+j}) is infinitesimally close to a_{r+i} (resp. b_{s+j}) in M.
Let $\overline{c} = (c_1,\dots,c_n)$ and $\overline{d} = (d_1,\dots,d_n)$. Then $(\overline{c},\overline{d}) \in N$ belongs to the m-open set U and furthermore, the algebraic independence of $t_1,\dots,t_r,u_1,\dots,u_s$ ensure that it is a generic point of W.
By Lemma 1.2.1, the derivation D on M extends to a derivation D^* on $M(\overline{c},\overline{d})$ with $D^*(\overline{c}) = \overline{d}$.
Using a transfinite induction and the saturation of N, one can build an extension of M which is a model of $CODF_m$. □

It remains now to prove the model completeness of $CODF_m$.
For this we use the **Robinson's test** (see [7, Ch.I, (2.17)]) and the problem reduces to showing that any model of $CODF_m$ is an existentially closed m-ordered field.

Proof. of (B):
Let (M,D) be a model of $CODF_m$ and consider an extension $\overline{M}$ of M which is a model of ODF_m. We keep the

same notation D for the derivation on M and for its extension to $\overline{M}$.
Let $\overline{a} \subset \overline{M}$ and $\varphi(\overline{X})$ be a quantifier free L^D-formula such that

$$\overline{M} \models \varphi(\overline{a}).$$

Remark that $\varphi(\overline{X})$ can be interpreted as the L^D-formula

$$\varphi_L(\overline{X}_0, \overline{X}_1, \ldots, \overline{X}_r) \wedge \overline{X}_1 = D(\overline{X}_0) \wedge \ldots \wedge \overline{X}_r = D(\overline{X}_{r-1})$$

where φ_L is a L-formula.
Define $\phi_L(\overline{X}, \overline{Y})$ to be the following L-formula:

$$\phi_L(\overline{X}, \overline{Y}) \equiv \varphi_L(\overline{X}_0, \overline{X}_1, \ldots, \overline{X}_{r-1}, \overline{Y}_{r-1}) \wedge \overline{X}_1 = \overline{Y}_0 \wedge \ldots \wedge \overline{X}_{r-1} = \overline{Y}_{r-2}.$$

Let

$$\overline{c} = (\overline{a}, D(\overline{a}), \ldots, D^{r-1}(\overline{a})),$$

$$V = V(I(\overline{c})) \text{ and } W = V(I(\overline{c}, D(\overline{c})))$$

(so that W projects generically onto V).
Since $\varphi(\overline{X})$ is a quantifier free L^D-formula, the set defined by ϕ_L in M is a finite union of subsets which are the intersection of a M-variety and a M-definable m-open set.
More precisely,

$$\phi_L(\overline{X}, \overline{Y}) \equiv \bigvee \Big(\underbrace{\bigwedge_h f_h(\overline{X}, \overline{Y}) = 0}_{\text{defines a } M\text{-variety}} \wedge \underbrace{\bigwedge_{i=1}^{m} (\bigwedge_{j=1}^{l_i} g_{ij}(\overline{X}, \overline{Y}) > 0)}_{\text{defines a } <_i\text{-open set}} \Big).$$

Let $\hat{M}$ be a model of COF_m extending $\overline{M}$ ($\hat{M}$ exists since COF_m is the model companion of OF_m). The semi-generic point $(\overline{c}, D(\overline{c}))$ of W belongs to one of the m-open sets defined by ϕ_L. Let us denote this latter by U.
Since M and $\hat{M}$ are models of the model complete theory COF_m, there exists a semi-generic point $(\overline{d}, \overline{e})$ of W in $U\mid_M$.
Furthermore, since $M \models CODF_m$, there exists $\overline{b} \subset M$ such that

$$M \models \phi_L(\overline{b}, D(\overline{b}))$$

and then

$$M \models \phi(\overline{b_0})$$

where $\overline{b_0}$ is the initial sub-tuple of $\overline{b}$ whose length is equal to the length of $\overline{a}$. □

We end this section with two easy applications of the axiomatization of $CODF_m$. Again, these results generalize well-known facts for $CODF$ (see [6, Ch. 2]).

