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Decision Aiding

Multiobjective fuzzy linear programming problems with fuzzy decision variables

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Abstract

In this paper, a multiobjective decision-making process is modeled by a multiobjective fuzzy linear programming problem with fuzzy coefficients for the objectives and the constraints. Moreover, the decision variables are linked together because they have to sum up to a constant. Most of the time, the solutions of a multiobjective fuzzy linear programming problem are compelled to be crisp values. Thus the fuzzy aspect of the decision is partly lost and the decision-making process is constrained to crisp decisions. We propose a method that uses fuzzy decision variables with a joint membership function instead of crisp decision variables. First, we consider lower-bounded fuzzy decision variables that set up the lower bounds of the decision variables. Then, the method is generalized to lower–upper-bounded fuzzy decision variables that set up also the upper bounds of the decision variables. The results are closely related to the special type of problem we are coping with, since we embed a sum constraint in the joint membership function of the fuzzy decision variables. Numerical examples are presented in order to illustrate our method.

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1. Introduction

Complex decision-making problems involving multiple criteria are one of the important daily human activities. The methodologies proposed in the field of multicriteria decision aid (MCDA) assist the decision-makers in exploring various potential solutions, but they do not assume the decision-making tasks. Several approaches have been proposed in the literature: multiattribute utility theory (MAUT) methods [9,12],

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Bayesian theory methods [25], outranking methods [23], analytic hierarchy process methods [11,24], goal programming methods [29], etc. Most of the real decision problems often take place in an uncertain environment, since some coefficients of the objectives' and the constraints' functions cannot be exactly assessed, are imprecise, unreliable or vague, etc. Indeed in many practical situations, the lack of samples or of an underlying statistical model determines inefficient statistical estimations. Consequently, the use of a given crisp decision model may lead to propose non-realistic solutions. In these conditions fuzzy logic theory, first introduced by [31], allows for a conceptual and theoretical framework for dealing with these uncertainties.

This paper associates fuzzy logic concepts to optimization concepts within a linear mathematical programming approach, in order to support a multiobjective decision-making process. A special type of multiobjective fuzzy linear programming problem, where the decision variables sum up to a constant, is considered. Indeed, many real decision problems dealing with the allocation of resources include such a constraint expressing the limited amount of the resources: resources' allocation problems [17], portfolio problems [16], knapsack problems [20], etc.

Various fuzzy linear programming techniques are surveyed in [14,21]. For the sake of simplicity, usually these techniques consider only crisp solutions of the fuzzy problems. Thus the decision-making process is constrained to crisp decisions that hide the fuzzy aspect of the problem. In Section 3, the pertinence of the use of fuzzy decision variables instead of crisp decision variables is justified. Hence finding fuzzy solutions that provide ranges of flexibility to the decision-maker looks more attractive [6]. This paper proposes a method that uses fuzzy decision variables instead of crisp ones and that supplies fuzzy solutions to the decision-maker. However since the decisions taken by the decision-maker are always crisp, our methodology will also assist the decision-maker in the choice of these crisp decisions among the fuzzy solutions.

In Section 2, the fuzzy linear modeling of a multiobjective decision problem is presented. An imprecise probability interpretation [7,10] of the fuzzy coefficients of the problem is considered.

In Section 3.1, lower-bounded fuzzy decision variables are defined. We will see that it is not very difficult to manage these fuzzy variables, thanks to the special type of problem we are coping with. That is, the sum constraint on the decision variables allows us to define a joint membership function of the fuzzy decision variables.

In general, the fuzzy objectives and the fuzzy constraints are defuzzified in order to transform the fuzzy problem into an equivalent crisp problem that can be solved using classical multicriteria decision-making techniques. The defuzzification of the objectives by means of the area compensation method [3,13,10] and of the constraints by a *worst case* approach is presented in Sections 3.2 and 3.3. Then, the equivalent multiobjective crisp programming problem is solved in Section 3.4 through an interactive and iterative MAUT method. Section 3 ends up with a numerical example.

In Section 4, some problems that can sometimes arise with the application of the proposed method are discussed. Moreover if necessary, the fuzzy model of the decision problem is tuned in order to obtain realistic solutions that represent a coherent advice to the decision-maker. Numerical examples illustrate also the tuning procedure.

The generalization of our method to lower–upper-bounded fuzzy decision variables is proposed in Section 5. First, lower–upper-bounded fuzzy decision variables with a joint membership function are defined. Then, the objectives' and the constraints' functions of the decision problem are adjusted to this generalized case.

Finally, the paper ends with conclusions in Section 6.

2. Multiobjective fuzzy linear programming problems

A multiobjective fuzzy linear programming problem where the decision variables sum up to a constant is defined as follows:

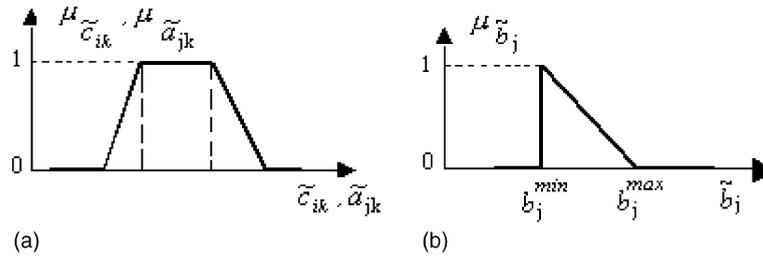


Fig. 1. Fuzzy coefficients of the problem.

$$\begin{aligned}
 \min_{\mathbf{x}} \quad & \tilde{f}_i(\mathbf{x}) = \min_{\mathbf{x}} \tilde{\mathbf{c}}_i \mathbf{x} = \min_{\mathbf{x}} \sum_{k=1}^n \tilde{c}_{ik} x_k, \quad i = 1, \dots, q, \\
 \text{s.t.} \quad & \tilde{\mathbf{a}}_j \mathbf{x} = \sum_{k=1}^n \tilde{a}_{jk} x_k \leq \tilde{b}_j, \quad j = 1, \dots, m, \\
 & \sum_{k=1}^n x_k = K, \\
 & x_k \geq 0, x_k \in \mathfrak{R}, \quad k = 1, \dots, n,
 \end{aligned} \tag{1}$$

with $\mathbf{x} = (x_1, \dots, x_n)$, the n -dimension vector of the crisp decision variables; $\tilde{f}_1(\mathbf{x}), \dots, \tilde{f}_q(\mathbf{x})$, the fuzzy linear objective functions; $\tilde{\mathbf{c}}_i = (\tilde{c}_{i1}, \dots, \tilde{c}_{in})$, the fuzzy coefficients of the objective functions; $\tilde{\mathbf{a}}_j = (\tilde{a}_{j1}, \dots, \tilde{a}_{jn})$, the fuzzy coefficients of the left side of the fuzzy linear constraints; \tilde{b}_j , the fuzzy coefficient of the right side of the fuzzy linear constraints; K , a real positive constant.

The set of decision variables that minimize, in some sense that will be explained in Section 3.4, the set of the objective functions while satisfying the constraints represents the solution of this multiobjective problem.

Fuzzy coefficients of the problem are described by flat fuzzy numbers (fuzzy intervals) [8]. The definition of a flat fuzzy number is recalled in Appendix A. As shown in Fig. 1a, we use fuzzy numbers of L-R-type [5] to describe the fuzzy coefficients of the objectives, \tilde{c}_{ik} , and of the left side of the constraints, \tilde{a}_{jk} . The specific feature of a L-R-type fuzzy number is its trapezoidal membership function. The decision-maker can modify this membership function according to its information and interpretation of the fuzzy data. In the sequel, a L-R-type fuzzy number \tilde{v} is denoted by $\tilde{v} = [v^{\min}; v^l; v^u; v^{\max}]$, where $v^{\min}, v^l, v^u, v^{\max} \in \mathfrak{R}$, $v^{\min} \leq v^l \leq v^u \leq v^{\max}$, $[v^{\min}, v^{\max}]$ is the support of \tilde{v} and $[v^l, v^u]$ is the core of \tilde{v} .

The fuzzy coefficients of the right side of the constraints, \tilde{b}_j , are modeled by fuzzy numbers with membership functions shown in Fig. 1b. In this paper, a fuzzy coefficient \tilde{b}_j is denoted by $\tilde{b}_j = [b_j^{\min}; b_j^{\max}]$, where $b_j^{\min}, b_j^{\max} \in \mathfrak{R}$ and $b_j^{\min} \leq b_j^{\max}$.

In the approach proposed in this paper, the fuzzy coefficients of the problem can be interpreted as generalized intervals and the fuzzy membership functions can be viewed as *imprecise probability distributions* [7,10]. According to this interpretation, it makes sense to consider the defuzzification procedure of the fuzzy problem by means of the area compensation method [3,10] (see Section 3.2).

3. Solutions with lower-bounded fuzzy decision variables

Usually fuzzy linear programming techniques consider only crisp solutions—i.e. crisp variables, for two main reasons. First, the decisions taken by the decision-maker are crisp decisions, by all means. Hence there is a semantic and meaningful interest. Second, it is easier to defuzzify a problem with fuzzy coefficients and

crisp variables instead of fuzzy variables, especially when the problem is linear. Hence there is also a practical interest in the consideration of crisp variables. But we show next that there are semantic and practical interests too, in the use of non-crisp decision variables, so-called fuzzy decision variables, in fuzzy linear programming problems.

