Weyl gravity and Cartan geometry

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Abstract

We point out that the Cartan geometry known as the second-order conformal structure provides a natural differential geometric framework underlying gauge theories of conformal gravity. We are concerned by two theories: the first one will be the associated Yang-Mills-like Lagrangian, while the second, inspired by [1], will be a slightly more general one which will relax the conformal Cartan geometry. The corresponding gauge symmetry is treated within the BRST language. We show that the Weyl gauge potential is a spurious degree of freedom, analogous to a Stueckelberg field, that can be eliminated through the dressing field method. We derive sets of field equations for both the studied Lagrangians. For the second one, they constrain the gauge field to be the 'normal conformal Cartan connection'. Finally, we provide in a Lagrangian framework a justification of the identification, in dimension 4, of the Bach tensor with the Yang-Mills current of the normal conformal Cartan connection, as proved in [2].

Keywords : Conformal field theory, differential geometry, gauge field theories, Cartan connections.

PACS numbers: 11.15.-q, 02.40.Ma, 12.38.-t, 14.20.Dh.

Introduction

In 1918-19, H. Weyl, trying to devise a "truly infinitesimal geometry" that generalizes Riemann's,[1] came up with a spacetime manifold equipped with what today we would call a conformal class of metric: a metric defined up to positive local rescalings. The natural scale-invariant Lagrangian he proposed (of Yang-Mills type, as we could say anachronistically) intended to unify gravity and electromagnetism [3; 4]. The theory turned out to be incompatible with the basic experimental fact of the stability of atomic spectra. But still to this day, scale invariance retains theoretical interest, as witnessed by its importance *e.g.* in string theory and conformal field theory, among many other topics.

In particular, the Lagrangian for Weyl gravity

$$\mathcal{L}_{\text{Weyl}} = -\operatorname{Tr}(W \wedge *W) = -\tfrac{1}{2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} dV \quad (1)$$

introduced by Bach in 1921 [5] and constructed with the Weyl tensor W, is still actively investigated. Solutions of its field equation, the Bach equation, are under study to connect the theory to empirical data and see if it can rival General Relativity (GR). In particular its viability as an alternative to dark matter and dark energy is still under scrutiny, as is its viability as a quantum gravity theory. See the reviews [6; 7] and references therein to get only a sample of the significant literature on the subject.

After the 1956 pioneering work of Utiyama on the gauging of an arbitrary Lie group and its first treatment of gravitation as a gauge theory of the Lorentz group [8], in the late 1970's, several authors

*Supported by a Riemann fellowship.

[1] H. Weyl did so by requiring that not only the directions of vectors at distant points a manifold couldn't be compared without a non-canonical choice of connection, as in Riemann's geometry, but also that neither could be their lengths. This he called "scale freedom" and then "gauge freedom". He thereby originated the notion of gauge symmetry, which would reveal its deepness within quantum mechanics few years later, with the posterity we know.

investigated the question of the gauge structure of gravity (and supergravity) [9–13]. During the same period, some of them studied the gauging of the 15-parameter conformal group extending the Poincaré group, and its supersymmetric counterpart as well [14–18]. For a general review see *e.g.* [19].

Following a more abstract differential geometric approach, authors [20–22] already gave a gauge formulation of conformal gravity within the framework of higher-order frame bundles [23]. The relevant geometry is known as the second-order conformal structure. However, it is better to use an equivalent formulation in terms of Cartan geometry [23; 24], which allows a matrix treatment much closer to the usual gauge field framework familiar to physicists.

As is well known, the geometry of connections on principal fiber bundles is an appropriate mathematical setting for dealing with Yang-Mills gauge theories. Because of its strong link to the space-time manifold $\mathcal{M}$, Cartan geometry provides a natural framework that properly addresses the peculiarity of gravitation among the other interactions. Thus, it would perfectly fit the geometry underlying gauge theories of gravitation, in particular that of Weyl gravity. Accordingly, our aim is to show that the second-order conformal structure is the Cartan geometry underlying a genuine gauge formulation of conformal gravity containing Weyl gravity as a special case.