Corollary 2.1.5
Let $M \models CODF_m$, then

(i) *for any $(n_1, \ldots, n_k) \in \mathbb{N}^k$, the $(n_1, \ldots, n_k)$-jet-space of M^k*

$$J_{(n_1,\ldots,n_k)}(M) := \{(x_1, D(x_1), \ldots, D^{n_1}(x_1), \ldots, x_k, D(x_k), \ldots, D^{n_k}(x_k)) \mid (x_1, \ldots, x_n) \in M^k\}$$

is dense in $M^{(n_1+1)+\cdots+(n_k+1)}$ w.r.t. the topology τ_m introduced in Fact 1.1.2;

(ii) *the subfield $M_0 := \{x \in M \mid D(x) = 0\}$ is dense in M w.r.t. τ_m.*

P r o o f.

(i) The proof faithfully follows the proof of [2, Theorem 4.2], we include it for the sake of completeness.
Since $J_{(n_1,\dots,n_k)}(M) = J_{n_1}(M) \times \cdots \times J_{n_k}(M)$ and density is preserved by direct products of topological spaces it is sufficient to prove that, for any positive integer $n \geq 0$, the (n)-jet-space of M is dense in M^{n+1}.
Let U be a M-definable m-open subset of M^{n+1} and consider the differential polynomial $f(X) = D^{n+1}(X)$ whose separant is equal to the constant polynomial 1.
Consider also the following M-varieties:

$$\begin{cases} V := V((0)) = \tilde{N}^{n+1} \\ W := \{(\overline{x},\overline{y}) \mid f(x_0,\dots,x_n,y_n) = 0 \wedge y_0 = x_1 \wedge \dots \wedge y_{n-1} = x_n\} \end{cases}$$

Remark that W projects generically over V.
The set $\widetilde{U} := \{(u_o,\dots,u_n,u_1,\dots,u_n,0) \mid (u_o,\dots,u_n) \in U\}$ is a M-definable m-open subset of W and each point of $\widetilde{U}$ is semi-generic for W. By Theorem 2.1.1, there exists a differential point $(u,D(u),\dots,D^n(u))$ in U.
This proves the density of the jet-spaces.

(ii) Assume U is a m-open subset of M and let $\tilde{U} := U \times M$.
Consider the M-varieties $V := \tilde{N}$ and $W := \{(x_0,x_1) \in M^2 \mid x_1 = 0\}$. Remark that W projects generically on V and that $\tilde{U} \cap W$ is a non-empty m-open subset of W which contains semi-generic points of W.
By Theorem 2.1.1, there exists a point $(u,D(u))$ in $\tilde{U} \cap W$. In other words, there exists $u \in U$ such that $D(u) = 0$.
This proves that any m-open subset of M contains an element of M_0. □

Remark 2.1.6. The construction of some kind of "natural" or "canonical" model of the model companion of a given theory of differential fields is a longstanding open problem.
In fact, even for the theory of differentially closed fields or for *CODF*, such models are not known.
In this latter case, as well as in the case of $CODF_m$, the results of density stated above implies very strong conditions on the models. As an example, it must exist in these models arbitrarily large constants (even w.r.t. to each orderings in the case of $CODF_m$). In particular this shows that fields of germs of functions over $\mathbb{R}$ (or Hardy fields) are not models of *CODF*; or of $CODF_m$ if we want to consider the asymptotic behaviour of real functions in several different points (e.g. in $-\infty$ and $+\infty$).