One of the drawbacks to consider only crisp solutions is that the fuzzy aspect of the problem is partly lost by constraining the decision-making process to crisp decisions. There is also a twofold loss of information related to the fuzzy problem. First, the fuzzy problem is defuzzified in order to reduce the fuzzy model into a surrogate crisp model. Second, crisp solutions are supplied to the decision-maker by solving this equivalent crisp problem. In other words, whatever the uncertainties on the problem parameters, such an approach relying on crisp variables claims, in some sense, to be able to provide a deterministic optimal solution. Even if a crisp and deterministic decision is the final result of a decision process, we should not grant the optimization procedure the right to univocally determine it. The uncertainties on the parameters have to be reflected in the results and allows for a final decision step performed by the actual decision-maker.

Furthermore, it may happen that the decision-maker can not exactly adopt the suggested crisp decisions. For example, let us consider an agricultural problem of land allocation to different crops in a farm, that is ruled by ecological and economic objectives and constraints. The difficulty that can appear with crisp solutions is the impossibility to implement these decisions because of the natural divisions of the field. In these conditions, it is pertinent to supply fuzzy solutions for the decision problem and to assist the decision-maker to take the appropriate decisions according (or “inside”) the proposed fuzzy solutions. Hence in our methodology, besides the optimal solutions of the problem, regions around them containing potential “satisfactory” solutions are supplied to the decision-maker. Moreover, the direction of “more advisable” solutions than others among these regions is proposed. Indeed, Section 1 has already stressed that the methodologies proposed in the MCDA discipline and thus in this paper support the decision-making process, but do not assume the decision-making tasks.

3.1. Lower-bounded fuzzy decision variables with a joint membership function

Our first idea is to supply the decision-maker with not only an optimal solution, but also with a region around this solution that contains potential “satisfactory” solutions. These new solutions could be appropriate decisions if the optimal solution is not convenient—e.g., because of constraints that have not been expressed in the mathematical model of the decision problem.

3.1.1. The satisfactory region

Hence, the key question now is to describe the shape of the region around the optimal solution. In most resource allocation problems, it is interesting to supply two values for each decision variable: the optimal value and the lower satisfactory bound for this variable. Therefore, we provide the decision-maker with an enhanced proposal: if possible, we advise him to adopt the optimal value for each variable; if these values are not convenient, any other set of values that satisfies the lower bounds could be satisfactory.

In order to illustrate the region around the optimal solution, we consider a simple example with three decision variables x_1, x_2, x_3 , that sum up to a constant K (see relation (1)). In Fig. 2 the optimal solution (x_1, x_2, x_3) is represented with a star in the space of the decision variables. The admissibility domain of the solutions given by the sum constraint on the variables is bounded by the triangle drawn with dotted line. It is obvious that the region around the optimal solution that sets up a lower bound for each decision variable determines a triangle drawn with solid line in Fig. 2. We note that the upper bound of each variable implicitly results from the choice of the lower bounds for the other variables. Moreover, the triangle drawn with solid line is homothetic to the triangle drawn with dotted line. Hence the region defined around the optimal solution is homothetic to the admissibility domain of the solutions given by the sum constraint on the decision variables.

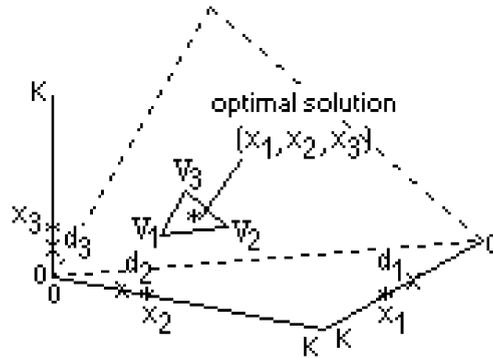


Fig. 2. Region around the optimal solution.

In Fig. 2, the coordinates of the optimal solution are represented with a star on each axis of the decision variables. The lower bound of each decision variable is represented with a cross on the axis corresponding to the variable. In order to express the distance between the optimal value and the lower bound for each variable, we introduce new decision variables, denoted d_1, d_2, d_3 . The coordinates of the vertices of the triangle obtained around the optimal solution are defined as follows:

$$\begin{aligned}
 V_1 &= (x_1 + (d_2 + d_3), x_2 - d_2, x_3 - d_3) \\
 V_2 &= (x_1 - d_1, x_2 + (d_1 + d_3), x_3 - d_3) \\
 V_3 &= (x_1 - d_1, x_2 - d_2, x_3 + (d_1 + d_2)).
 \end{aligned}
 \tag{2}$$

Since each of these vertices belongs to the admissibility domain of the solutions given by the sum constraint on the variables (see relation (1)), the sum of the coordinates for each vertex is also equal to the constant K and equal to the sum of the initial decision variables. For example, the vertex V_3 corresponds to the lower bounds of the variables x_1 and x_2 and has the first two coordinates $x_1 - d_1$ and $x_2 - d_2$. Taking into account the remark about the sum of the coordinates of a vertex, it is easy to infer the third coordinate of the vertex $V_3 = x_3 + (d_1 + d_2)$. Moreover, the inclusion of the small triangle into the big one requires the following constraints:

$$\begin{aligned}
 d_k &\leq x_k, \quad k = 1, 2, 3, \\
 d_k &\geq 0.
 \end{aligned}
 \tag{3}$$

Next, we present the generalization of the region around the optimal solution for n fuzzy decision variables.

Let us define the following column vectors:

$$\begin{aligned}
 \mathbf{X} &= \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ the vector of the decision variables } x_k, \quad k = 1, \dots, n, \\
 \mathbf{D} &= \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \text{ the vector of the decision variables } d_k, \quad k = 1, \dots, n,
 \end{aligned}$$

$$\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \text{ the unity vector} \tag{4}$$

and the following matrices:

$\mathbf{V} = (\mathbf{V}_1 \dots \mathbf{V}_n)$ the matrix whose columns are the coordinates of the vertices $V_p, p = 1, \dots, n$, of the region around the optimal solution,

$$\mathbf{I} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \text{ the identity matrix.} \tag{5}$$

The matrix \mathbf{V} is computed as follows (see Appendix B for computation details):

$$\mathbf{V} = \mathbf{X} * \mathbf{1}' - \mathbf{D} * \mathbf{1}' + \mathbf{1}' * \mathbf{D} * \mathbf{I}, \tag{6}$$

where the operator $\langle\langle \cdot \rangle\rangle$ transposes a vector.

Therefore, to characterize the optimal value as well as the hyper-triangle with n vertices of the satisfactory solutions, it is sufficient to consider the $2n$ decision variables $(x_k, d_k, k = 1, \dots, n)$ and the following additional constraints:

$$\begin{aligned} d_k &\leq x_k, & k = 1, \dots, n, \\ d_k &\geq 0. \end{aligned} \tag{7}$$

The lower bound of a fuzzy variable k is equal to $x_k - d_k$ and the upper bound is equal to $x_k + (d_1 + \dots + d_{k-1} + d_{k+1} + \dots + d_n)$ (see the matrix \mathbf{V} in Appendix B). Moreover, the inclusion of the region containing potential satisfactory solutions in the admissibility domain of the solutions given by the sum constraint on the variables is easily managed by the constraints (7). This very simple characterization advocates also for the use of this type of lower-bounded variables. It would not be so easy to consider upper-bounded fuzzy decision variables and to embed the satisfaction of the sum constraint in the satisfactory region.

3.1.2. The fuzzy solution

The meaning of the optimal solution and the interpretation we give to the region around it allow us to consider it as a fuzzy set and to define a joint membership function on the variables. We will now define this joint membership function by its 1-cut and 0-cut (see Appendix A for the definition of the α -cuts).

The *1-cut*: i.e. the core, of the fuzzy solution will be a single point: the optimal solution (x_1, \dots, x_n) . Thus the optimal solution corresponds to the maximum grade of membership for the fuzzy solution. This expresses that the optimal solution is the most advisable solution among the potential satisfactory solutions. The *0-cut*: i.e. the support, is the satisfactory region around the optimal solution that contains the set of potentially satisfactory solutions. All the points that do not belong to this region represent non-satisfactory decisions and have a zero grade of membership for the fuzzy solution.

For better understanding, we come back to the example with three fuzzy decision variables. The joint membership function of the variables determines a pyramid above the triangle defined around the optimal solution (see Fig. 3). The core of the joint membership function is equal to the optimal solution (x_1, x_2, x_3) . The support is equal to the region bounded by the triangle $V_1V_2V_3$.