Moreover, inspection of the explicit field equations obtained in [1] raises the issue whether the field variables could be pieced together into a single object, namely the conformal Cartan connection. Because of the possible geometry underlying Weyl gravity, it may be relevant to give an account of this aspects.

The paper is organized as follows. In section 1 we give a brief description of the second-order conformal structure. In section 2 we write the most natural Yang-Mills like Lagrangian, and a slightly generalized version. We show why the Weyl gauge potential of dilation can be considered as a spurious degree of freedom, and can be suppressed thanks to the so-called dressing field method; the latter is consistent with the locality principle. We also derive field equations and show that they single out the normal conformal Cartan connection as gauge field. In section 3 we make contact with some papers in the literature, in particular with [1] and in addition we will justify the equivalence between the Bach equation and the Yang-Mills current of the normal conformal Cartan connection as found in [2]. Then we conclude. Appendices give some details on how gauge invariance restricts the choices of Lagrangians, as well as a brief recap of the dressing field method.

1 Second-order conformal structure

We refer to [24] and to [23; 25] for a detailed mathematical presentations of Cartan geometry and higher-order frame bundles respectively. Here we just sketch the necessary material to follow our scheme.

The whole structure is modeled on the Klein pair of Lie groups (G, H) where $G = O(2, m)/\{\pm I_{m+2}\}$ and H is the isotropy group such that the corresponding homegeneous space is the compactified Minkowski space $(S^{m-1} \times S^1)/\mathbb{Z}^2 \simeq G/H$. The group H has the following factorized matrix presentation

$$H = K_0\, K_1 = \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & r & \frac{1}{2} r r^t \\ 0 & \mathbb{1} & r^t \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad (2)$$

where $z \in \mathsf{W} = \mathbb{R}^*_+$, $S \in SO(1, m-1)$ and $r \in \mathbb{R}^{m*}$. Here ^{t} stands for the η-transposition, namely for the row vector r one has $r^t = (r\eta^{-1})^T$ (the operation ^{T} being the usual matrix transposition), and $\mathbb{R}^{m*}$ is the dual of $\mathbb{R}^m$. We refer to W as the Weyl group of rescaling. Obviously $K_0 \simeq CO(1, m-1)$, and K_1 is the abelian group of inversions (or conformal boosts).

Infinitesimally we have the Klein pair $(\mathfrak{g}, \mathfrak{h})$ of graded Lie algebras [23]. They decompose respectively as, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \simeq \mathbb{R}^m \oplus \mathfrak{co} \oplus \mathbb{R}^{m*}$, and $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \simeq \mathfrak{co} \oplus \mathbb{R}^{m*}$. In matrix notation,

$$\mathfrak{g} = \left\{ \begin{pmatrix} \epsilon & \iota & 0 \\ \tau & v & \iota^t \\ 0 & \tau^t & -\epsilon \end{pmatrix} \right\} \supset \mathfrak{h} = \left\{ \begin{pmatrix} \epsilon & \iota & 0 \\ 0 & v & \iota^t \\ 0 & 0 & -\epsilon \end{pmatrix} \right\}$$

with $(v - \epsilon \mathbb{1}) \in \mathfrak{co}$, $\tau \in \mathbb{R}^m$, $\iota \in \mathbb{R}^{m*}$ and the η-transposition of the column vector τ is $\tau^t = (\eta\tau)^T$. The graded structure of the Lie algebras, $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, $i, j = 0, \pm 1$ with the abelian Lie subalgebras $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0 = [\mathfrak{g}_1, \mathfrak{g}_1]$, is automatically handled by the matrix commutator.