2.2 Completions of $CODF_m$ and quantifier elimination

We now show how to generalize Facts 1.1.3 and 1.1.4 to the case of $CODF_m$.
First, let us remark that for any $M \models CODF_m$, $alg(M)$ is a differential field equipped with the trivial derivation $D_0 : x \mapsto 0$ and that this latter is the only possible derivation on $alg(M)$.
Hence, $alg(M)$ is a model of the theory $ODF_{m,alg} \equiv ODF_m \cup OF_{m,alg}$.

Lemma 2.2.1
The theory $ODF_{m,alg}$ has the amalgamation property.

Proof. The proof is similar to the one of [7, Ch. II, Corollary (2.7)].
Let $(M,D),(M_1,D_1),(M_2,D_2) \models ODF_{m,alg}$ be such that there exist two L^D-embeddings $(M,D) \to (M_1,D_1)$ and $(M,D) \to (M_2,D_2)$. M is identified with a subfield of M_1 and M_2 via these embeddings.
Since M is algebraically closed in M_1 and M_2, these two fields can be L-embedded in a common field extension $\mathscr{L}$ in such a way that M_1 and M_2 are linearly disjoint over M.
By [7, Ch. II, Lemma (2.5)], M_1 and M_2 have a common m-ordered extension over M (namely the field M_1M_2 which is the smallest subfield of $\mathscr{L}$ containing both M_1 and M_2).
Since M_1 and M_2 are linearly disjoint over M, we can extend the derivations D_1 and D_2 to a common derivation D^* on M_1M_2. □

The previous lemma leads to the following theorem:

Theorem 2.2.2
Let $(M_1,D_{M_1},<_1\dots,<_m)$ and $(M_2,D_{M_2},\tilde{<}_1,\dots,\tilde{<}_m)$ be two models of $CODF_m$.
Then these two L^D-structures are L^D-elementary equivalent iff

$$(alg(M_1),D_0,<_1\dots,<_m) \cong_{L^D} (alg(M_2),D_0,\tilde{<}_1,\dots,\tilde{<}_m).$$

Note that, as in Fact 1.1.3, we used the notation $<_i$ (resp. $\tilde{<}_i$) for the orderings induced by M_1 (resp. M_2) on $alg(M_1)$ (resp. $alg(M_2)$).

Again the proof is extremely similar to the proof of Fact 1.1.3 in [7].

Proof. In order to simplify the notations we will denote by M_1 the L^D-structure $(M_1, D_{M_1}, <_1 \ldots, <_m)$ and by $alg(M_1)$ the L^D-structure $(alg(M_1), D_0, <_1 \ldots, <_m)$ (and similarly for M_2 and $alg(M_2)$).

. Assume that $alg(M_1) \cong_{L^D} alg(M_2)$ so that we can identify these two L^D-structures.
By Lemma 2.2.1 and since any model of $CODF_m$ is a model of $ODF_{m,alg}$, there exists a common extension $\mathscr{L}$ of M_1 and M_2 which is a model of $ODF_{m,alg}$.
By Theorem 2.1.1, we can assume that $\mathscr{L} \models CODF_m$ and then, since $CODF_m$ is model complete, $M_1 \preceq \mathscr{L}$ and $M_2 \preceq \mathscr{L}$.
It follows that $M_1 \equiv_{L^D} M_2$.

. Assume now that $M_1 \equiv_{L^D} M_2$ and consider the set of L^D-sentences

$$S := CODF_m \cup Diag(M_1) \cup Diag(M_2)$$

where $Diag(M_i)$ denotes the set of all atomic and negation of atomic L^D-sentences which are true in M_i ($i = 1, 2$).
Since $M_1 \equiv_{L^D} M_2$, each finite fragment of S is satisfied in M_1 (and in M_2). By compactness, there exists $\mathscr{L} \models S$ and we can consider M_1 and M_2 as L^D-substructures of $\mathscr{L}$.
Since $CODF_m$ is model complete, M_1 and M_2 are elementary L^D-substructure of $\mathscr{L}$ and then