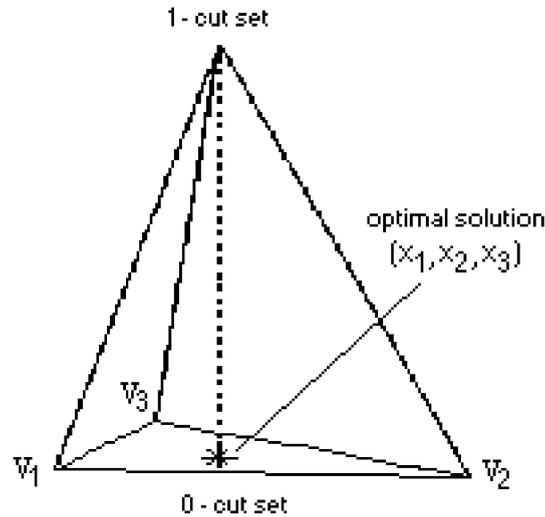


Fig. 3. Joint membership function of the fuzzy variables.

In conclusion, we have defined dependent lower-bounded fuzzy variables linked by a sum constraint by means of a joint membership function. Further, among the solutions contained in the satisfactory region, we allow the optimization to consider a more plausible one. This one will thus be the most advisable solution in this region.

3.2. Defuzzification of the objectives

The definition of a fuzzy objective was already given in Section 2 (see relation (1)), for the case of crisp decision variables. In this section, we determine first the changes in this definition when lower-bounded fuzzy decision variables are considered. For this purpose, the membership function of a fuzzy objective is computed by associating the membership functions of the fuzzy coefficients of this objective and the joint membership function of the variables. Then, the defuzzification procedure of a fuzzy objective is reminded. This defuzzification is consistent with the area compensation method, that supports the imprecise probability interpretation of the fuzzy parameters of our problem.

3.2.1. The fuzzy objective

The membership function of a fuzzy objective is defined by its α -cuts as a natural extension of the concept of mapping sets, which is called the *extension principle* [32]. For this purpose, the extension principle associates the α -cuts of the membership functions of the fuzzy coefficients of the objective (see Fig. 1a) to the corresponding α -cuts of the joint membership function of the variables (see Fig. 3 for three fuzzy decision variables).

We assume a trapezoidal membership function like in Fig. 1a for a fuzzy objective \tilde{f}_i , since this piecewise linear membership function seems to be a good approximation for the fuzzy objectives [1]. The fuzzy number describing the fuzzy objective \tilde{f}_i is denoted $[f_i^{\min}, f_i^l, f_i^u, f_i^{\max}]$ (see Section 2), where $[f_i^l, f_i^u]$ is the 1-cut of \tilde{f}_i and $[f_i^{\min}, f_i^{\max}]$ is the 0-cut of \tilde{f}_i .

The 1-cut of the fuzzy objective \tilde{f}_i is obtained associating the 1-cuts of the fuzzy coefficients \tilde{c}_{ik} , $k = 1, \dots, n$ (that is $[c_{ik}^l, c_{ik}^u]$) and the 1-cut of the fuzzy decision variables (that is $\{(x_1, \dots, x_n)\}$), as follows:

$$f_i^l = \sum_{k=1}^n c_{ik}^l x_k, \quad f_i^u = \sum_{k=1}^n c_{ik}^u x_k. \tag{8}$$

The 0-cut of the fuzzy objective \tilde{f}_i is obtained associating the 0-cuts of the fuzzy coefficients \tilde{c}_{ik} , $k = 1, \dots, n$ (that is $[c_{ik}^{\min}, c_{ik}^{\max}]$) and the 0-cut of the fuzzy decision variables (that is the satisfactory region with the vertices V_p , $p = 1, \dots, n$ (see Section 3.1)), as follows:

$$f_i^{\min} = \min_{p/V_p \text{ vertex}} \left\{ \sum_{k=1}^n c_{ik}^{\min} V_{kp} \right\}, \quad f_i^{\max} = \max_{p/V_p \text{ vertex}} \left\{ \sum_{k=1}^n c_{ik}^{\max} V_{kp} \right\}, \tag{9}$$

where

$$V_p = \begin{pmatrix} V_{1p} \\ \vdots \\ V_{np} \end{pmatrix}.$$

The relation (9) means that f_i^{\min} and respectively f_i^{\max} are computed for each vertex V_p , $p = 1, \dots, n$, and then the minimum value of f_i^{\min} and respectively the maximum value of f_i^{\max} are kept. In fact, we show in Appendix C that the formulas in the relation (9) are equivalent to:

$$f_i^{\min} = \sum_{k=1}^n c_{ik}^{\min} (x_k - d_k) + \min_k \{c_{ik}^{\min}\} \sum_{k=1}^n d_k, \quad f_i^{\max} = \sum_{k=1}^n c_{ik}^{\max} (x_k - d_k) + \max_k \{c_{ik}^{\max}\} \sum_{k=1}^n d_k. \tag{10}$$

3.2.2. The defuzzification procedure

Various defuzzification methods for the fuzzy objectives have been proposed in the literature [19,21]. These defuzzification methods substitute a fuzzy objective by either one “equivalent” (“compromise”) crisp objective or several crisp objectives using α -cuts of that fuzzy objective. Since the fuzziness dissolution implies many simplifications of the initial fuzzy information, many objectives’ defuzzification methods are interactive and iterative, in order to allow and to take into account the feedback of the decision-maker regarding the proposed solutions.

In [22] an intuitive comparison method of two fuzzy numbers based on the area compensation determined by the membership functions of the fuzzy numbers is proposed. This comparison method is revised in [10] in order to explicitly obtain a ranking relation between fuzzy numbers as well as a defuzzification procedure of a fuzzy number. According to this defuzzification procedure, a fuzzy number \tilde{v} is defuzzified by means of the following integral:

$$v = \mathcal{F}(\tilde{v}) = \frac{1}{2} \int_0^1 (v_x^{\min} + v_x^{\max}) d\alpha, \tag{11}$$

where v is the defuzzified value, \mathcal{F} is the defuzzification function (a mapping function from the set of fuzzy numbers to the set of real numbers), v_x^{\min} and v_x^{\max} are defined in Appendix A. This defuzzified value is the arithmetic mean of the two areas A_L and A_R defined by the vertical axis and respectively by the left and the right slope of the fuzzy number \tilde{v} (see Fig. 4). Both areas describe how much the fuzzy number is greater than zero: A_L in a pessimistic way and A_R in an optimistic way. Moreover in the context of imprecise probability distribution considered for the fuzzy parameters of our problem, it was proved [10] that the relation (11) gives also the center of the mean value of the fuzzy number viewed in the sense of the Dempster–Shafer theory [7,4,25]. According to this theory, a fuzzy set can be viewed as describing a set of admissible probability measures.

Hence, the defuzzification function based on the area compensation method is applied to each fuzzy objective described by the fuzzy number $\tilde{f}_i = [f_i^{\min}; f_i^l; f_i^u; f_i^{\max}]$, where f_i^l, f_i^u are given by the relation (8)

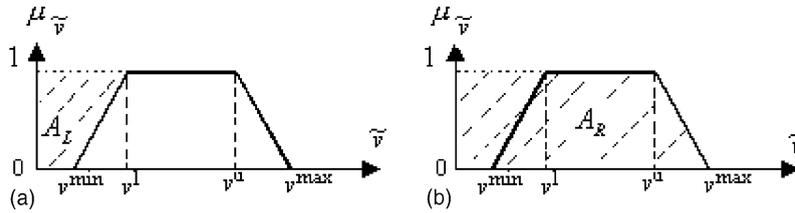


Fig. 4. Mapping a fuzzy number to the set of real numbers.

and f_i^{\min}, f_i^{\max} are given by the relation (10). After computing the integral of the relation (11), the following defuzzified value is obtained for each fuzzy objective $\tilde{f}_i, i = 1, \dots, q$:

$$f_i(\mathbf{x}, \mathbf{d}) = \mathcal{F}(\tilde{f}_i(\mathbf{x}, \mathbf{d})) = \frac{f_i^{\min} + f_i^l + f_i^u + f_i^{\max}}{4} = \sum_{k=1}^n \frac{c_{ik}^{\min} + c_{ik}^l + c_{ik}^u + c_{ik}^{\max}}{4} x_k - \sum_{k=1}^n \frac{c_{ik}^{\min} + c_{ik}^{\max}}{4} d_k + \frac{\min_k \{c_{ik}^{\min}\} + \max_k \{c_{ik}^{\max}\}}{4} \sum_{k=1}^n d_k. \quad (12)$$

In conclusion, the fuzzy objectives $\tilde{f}_i, i = 1, \dots, q$, are substituted by the corresponding defuzzified values $f_i(\mathbf{x}, \mathbf{d})$ given by the relation (12).

We note that this defuzzification procedure of the fuzzy objectives, that considers only a kind of “mean values” of the fuzzy coefficients of the objectives, can be refined in order to take into account the uncertainties—i.e. imprecision, vagueness and variety—related to the fuzzy coefficients of the objectives [27,28]. But this refinement is not necessary when fuzzy decision variables are used, since the support of the fuzzy solution implicitly considers the uncertainties related to the fuzzy coefficients of the objectives.