The second-order conformal structure is a Cartan geometry $(\mathcal{P}, \varpi)$ where $\mathcal{P} = \mathcal{P}(\mathcal{M}, H)$ is a principal bundle over $\mathcal{M}$ with structure group $H =$

K_0K_1, and $\varpi \in \Omega^1(\mathcal{U}, \mathfrak{g})$ is a (local) Cartan connection 1-form on $\mathcal{U} \subset \mathcal{M}$. The curvature of ϖ is given by the structure equation, $\Omega = d\varpi + \frac{1}{2}[\varpi, \varpi] = d\varpi + \varpi^2 \in \Omega^2(\mathcal{U}, \mathfrak{g})$ (the wedge product is tacit: $\varpi^2 = \varpi \wedge \varpi$). Both have matrix representations

$$\varpi = \begin{pmatrix} a & \alpha & 0 \\ \theta & A & \alpha^t \\ 0 & \theta^t & -a \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} f & \Pi & 0 \\ \Theta & F & \Pi^t \\ 0 & \Theta^t & -f \end{pmatrix}. \quad (3)$$

One can single out the so-called *normal* conformal Cartan connection (which is unique) by imposing the constraints

$$\Theta = 0 \ \ \text{(torsion free)} \quad \text{and} \quad {F^a}_{bad} = 0. \quad (4)$$

Together with the $\mathfrak{g}_{-1}$-sector of the Bianchi identity $d\Omega + [\varpi, \Omega] = 0$, (4) implies $f = 0$ (trace free), so that the curvature of the normal Cartan connection reduces to

$$\Omega = \begin{pmatrix} 0 & \Pi & 0 \\ 0 & F & \Pi^t \\ 0 & 0 & 0 \end{pmatrix}.$$

From the normality condition ${F^a}_{bad} = 0$ in (4), follows that α has components (in the θ basis of $\Omega^\bullet(\mathcal{U})$)

$$\alpha_{ab} = -\frac{1}{(m-2)}\left(R_{ab} - \frac{R}{2(m-1)}\eta_{ab}\right) \quad (5)$$

where R and R_{ab} are the Ricci scalar and Ricci tensor associated with the 2-form $R = dA + A^2$. In turn, from (5) follows that

$$F := R + \theta\alpha + \alpha^t\theta^t = W$$

is the Weyl 2-form. By the way, in the gauge $a = 0$, $\Pi := d\alpha + \alpha A = D\alpha$ looks like what we can call the Cotton 2-form.

The principal bundle $\mathcal{P}(\mathcal{M}, H)$ is a second order G-structure, a reduction of the second order frame bundle $L^2\mathcal{M}$; it is thus a "2-stage bundle". The bundle $\mathcal{P}(\mathcal{M}, H)$ over $\mathcal{M}$ can also be seen as a principal bundle $\mathcal{P}_1 := \mathcal{P}(\mathcal{P}_0, K_1)$ with structure group K_1 over $\mathcal{P}_0 := \mathcal{P}(\mathcal{M}, K_0)$.

2 Conformal gauge theories

Yang-Mills conformal Lagrangian

In this geometrical setting given by the above principal bundle $\mathcal{P}(\mathcal{M}, H)$, consider the Cartan connection (3) ϖ as gauge field and its curvature Ω as field strength. A physical theory for treating the dynamics of the gauge field is given by a choice of gauge invariant action functional. For instance, one may take a $\mathcal{H}$-gauge invariant Lagrangian m-form, with $m = \dim\mathcal{M}$ and $\mathcal{H} := \{\gamma : \mathcal{U} \subset \mathcal{M} \to H\}$ the gauge group.

The most obvious and natural choice is to write the Yang-Mills prototype Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{YM}}(\varpi) &= \text{Tr}(\Omega \wedge *\Omega), \quad (6) \\ &= \text{Tr}(F \wedge *F) + 4\Pi \wedge *\Theta + 2f \wedge *f. \end{aligned}$$

At this stage some care is required. Indeed when $\mathcal{H}$ acts, so too does the Weyl gauge group of rescalings, $\mathcal{W} := \{z : \mathcal{U} \subset \mathcal{M} \to \mathsf{W}\}$. In particular, its action on ϖ implies $\theta^W = z\theta$. Hence, given a p-form B, the Hodge operator transforms under $\mathcal{W}$ according to

$$(*B)^W = z^{m-2p} * B.$$

Therefore, the $\mathcal{H}$-invariance of the Lagrangian (6) requires to restrict oneself to a spacetime $\mathcal{M}$ of dimension $m = 4$; [2] this will be assumed throughtout the rest of the paper.