$$alg(M_1) = alg(\mathscr{L}) \text{ and } alg(M_2) = alg(\mathscr{L}).$$

Hence $alg(M_1) \cong_{L^D} alg(M_2)$. $\square$

Consider now the extension $\overline{L^D}$ of the language L^D by the new predicate symbols $W_{d,k_1,\ldots,k_m}$ and let $\overline{CODF_m}$ be the $\overline{L^D}$-theory $CODF_m \cup \overline{COF_m}$.
We want to show that $\overline{CODF_m}$ admits quantifier elimination in $\overline{L^D}$.
For this, according to [1, Ch. 4, Lemma 12], we have to prove that $\overline{CODF_m}$ is model complete and that $(\overline{CODF_m})_\forall$ has the amalgamation property.
Remark that since $\overline{CODF_m}$ is an extension by definition of $CODF_m$ it is still model complete and then it suffices to prove the following lemma:

Lemma 2.2.3
The theory $(\overline{CODF_m})_\forall$ has the amalgamation property.

Proof. Let $(M,D), (M_1,D_1), (M_2,D_2)$ be three models of $(\overline{CODF_m})_\forall$ such that there exist two $\overline{L^D}$-embeddings $\sigma_1 : (M,D) \to (M_1,D_1)$ and $\sigma_2 : (M,D) \to (M_2,D_2)$.
Let $\overline{ODF_{m,alg}} := ODF_{m,alg} \cup (\overline{CODF_m})_\forall$ and remark that

$$(\overline{ODF_{m,alg}})_\forall = (\overline{CODF_m})_\forall.$$

For any model A of $(\overline{CODF_m})_\forall$, define

$$\tilde{A} := \{a \in \bar{A} \mid a \text{ is algebraic over } \mathbb{Q}(A)\}$$

where $\bar{A}$ is a model of $\overline{CODF_m}$ extending A.
By [7, Ch. II, Lemma (2.13) (ii)], $\tilde{A}$ is a prime extension of A and is a model of $\overline{ODF_{m,alg}}$.
Hence σ_1 and σ_2 induce $\bar{L}$-embeddings

$$\tilde{\sigma_1} : \tilde{M} \to \tilde{M_1} \text{ and } \tilde{\sigma_2} : \tilde{M} \to \tilde{M_2}.$$

Furthermore, since $\tilde{M}$ (resp. $\tilde{M_1}, \tilde{M_2}$) is an algebraic extension of M (resp. M_1, M_2), D (resp. D_1, D_2) extends to an unique derivation $\tilde{D}$ (resp. $\tilde{D_1}, \tilde{D_2}$) on $\tilde{M}$ (resp. $\tilde{M_1}, \tilde{M_2}$).
Hence $\tilde{\sigma_1}$ and $\tilde{\sigma_2}$ are $\overline{L^D}$-embeddings of models of $\overline{ODF_{m,alg}}$.

But, by [7, Ch. II, Lemma (2.13) (i)], each model of $ODF_{m,alg}$ has an unique expansion to a model of $\overline{ODF_{m,alg}}$. It follows, from Lemma 2.2.1, that $\overline{ODF_{m,alg}}$ has the amalgamation property. Hence, there exists a model $\mathscr{L}$ of $\overline{ODF_{m,alg}}$ such that $\tilde{M}_1$ and $\tilde{M}_2$ $\overline{L^D}$-embeds into $\mathscr{L}$ over $\tilde{M}$.
This gives us $\overline{L^D}$-embeddings of M_1 and M_2 over M in $\mathscr{L}$ which is a model of $(\overline{CODF_m})_\forall$. □

We can then conclude with the following theorem:

Theorem 2.2.4
The theory $\overline{CODF_m}$ has quantifier elimination in $\overline{L^D}$.

Acknowledgment

The results presented here were part of my PhD. thesis. I would like to thank my advisor C. Michaux as well as the members of the jury for their various remarks on this work. I also thank the referee for his/her suggestions and comments on the presentation of this paper.

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