3.3. Defuzzification of the constraints

The definition of a fuzzy constraint was already given in Section 2 (see relation (1)), for the case of crisp decision variables. In the same way as for the fuzzy objectives, in this section we determine first the changes in this definition when lower-bounded fuzzy decision variables are considered. These changes occur in the left hand side of the fuzzy constraints, that depends on the decision variables. While the defuzzification of a fuzzy objective considers a kind of “mean value”, we choose for the defuzzification of a fuzzy constraint a worst case approach. There are, of course, other defuzzification methods with the compensation of the fuzzy constraints or not [14,18], but for safety reasons and for the sake of simplicity, we defuzzify the fuzzy constraints by a worst case approach. In some contexts, this defuzzification could be improved, but this is beyond the scope of our paper. Our main contribution is the use of fuzzy decision variables in multi-objective fuzzy problems.

In the same way as for a fuzzy objective, we assume a trapezoidal membership function like in Fig. 1a for the left hand side of a fuzzy constraint $j, j = 1, \dots, m$. This membership function, computed by associating the membership functions of the fuzzy coefficients of the constraint j and the joint membership function of the variables, is defined by its α -cuts, as follows (the same reasoning as in Section 3.2 is applied):

$$\begin{aligned} \text{1-cut:} & \left[\sum_{k=1}^n a_{jk}^l x_k, \sum_{k=1}^n a_{jk}^u x_k \right], \\ \text{0-cut:} & \left[\sum_{k=1}^n a_{jk}^{\min} (x_k - d_k) + \min_k \{a_{jk}^{\min}\} \sum_{k=1}^n d_k, \sum_{k=1}^n a_{jk}^{\max} (x_k - d_k) + \max_k \{a_{jk}^{\max}\} \sum_{k=1}^n d_k \right]. \end{aligned} \quad (13)$$

The defuzzification of a fuzzy constraint by a worst case approach ensures the satisfaction of the constraint for all values of its fuzzy coefficients and the satisfactory ones of the decision variables. For this purpose, the fuzzy number describing the left hand side of the fuzzy constraint must be completely on the left of the fuzzy number describing the right hand side of the fuzzy constraint. Hence the following crisp constraint is imposed:

$$\sum_{k=1}^n a_{jk}^{\max}(x_k - d_k) + \max_k \{a_{jk}^{\max}\} \sum_{k=1}^n d_k \leq b_j^{\min}. \tag{14}$$

Thus in a worst case approach, the fuzzy number describing the right hand side of the fuzzy constraint—i.e. $\tilde{b}_j = [b_j^{\min}; b_j^{\max}]$ —could be replaced by a crisp value—i.e. b_j^{\min} .

In conclusion, each fuzzy constraint $j, j = 1, \dots, m$, is substituted by the corresponding crisp constraint given by the relation (14).

3.4. Solutions by the multiattribute utility theory

After the defuzzification step, an equivalent multiobjective crisp problem is obtained:

$$\begin{aligned} \min_{\mathbf{x}, \mathbf{d}} \quad & f_i(\mathbf{x}, \mathbf{d}) = \min_{\mathbf{x}, \mathbf{d}} \sum_{k=1}^n \frac{c_{ik}^{\min} + c_{ik}^l + c_{ik}^u + c_{ik}^{\max}}{4} x_k - \sum_{k=1}^n \frac{c_{ik}^{\min} + c_{ik}^{\max}}{4} d_k \\ & + \frac{\min_k \{c_{ik}^{\min}\} + \max_k \{c_{ik}^{\max}\}}{4} \sum_{k=1}^n d_k, \quad i = 1, \dots, q, \\ \text{s.t.} \quad & \sum_{k=1}^n a_{jk}^{\max}(x_k - d_k) + \max_k \{a_{jk}^{\max}\} \sum_{k=1}^n d_k \leq b_j^{\min}, \quad j = 1, \dots, m, \\ & d_k \leq x_k, \quad k = 1, \dots, n, \\ & \sum_{k=1}^n x_k = K, \\ & x_k \geq 0, \quad d_k \geq 0, \quad x_k, d_k \in \mathfrak{R}. \end{aligned} \tag{15}$$

Various interactive and iterative methods to determine efficient (pareto-optimal) solutions of a multi-objective linear programming problem are proposed in the literature [26,29,30]. They consist of successive mono-objective optimizations of a weighted aggregated objective. At each step, a new efficient solution is computed and presented to the decision-maker, who can relax the objective requirements and/or modify the weight vector. We choose a very basic approach to solve the multiobjective linear problem (15), relying on the MAUT [12,9]. This approach is easily understood by the decision-makers and allows the decision-makers to explore various efficient solutions depending on their preferences for the problem objectives.

According to the MAUT approach, a utility function u_i is defined for each objective $f_i(\mathbf{x}, \mathbf{d}), i = 1, \dots, q$. We choose an exponential utility function, as follows (see also Fig. 5a):

$$u_i(f_i(\mathbf{x}, \mathbf{d})) = \frac{1 - \exp(\gamma_i(f_i^* - f_i(\mathbf{x}, \mathbf{d})))}{1 - \exp(\gamma_i(f_i^* - f_{i*}))}, \tag{16}$$

with f_i^* , the less satisfying value of the objective f_i ; f_{i*} , the full satisfying value of the objective f_i ; $\gamma_i < 0$, a coefficient controlling the curvature of the utility function u_i .

The less and the full satisfying values of the objectives are first obtained by individual optimizations of these objectives.

Next, the multiobjective problem (15) is transformed into a mono-objective problem by maximizing a global additive utility function, that sums the utility functions of all objectives:

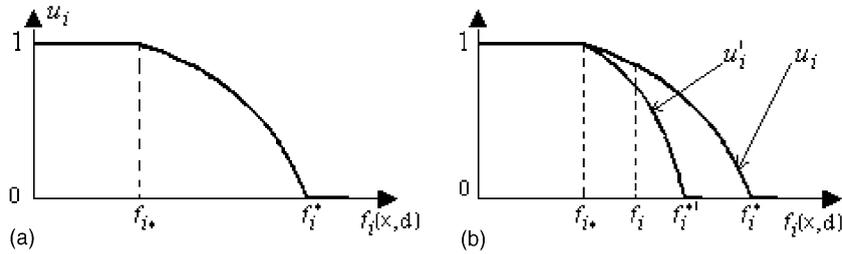


Fig. 5. Utility function of an objective.

$$\begin{aligned}
 \max_{\mathbf{x}, \mathbf{d}} \quad & U(\mathbf{x}, \mathbf{d}) = \max_{\mathbf{x}, \mathbf{d}} \sum_{i=1}^q u_i(f_i(\mathbf{x}, \mathbf{d})), \\
 \text{s.t.} \quad & \sum_{k=1}^n a_{jk}^{\max} (x_k - d_k) + \max_k \{a_{jk}^{\max}\} \sum_{k=1}^n d_k \leq b_j^{\min}, \quad j = 1, \dots, m, \\
 & d_k \leq x_k, \quad k = 1, \dots, n, \\
 & \sum_{k=1}^n x_k = K, \\
 & x_k \geq 0, \quad d_k \geq 0, \quad x_k, d_k \in \mathfrak{R}.
 \end{aligned} \tag{17}$$

The expression of the global utility function supposes two hypotheses [9]:

- the preference relation between any two objectives is independent of the values of the other objectives,
- the utility of any objective is independent of the values of the other objectives.

The preferences for the objectives can be modified through an adjustment of the bounds f_i^* and f_{i*} for each objective. By this approach indeed, it is possible to model the preferences of the decision-maker by interactively and iteratively adapting f_i^* and f_{i*} for each objective f_i [2]. A smaller value for f_i^* , denoted $f_i^{*'}$ (see Fig. 5b), is equivalent to a greater preference of the decision-maker for the objective f_i .

The mono-objective non-linear problem (17) is solved in an interactive and iterative way using the computer software CFSQP [15], that implements an algorithm based on sequential quadratic programming for solving constrained non-linear optimization problems.

3.5. Numerical example

In order to illustrate our methodology, we consider the following example (see Section 2 for notations):

$$\begin{aligned}
 \min_{\mathbf{x}} \quad & \tilde{f}_1(\mathbf{x}) = \min_{\mathbf{x}} [0.5; 2; 3; 4]x_1 + [2.5; 4; 5; 6]x_2 + [6; 7; 8; 9]x_3, \\
 \min_{\mathbf{x}} \quad & \tilde{f}_2(\mathbf{x}) = \min_{\mathbf{x}} [-4; -3; -2; -1]x_1 + [-6; -5; -4; -3.5]x_2 + [-9.5; -8; -7; -6.5]x_3, \\
 \text{s.t.} \quad & [1.5; 2; 3; 4]x_1 + [-1; 0; 1; 2]x_2 + [-5; -4; -3; -2]x_3 \leq [100; 110], \\
 & x_1 + x_2 + x_3 = 100, \\
 & x_1, x_2, x_3 \geq 0.
 \end{aligned} \tag{18}$$

First, three new decision variables d_1, d_2, d_3 , that must satisfy the constraints of the relation (3), are introduced. After the objectives' and the constraints' defuzzification step, the following equivalent multi-objective crisp problem is obtained (see relation (15)):

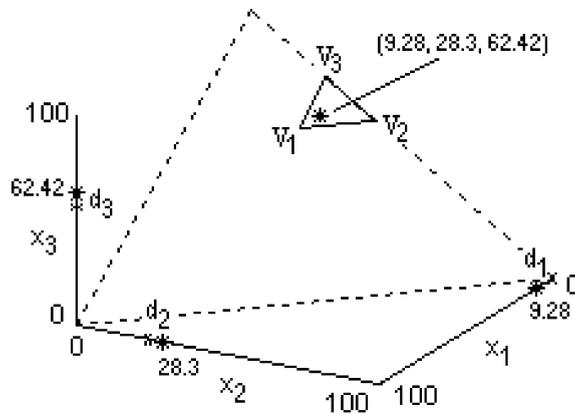


Fig. 6. Solutions of the example (18).

$$\begin{aligned}
 \min_{\mathbf{x}, \mathbf{d}} \quad & f_1(\mathbf{x}, \mathbf{d}) = \min_{\mathbf{x}, \mathbf{d}} \frac{9.5}{4}x_1 + \frac{17.5}{4}x_2 + \frac{30}{4}x_3 - \frac{4.5}{4}d_1 - \frac{8.5}{4}d_2 - \frac{15}{4}d_3 + \frac{9.5}{4}(d_1 + d_2 + d_3), \\
 \min_{\mathbf{x}, \mathbf{d}} \quad & f_2(\mathbf{x}, \mathbf{d}) = \min_{\mathbf{x}, \mathbf{d}} -\frac{10}{4}x_1 - \frac{18.5}{4}x_2 - \frac{31}{4}x_3 + \frac{5}{4}d_1 + \frac{9.5}{4}d_2 + \frac{16}{4}d_3 - \frac{10.5}{4}(d_1 + d_2 + d_3), \\
 \text{s.t.} \quad & 4(x_1 - d_1) + 2(x_2 - d_2) - 2(x_3 - d_3) + 4(d_1 + d_2 + d_3) \leq 100, \\
 & d_1 \leq x_1, \quad d_2 \leq x_2, \quad d_3 \leq x_3, \\
 & x_1 + x_2 + x_3 = 100, \\
 & x_1, x_2, x_3 \geq 0, \quad d_1, d_2, d_3 \geq 0.
 \end{aligned} \tag{19}$$

Then, the less and the full satisfying values of the objectives necessary to the MAUT method (see Section 3.4) are obtained by individual optimizations of the objectives. Finally, the efficient solution computed for the problem (19) by the MAUT approach is given by:

$$\begin{aligned}
 (x_1, x_2, x_3) &= (9.28, 28.3, 62.42), \\
 (d_1, d_2, d_3) &= (9.28, 3.51, 3.12).
 \end{aligned} \tag{20}$$

These results are represented in Fig. 6, where the notations are similar to the notations of Fig. 2 (see also the explanations in Section 3.1).

Hence, the region bounded by the triangle $V_1V_2V_3$ contains potential satisfactory solutions for the decision-maker. Among these solutions, the first and the most advisable solution to the decision-maker is given by $(x_1, x_2, x_3) = (9.28, 28.3, 62.42)$ and represented with a star in the triangle. If this value is not convenient, the decision-maker can take an appropriate decision among the other satisfactory solutions.

4. Tuning the fuzzy model

In Section 3, we have proposed a method that solves multiobjective fuzzy linear programming problems with lower-bounded fuzzy decision variables. This method gives good results for most of the problems. In this section, we analyze some difficulties arising with the use of our proposed methodology.

4.1. Degenerated satisfactory region

These difficulties are in fact related to the dimension of the region containing potential satisfactory solutions for the decision-maker. One can observe in very particular cases that the satisfactory region collapses to a single solution, which is the optimal one. This degenerated region does not allow to provide the decision-maker with other potential satisfactory solutions than the optimal one. Let us remind that the aim of the method is to find a satisfactory region larger than the optimal solution. Therefore, one has to derive additional constraints to be added to the problem model, in order to enforce a non-degenerated satisfactory region, keeping in mind the consistency of the approach and of the advice to the decision-maker.

Let us consider the following example:

$$\begin{aligned}
 \min_{\mathbf{x}} \quad & \tilde{f}_1(\mathbf{x}) = \min_{\mathbf{x}} [1; 2; 3; 5]x_1 + [3; 4; 5; 6]x_2 + [5; 6; 7; 8]x_3, \\
 \min_{\mathbf{x}} \quad & \tilde{f}_2(\mathbf{x}) = \min_{\mathbf{x}} [-7; -5; -4; -3]x_1 + [-14; -13; -11; -10]x_2 + [-9; -8; -6; -5]x_3, \\
 \text{s.t.} \quad & [-3; -2; -1; 0]x_1 + [2; 3; 4; 5]x_2 + [-4; -3; -2; -1]x_3 \leq [120; 130], \\
 & x_1 + x_2 + x_3 = 100, \\
 & x_1, x_2, x_3 \geq 0.
 \end{aligned} \tag{21}$$

After applying the method proposed in Section 3, the following results are obtained (see also Fig. 7):

$$\begin{aligned}
 (x_1, x_2, x_3) &= (62.75, 26.21, 11.04), \\
 (d_1, d_2, d_3) &= (0, 0, 0).
 \end{aligned} \tag{22}$$

Hence, a degenerated satisfactory region that contains only the optimal solution is obtained. Moreover, this solution belongs to the segment—represented with dashed line in Fig. 7—bounding the intersection region between the admissibility domains of the solutions given by the sum constraint and by the inequality constraint on the variables \mathbf{x} . This intersection region is found on the left of this segment.

The degeneracy of the satisfactory region is explained by the linearity of the problem and by the fact that in certain particular cases, there are not enough constraints on the variables, which implies too many degrees of freedom for the variables. Moreover, the values of the variables \mathbf{d} depend also on the membership functions of the fuzzy coefficients of the problem. Hence, particular membership functions can imply values equal to zero for the variables \mathbf{d} .

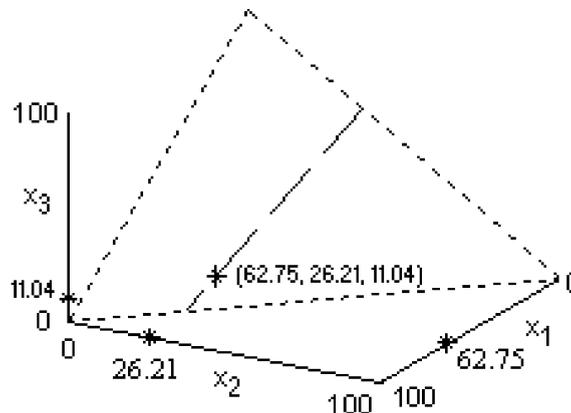


Fig. 7. Solution of the example (21).

In these situations, it is not counter-intuitive to impose minimum values different from zero for the variables \mathbf{d} , since these variables determine the dimension of the satisfactory region. For this purpose, new constraints are added to the problem model, as follows:

$$d_k \geq p\% \alpha x_k, \quad k = 1, \dots, n. \tag{23}$$

The percentage value p can differ for each pair of variables $d_k, x_k, k = 1, \dots, n$. For the sake of simplicity, in this paper the same percentage value is considered for all pairs of variables.

Next, the example (21) for which a degenerated satisfactory region was found, is reconsidered by adding the constraints (23). The extended problem is solved with different percentage values p . The solutions computed with 10% and 20% are given bellow (see also Fig. 8):

$$\begin{aligned} p\% = 10\%: (x_1, x_2, x_3) &= (58.8, 19.83, 21.37) \\ (d_1, d_2, d_3) &= (5.88, 1.98, 2.14), \\ p\% = 20\%: (x_1, x_2, x_3) &= (53.04, 11.99, 34.97) \\ (d_1, d_2, d_3) &= (10.61, 2.4, 6.99). \end{aligned} \tag{24}$$

We note that the solution obtained for the initial problem (without the constraints (23)) corresponds to the solution of the extended problem (with the constraints (23)) when $p\% = 0\%$ (see Fig. 8).

4.2. Non-coherent satisfactory regions

Fig. 8 brings to the fore two new problems that can sometimes arise when the dimension of the satisfactory region is imposed. The first problem concerns the optimal solution obtained when $p\% = 0\%$, that is not contained in the satisfactory regions for $p\% > 0\%$. The second problem is related to the fact that the satisfactory regions are not embedded. We observed that these difficulties arise very rarely and the very peculiar example (21) was chosen in order to analyze them.

These difficulties are due to the constraints (23), that impose the dimension of the satisfactory regions, and to the fact that the optimal solution obtained for the example (21) when $p\% = 0\%$ belongs to the boundary of the admissibility domain of the solutions. Indeed since the satisfactory regions must be contained in the admissibility domain of the solutions, these regions are shifted with respect to the optimal solution.

In order to overcome both difficulties and to supply coherent advises to the decision-maker, new restrictions are added to the problem model.

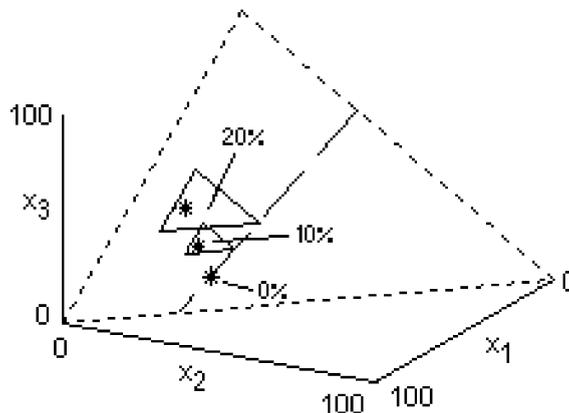


Fig. 8. Solution of the example (21) with 0%, 10%, 20%.

For this purpose, the optimal solution when $p^0\% = 0\%$ is first computed and the values obtained in this case for the variables \mathbf{x} are noted (x_1^0, \dots, x_n^0) . Then, it is not counter-intuitive to impose that this optimal solution (x_1^0, \dots, x_n^0) is contained in the satisfactory regions. That is, each value x_k^0 , $k = 1, \dots, n$, must be found between the lower and the upper bounds of the corresponding fuzzy variable k . These restrictions are ensured by the following new constraints:

$$x_k - d_k \leq x_k^0, \quad k = 1, \dots, n, \tag{25}$$

imposing for each fuzzy variable k that its lower bound (see Section 3.1) is smaller than x_k^0 . Moreover, the constraints (25) ensure that x_k^0 is smaller than the upper bound of the fuzzy variable k (see Appendix D for proof).

Next, we come back to the extended example (21) (where the constraints (23) were added) and we add the constraints (25), with $(x_1^0, x_2^0, x_3^0) = (62.75, 26.21, 11.04)$ (see relation (22)). The solutions obtained with 10% and 20% (for $p\%$ in the constraints (23)) are given below (see also Fig. 9):

$$\begin{aligned} p^0\% = 10\%: (x_1, x_2, x_3) &= (78.7, 9.04, 12.26) \\ (d_1, d_2, d_3) &= (15.95, 0.9, 1.22), \\ p^0\% = 20\%: (x_1, x_2, x_3) &= (82.29, 3.92, 13.79) \\ (d_1, d_2, d_3) &= (19.54, 0.78, 2.75). \end{aligned} \tag{26}$$

Fig. 9 presents to the decision-maker the most advisable solution, that is the optimal solution, denoted 0%. The regions bounded by the triangles drawn with solid line, denoted 10% and 20%, contain potential satisfactory solutions for the decision-maker. The stars in the satisfactory regions correspond to the optimal values obtained for the variables \mathbf{x} when these regions are computed (see relation (26)). Hence, these stars give the direction of the more advisable solutions than others among the satisfactory regions. That is, if the optimal solution is not convenient for the decision-maker, he is advised first to look for a satisfactory solution lying in the indicated direction among the satisfactory regions. Then, if the decision-maker does still not find a satisfactory solution, he must investigate the other potential satisfactory solutions contained in the satisfactory regions.

Moreover, Fig. 9 shows that the optimal solution 0% belongs to the satisfactory regions and the satisfactory region 10% is contained in the satisfactory region 20%. Indeed, for all tested examples (where the constraints (23) and (25) were added) we have observed that the satisfactory regions are contained in the subsequent ones, going from the small regions to the larger ones. We do not have a rigorous proof for this fact, but this statement relies on the following observations about the satisfactory regions:

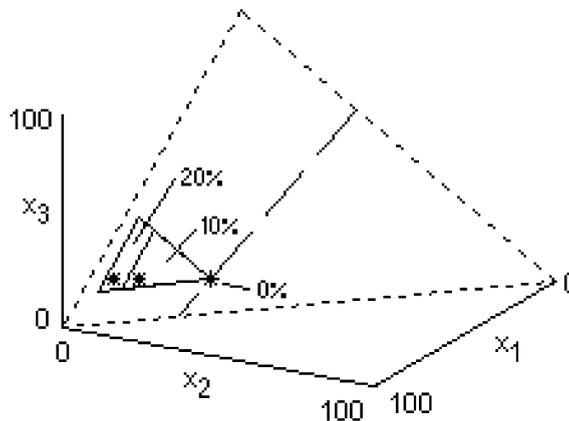


Fig. 9. Solution of the example (21) with the optimal solution contained in the satisfactory regions.

- they contain a same solution, that is the optimal solution 0%,
- they are homothetic (see Section 3.1),
- they always extend in the same direction, since the direction of the gradient (that is the search direction of the optimal solution) in the optimization procedure is the same for all satisfactory regions. In fact, the only differences between the optimization procedures for the different satisfactory regions are the bounds for the decision variables, due to the different dimensions of these regions, set by different percentage values p (see relation (23)).

5. Generalization to lower–upper-bounded fuzzy decision variables

In the previous sections, lower-bounded fuzzy decision variables were considered in order to solve multiobjective fuzzy linear programming problems. According to the presented methodology, the lower bounds for the decision variables are chosen and their upper bounds implicitly result from this choice of the lower bounds (see Section 3.1). Hence the decision-maker does not manage the upper bounds of the decision variables. In this section, the generalization of our methodology to lower–upper-bounded fuzzy decision variables is proposed. For this purpose, not only the lower bounds of the decision variables are chosen, but also their upper bounds. Thus, the decision-maker is able to better manage the region containing potential satisfactory solutions around the optimal solution.

In Section 5.1, lower–upper-bounded fuzzy decision variables with a joint membership function are defined. The type of the satisfactory region around the optimal solution is also investigated.

In Section 5.2, the changes in the fuzzy objectives and the fuzzy constraints when considering lower–upper-bounded fuzzy decision variables instead of lower-bounded ones are presented.

The procedure and the methods applied to solve the fuzzy problem are the same as in the case of lower-bounded fuzzy decision variables: first, the objectives' defuzzification by means of the area compensation method and the constraints' defuzzification by a worst case approach, next, the solution of the equivalent crisp problem by an interactive and iterative MAUT approach and finally if necessary, the tuning of the fuzzy model. All these methods have been already presented and will not be described again in this section.

5.1. Lower–upper-bounded fuzzy decision variables with a joint membership function

The idea is to supply three values for each decision variable: the optimal value, the lower satisfactory bound and the upper satisfactory bound of the variable. Thus, the advice concerning a decision variable given to the decision-maker is its optimal value, first. If this value is not convenient, any other satisfactory value has to be larger than the lower bound of the variable and smaller than the upper bound of the variable. As for the lower-bounded fuzzy decision variables, a region containing potential satisfactory solutions is thus defined around the optimal solution.

In order to illustrate the satisfactory region in the case of lower–upper-bounded fuzzy decision variables, a simple example with three decision variables x_1, x_2, x_3 , that sum up to a constant K (see relation (1)) is considered. The optimal solution (x_1, x_2, x_3) is represented with a star in Fig. 10. It is obvious that the region around the optimal solution that sets up the lower and the upper bounds for the decision variables determines a hexagon (drawn with solid line in Fig. 10). This hexagon is the intersection between the triangles $V_1V_2V_3$ and $U_1U_2U_3$ determined around the optimal solution when only the lower bounds and respectively only the upper bounds of the decision variables are set. The axes in Fig. 10 are similar to the axes in Fig. 2. Moreover, besides the decision variables d_1, d_2, d_3 expressing the distance between the optimal value and the lower bound for each variable, new decision variables, denoted s_1, s_2, s_3 , are introduced in order to express the distance between the optimal value and the upper bound for each variable.

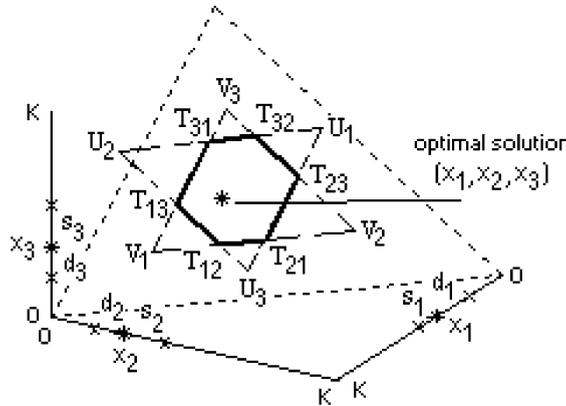


Fig. 10. New region around the optimal solution.

We note that the sum of the coordinates for each vertex of the hexagon obtained around the optimal solution is equal to the sum of the decision variables \mathbf{x} and equal to the constant K (see relation (1) and Section 3.1). Hence for each vertex of the hexagon, one coordinate implicitly results from the values of the other coordinates. Moreover for each vertex T_{rl} , $1 \leq r \leq 3$, $1 \leq l \leq 3$, $r \neq l$, of the hexagon, the index r corresponds to the index of the variable reaching its upper bound in T_{rl} and the index l corresponds to the index of the variable implicitly resulting from the values of the other variables, as follows:

$$\begin{aligned}
 \mathbf{T}_{12} &= (x_1 + s_1, x_2 + (d_3 - s_1), x_3 - d_3), & \mathbf{T}_{13} &= (x_1 + s_1, x_2 - d_2, x_3 + (d_2 - s_1)), \\
 \mathbf{T}_{21} &= (x_1 + (d_3 - s_2), x_2 + s_2, x_3 - d_3), & \mathbf{T}_{23} &= (x_1 - d_1, x_2 + s_2, x_3 + (d_1 - s_2)), \\
 \mathbf{T}_{31} &= (x_1 + (d_2 - s_3), x_2 - d_2, x_3 + s_3), & \mathbf{T}_{32} &= (x_1 - d_1, x_2 + (d_1 - s_3), x_3 + s_3).
 \end{aligned}
 \tag{27}$$

As for the lower-bounded fuzzy decision variables, the inclusion of the satisfactory region into the big triangle bounding the admissibility domain of the solutions given by the sum constraint on the variables requires the following constraints:

$$\begin{aligned}
 d_k &\leq x_k, \quad k = 1, 2, 3, \\
 d_k &\geq 0.
 \end{aligned}
 \tag{28}$$

Moreover, in order to obtain a hexagon by the intersection of the triangles $V_1V_2V_3$ and $U_1U_2U_3$, the constraints given below are imposed:

$$\begin{aligned}
 s_1 &\leq d_2 + d_3, \quad s_2 \leq d_1 + d_3, \quad s_3 \leq d_1 + d_2, \\
 d_1 &\leq s_2 + s_3, \quad d_2 \leq s_1 + s_3, \quad d_3 \leq s_1 + s_2, \\
 s_1, s_2, s_3 &\geq 0.
 \end{aligned}
 \tag{29}$$

Let us now generalize these results to the case of n decision variables.

We recall that in the case of lower-bounded fuzzy decision variables, in each vertex of the satisfactory region, $(n - 1)$ fuzzy variables reach their lower bounds and one fuzzy variable implicitly results from the values of the other variables (see the matrix \mathbf{V} in Appendix B). By analogy, in the case of lower-upper-bounded fuzzy decision variables, in each vertex of the satisfactory region, one fuzzy variable reaches its upper bound, $(n - 2)$ fuzzy variables reach their lower bounds and one fuzzy variable implicitly results from the values of the other variables. Thus, in this case the satisfactory region around the optimal solution has $n(n - 1)$ vertices, denoted T_{rl} , $1 \leq r \leq n$, $1 \leq l \leq n$, $r \neq l$. The meaning of the indices r and l has already been explained in the case of three fuzzy decision variables.

Hence, there are $3n$ decision variables $(x_k, d_k, s_k, k = 1, \dots, n)$ and the following additional constraints imposed by the geometric restrictions (see also relations (28) and (29)):

$$\begin{aligned}
 d_k &\leq x_k, \quad k = 1, \dots, n, \\
 s_k &\leq d_1 + \dots + d_{k-1} + d_{k+1} + \dots + d_n, \\
 d_k &\leq s_1 + \dots + s_{k-1} + s_{k+1} + \dots + s_n, \\
 d_k, s_k &\geq 0.
 \end{aligned}
 \tag{30}$$

The lower bound of a fuzzy variable k is equal to $x_k - d_k$ and the upper bound is equal to $x_k + s_k$.

Once the satisfactory region around the optimal solution has been defined in the case of lower–upper-bounded fuzzy decision variables, a joint membership function on the variables is considered. This joint membership function is defined by its 1-cut and 0-cut, in the same way as the joint membership function of the lower-bounded fuzzy decision variables (see Section 3.1). Hence, the 1-cut contains only the optimal solution (x_1, \dots, x_n) and the 0-cut is equal to the satisfactory region around the optimal solution. For example, in the case of three lower–upper-bounded fuzzy decision variables, the joint membership function of the variables determines a pyramid above the hexagon defined around the optimal solution.

In conclusion, in order to generalize the lower-bounded fuzzy decision variables, dependent lower–upper-bounded fuzzy decision variables linked by a sum constraint and by a joint membership function have been defined.

5.2. Membership functions of the objectives and the constraints

In this section, we present the changes in the fuzzy objectives and in the fuzzy constraints when lower–upper-bounded fuzzy variables are considered instead of lower-bounded fuzzy variables.

Sections 3.2 and 3.3 show that the membership functions of the fuzzy objectives and of the left side of the fuzzy constraints are computed by associating the α -cuts of the objectives’ coefficients, respectively the α -cuts of the constraints’ left hand side coefficients to the α -cuts of the joint membership function of the variables. Moreover, Section 5.1 shows that the only difference between the lower–upper-bounded fuzzy variables and the lower-bounded fuzzy variables is the type of the satisfactory region around the optimal solution. Since this satisfactory region is equal to the 0-cut of the joint membership function of the variables, only the 0-cuts of the fuzzy objectives and of the left hand side of the fuzzy constraints differ when lower–upper-bounded fuzzy variables are used instead of lower-bounded fuzzy variables.

Next, we present the 0-cut of a fuzzy objective \tilde{f}_i , denoted $[f_i^{\min}, f_i^{\max}]$, when lower–upper-bounded fuzzy variables are considered. As in the case of lower-bounded fuzzy variables (see Section 3.2), f_i^{\min} and respectively f_i^{\max} are computed for each vertex T_{rl} , $1 \leq r \leq n$, $1 \leq l \leq n$, $r \neq l$, of the satisfactory region around the optimal solution and then the minimum value of f_i^{\min} and respectively the maximum value of f_i^{\max} are kept. Hence, we have the following relation:

$$f_i^{\min} = \min_{r,l/T_{rl}\text{vertex}} \{ \mathbf{c}_i^{\min} \mathbf{T}_{rl} \}, \quad f_i^{\max} = \max_{r,l/T_{rl}\text{vertex}} \{ \mathbf{c}_i^{\max} \mathbf{T}_{rl} \},
 \tag{31}$$

where $1 \leq r \leq n$, $1 \leq l \leq n$, $r \neq l$, $\mathbf{c}_i^{\min} = (c_{i1}^{\min} \dots c_{in}^{\min})$, $\mathbf{c}_i^{\max} = (c_{i1}^{\max} \dots c_{in}^{\max})$ and the vertices \mathbf{T}_{rl} are defined in Section 5.1. These formulas are equivalent to (the proof is similar to the proof presented in Appendix C):

$$\begin{aligned}
 f_i^{\min} &= \sum_{k=1}^n c_{ik}^{\min} (x_k - d_k) + \min_{r,l} \left\{ c_{il}^{\min} \sum_{k=1}^n d_k + (c_{ir}^{\min} - c_{il}^{\min})(d_r + s_r) \right\}, \\
 f_i^{\max} &= \sum_{k=1}^n c_{ik}^{\max} (x_k - d_k) + \max_{r,l} \left\{ c_{il}^{\max} \sum_{k=1}^n d_k + (c_{ir}^{\max} - c_{il}^{\max})(d_r + s_r) \right\},
 \end{aligned}
 \tag{32}$$

where $1 \leq r \leq n$, $1 \leq l \leq n$, $r \neq l$.

Concerning the left hand side of a fuzzy constraint, its 0-cut is determined using the same reasoning as for a fuzzy objective. The bounds of the 0-cut of a fuzzy constraint j are given by formulas analogous to the formulas in the relation (32), where the coefficients \mathbf{c}_i are replaced by the coefficients \mathbf{a}_j of the constraint j .

As already mentioned, the procedure and the methods applied then to solve the fuzzy problem with lower–upper-bounded fuzzy variables are the same as in the case of lower-bounded fuzzy variables.

6. Conclusions

In this paper, we have proposed a new and original methodology that considers fuzzy decision variables for solving multiobjective fuzzy linear programming problems. In the literature, fuzzy linear programming techniques do not usually consider fuzzy decision variables. Even if fuzzy variables for the problem were considered, they were quickly abandoned because of crisp variables' advantages.

Besides the optimal solutions of the problem, our method supplies to the decision-maker regions containing potential satisfactory solutions around these optimal solutions. Since the final decisions taken by the decision-maker are always crisp, our methodology assists in the choice of these crisp decisions among the fuzzy solutions. Moreover, the results are closely related to the special type of problems we are coping with. In fact, multiobjective fuzzy linear programming problems where the decision variables sum up to a constant are considered. Hence the application field of the proposed method is very large. It concerns real decision problems dealing with a limited amount of resources: resources' allocation problems, portfolio problems, knapsack problems, etc.

The sum constraint on the variables has allowed us to define dependent fuzzy decision variables with a joint membership function. First, lower-bounded fuzzy variables have been considered and then, the method has been generalized to lower–upper-bounded fuzzy variables. The objectives have been defuzzified by means of the area compensation method, that supports the imprecise probability interpretation of the fuzzy parameters of the problem. The constraints have been defuzzified by a worst case approach. Then, the equivalent crisp problem has been solved by an interactive and iterative MAUT method. Finally if necessary, the fuzzy model can be tuned. Simple numerical examples with three decision variables have been discussed in order to illustrate the proposed methodology. The satisfactory region proposed to the decision-maker in this case is a convivial, attractive on view and easily understandable one. By the way, such a real-life application, that consists in supporting the choice of a sustainable heating system for a given house, is presented in [28]. But our methodology can be used in order to solve real-life decision problems with more than three decision variables. Indeed, we provide the decision-maker with an enhanced proposal: if possible, we advice him to adopt the optimal value for each decision variable; if these values are not convenient, any other set of values in the satisfactory region could be an appropriate choice.

Appendix A. Definition of a fuzzy number

Fuzzy numbers are fuzzy sets on the real line that generalize crisp real numbers. More precisely [5], a fuzzy number \tilde{v} is a normalized convex fuzzy set on the real line \mathfrak{R} defined by a membership function $\mu_{\tilde{v}} : \mathfrak{R} \rightarrow [0, 1]$, i.e. $\tilde{v} = \{(x, \mu_{\tilde{v}}(x)) / x \in \mathfrak{R}\}$, such that:

- there exists exactly one $x_o \in \mathfrak{R}$ with $\mu_{\tilde{v}}(x_o) = 1$,
- $\mu_{\tilde{v}}(x)$ is piecewise continuous in \mathfrak{R} ,
- $\mu_{\tilde{v}}(\lambda x + (1 - \lambda)y) \geq \min\{\mu_{\tilde{v}}(x), \mu_{\tilde{v}}(y)\}, \forall x, y \in \mathfrak{R}, \forall \lambda \in [0, 1]$.

This function $\mu_{\tilde{v}}$ assigns a membership grade in the fuzzy set of \tilde{v} to each real number value.

The most common normalized convex fuzzy sets used in fuzzy mathematical programming models are the flat fuzzy numbers (fuzzy intervals) [8]. A normalized convex fuzzy set $\tilde{v} = \{(x, \mu_{\tilde{v}}(x))/x \in \mathfrak{R}\}$, $\mu_{\tilde{v}} : \mathfrak{R} \rightarrow [0, 1]$, is called flat fuzzy number if:

- there exist more than one $x \in \mathfrak{R}$ with $\mu_{\tilde{v}}(x) = 1$,
- $\mu_{\tilde{v}}(x)$ is piecewise continuous in \mathfrak{R} ,
- $\mu_{\tilde{v}}(\lambda x + (1 - \lambda)y) \geq \min\{\mu_{\tilde{v}}(x), \mu_{\tilde{v}}(y)\}$, $\forall x, y \in \mathfrak{R}, \forall \lambda \in [0, 1]$.

The set of real number values having a membership grade equal to 1 defines the core of the fuzzy set. The set of real number values having a non-zero membership grade defines the support of the fuzzy set. Clearly, the smaller the known information about the fuzzy number is, the larger is the size of its fuzzy set support.

A crisp number is a particular case of a fuzzy number: (i) the fuzzy set support reduces to one single point that is the value of the crisp number and (ii) the membership function is equal to 1 on the fuzzy set support.

The α -cuts $\tilde{v}_\alpha = \{x \in \mathfrak{R} / \mu_{\tilde{v}}(x) \geq \alpha\}$, $\alpha \in (0, 1]$, of a fuzzy number \tilde{v} are convex subsets of \mathfrak{R} whose lower and upper limits are represented by:

$$\begin{aligned} v_\alpha^{\min} &= \inf_x \{x \in \mathfrak{R} / \mu_{\tilde{v}}(x) \geq \alpha\}, \\ v_\alpha^{\max} &= \sup_x \{x \in \mathfrak{R} / \mu_{\tilde{v}}(x) \geq \alpha\}. \end{aligned} \tag{A.1}$$

For the sake of simplicity, we suppose that both limits are finite.

Appendix B. Region around the optimal solution

The aim of this appendix is to show that the region around the optimal solution, with the vertices defined by the relation (2) in the case of three fuzzy decision variables, has the generalization, in the case of n fuzzy decision variables, given by the relation (6) (see relations (4) and (5) for notations).

The second term of the relation (6) becomes:

$$\begin{aligned} \mathbf{X} * \mathbf{1}' - \mathbf{D} * \mathbf{1}' + \mathbf{1}' * \mathbf{D} * \mathbf{I} &= \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (1 \cdots 1) - \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} (1 \cdots 1) + (1 \cdots 1) \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \\ &= \begin{pmatrix} x_1 - d_1 & \cdots & x_1 - d_1 \\ \vdots & & \vdots \\ x_n - d_n & \cdots & x_n - d_n \end{pmatrix} + \begin{pmatrix} d_1 + \cdots + d_n & & 0 \\ & \ddots & \\ 0 & & d_1 + \cdots + d_n \end{pmatrix} \\ &= \begin{pmatrix} x_1 + (d_2 + \cdots + d_n) & x_1 - d_1 & \cdots & x_1 - d_1 \\ x_2 - d_2 & x_2 + (d_1 + d_3 + \cdots + d_n) & \cdots & x_2 - d_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_n - d_n & x_n - d_n & \cdots & x_n + (d_1 + \cdots + d_{n-1}) \end{pmatrix} = \mathbf{V}. \end{aligned} \tag{B.1}$$

The columns of this last matrix are the coordinates of the vertices of the region around the optimal solution, in the case of n fuzzy decision variables. Indeed, these coordinates are the generalization of the coordinates of the three vertices given by the relation (2).

Appendix C. f_i^{\min} and f_i^{\max} values of a fuzzy objective \tilde{f}_i

In this appendix, we show that the f_i^{\min} and f_i^{\max} values of a fuzzy objective \tilde{f}_i given by the relation (9) are equivalent to the expressions given in the relation (10). The proof for the f_i^{\min} value is presented below:

$$\begin{aligned}
 f_i^{\min} &= \min_{p/V_p \text{ vertex}} \left\{ \sum_{k=1}^n c_{ik}^{\min} V_{kp} \right\} \\
 &= \min \{ c_{i1}^{\min}(x_1 + (d_2 + \dots + d_n)) + c_{i2}^{\min}(x_2 - d_2) + \dots + c_{in}^{\min}(x_n - d_n), \dots, c_{i1}^{\min}(x_1 - d_1) \\
 &\quad + \dots + c_{ik-1}^{\min}(x_{k-1} - d_{k-1}) + c_{ik}^{\min}(x_k + (d_1 + \dots + d_{k-1} + d_{k+1} + \dots + d_n)) + c_{ik+1}^{\min}(x_{k+1} - d_{k+1}) \\
 &\quad + \dots + c_{in}^{\min}(x_n - d_n), \dots, c_{i1}^{\min}(x_1 - d_1) + \dots + c_{in-1}^{\min}(x_{n-1} - d_{n-1}) + c_{in}^{\min}(x_n + (d_1 + \dots + d_{n-1})) \} \\
 &= \min \{ c_{i1}^{\min}(x_1 - d_1) + \dots + c_{in}^{\min}(x_n - d_n) + c_{i1}^{\min}(d_1 + \dots + d_n), \dots, c_{i1}^{\min}(x_1 - d_1) \\
 &\quad + \dots + c_{in}^{\min}(x_n - d_n) + c_{ik}^{\min}(d_1 + \dots + d_n), \dots, c_{i1}^{\min}(x_1 - d_1) + \dots + c_{in}^{\min}(x_n - d_n) \\
 &\quad + c_{in}^{\min}(d_1 + \dots + d_n) \} \\
 &= \min \left\{ \sum_{k=1}^n c_{ik}^{\min}(x_k - d_k) + c_{i1}^{\min} \sum_{k=1}^n d_k, \dots, \sum_{k=1}^n c_{ik}^{\min}(x_k - d_k) \right. \\
 &\quad \left. + c_{ik}^{\min} \sum_{k=1}^n d_k, \dots, \sum_{k=1}^n c_{ik}^{\min}(x_k - d_k) + c_{in}^{\min} \sum_{k=1}^n d_k \right\} = \sum_{k=1}^n c_{ik}^{\min}(x_k - d_k) + \min_k \{ c_{ik}^{\min} \} \sum_{k=1}^n d_k. \quad (C.1)
 \end{aligned}$$

The proof for the f_i^{\max} value is similar to that for the f_i^{\min} value.

Appendix D. Inclusion of the optimal solution in the satisfactory regions

The constraints (25) impose that the lower bound of each fuzzy variable $k, k = 1, \dots, n$, is smaller than the corresponding optimal value x_k^0 . This appendix shows that the constraints (25) ensure also that x_k^0 is smaller than the upper bound of the fuzzy variable k .

Using the constraints (25), the following inequality is inferred:

$$x_1 + \dots + x_{k-1} + x_{k+1} + \dots + x_n - (d_1 + \dots + d_{k-1} + d_{k+1} + \dots + d_n) \leq x_1^0 + \dots + x_{k-1}^0 + x_{k+1}^0 + \dots + x_n^0. \quad (D.1)$$

Moreover, the sum constraint on the decision variables \mathbf{x} (see relation (1)) ensures the equalities given below:

$$\begin{aligned}
 x_1 + \dots + x_{k-1} + x_{k+1} + \dots + x_n &= K - x_k, \\
 x_1^0 + \dots + x_{k-1}^0 + x_{k+1}^0 + \dots + x_n^0 &= K - x_k^0.
 \end{aligned} \quad (D.2)$$

These equalities are introduced in the inequality (D.1) and the following relation is obtained:

$$K - x_k - (d_1 + \dots + d_{k-1} + d_{k+1} + \dots + d_n) \leq K - x_k^0, \quad (D.3)$$

equivalent to:

$$x_k^0 \leq x_k + (d_1 + \dots + d_{k-1} + d_{k+1} + \dots + d_n). \quad (D.4)$$

Hence, x_k^0 is smaller than the upper bound (see Section 3.1) for the fuzzy variable k .

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