Along the line suggested by [1], one can also choose the slightly more general Lagrangian, which relaxes the conformal Cartan geometry

$$\begin{aligned} &\mathcal{L}_{\text{gen}}(\varpi) = \\ &\quad c_1 \,\text{Tr}(F \wedge *F) + c_3\Pi \wedge *\Theta + c_2 f \wedge *f \quad (7) \end{aligned}$$

with c_1, c_2 and c_3 arbitrary constants.

Some remarks are in order. First, the discrepency from the case $4c_1 = c_3 = 2c_2$ is not quite natural with respect to the underlying geometry. Second, let δ_0 and δ_1 be the infinitesimal actions of the gauge subgroups $\mathcal{K}_0$ and $\mathcal{K}_1$ respectively. One has (see appendix A) $\delta_0\mathcal{L}_{\text{gen}} = 0$ since each of the three terms in (7) is separately $\mathcal{K}_0$-invariant, but

$$\begin{aligned} &\delta_1\mathcal{L}_{\text{gen}} = \quad (8) \\ &\quad (4c_1 - c_3)\,\text{Tr}\left(\Theta\kappa \wedge *F\right) + (c_3 - 2c_2)\kappa\Theta \wedge *f \end{aligned}$$

where κ is the infinitesimal $\mathcal{K}_1$ parameter (i.e an infinitesimal conformal boost). This vanishes only if $\Theta = 0$, or if $4c_1 = c_3 = 2c_2$. The latter case is of course $\mathcal{L}_{\text{gen}} = c_1\mathcal{L}_{\text{YM}}$, the natural choice dictated by the conformal geometry. Confronted with this problem one can adopt three strategies.

[2] This peculiarity of dimension $m = 4$ is very similar to the requirement of the conformal invariance of the Maxwell Lagrangian density $\mathcal{L}_{\text{Maxwell}}(F, g) = F *_g F$. Indeed, $\mathcal{L}_{\text{Maxwell}}(F, z^2g) = z^{m-4}\mathcal{L}_{\text{Maxwell}}(F, g)$ implies $\mathcal{L}_{\text{Maxwell}}(F, z^2g) = \mathcal{L}_{\text{Maxwell}}(F, g)$ for all $z \in \mathcal{W}$ if $m = 4$.

First, one could restore full $\mathcal{H}$-invariance by restricting to a torsion free geometry $\Theta = 0$ from the very beginning. This reduces the Lagangian (7) to

$$\mathcal{L}_{\mathrm{W}}(\varpi) := c_1 \operatorname{Tr}(F \wedge *F) + c_2\ f \wedge *f. \tag{9}$$

As well, if one is willing to allow for torsion, one could *state* that the $\mathcal{K}_1$ gauge group of conformal boost doesn't act, thus breaking by hand the gauge symmetry from $\mathcal{H}$ to $\mathcal{K}_0$.

Finally, the third route consists in erasing the $\mathcal{K}_1$ gauge symmetry by means of the so-called K_1-valued *dressing field* u_1, as described in [26] (see appendix B). This amounts to a local reduction of $\mathcal{P}(\mathcal{M}, H)$ to the subbundle $\mathcal{P}(\mathcal{M}, K_0)$. The dressing of ϖ and Ω respectively gives

$$\varpi_1 := u_1^{-1}\varpi u_1 + u_1^{-1}du_1 = \begin{pmatrix} 0 & \alpha_1 & 0 \\ \theta & A_1 & \alpha_1^t \\ 0 & \theta^t & 0 \end{pmatrix}, \tag{10}$$

$$\Omega_1 := u_1^{-1}\Omega u_1 = d\varpi_1 + \varpi_1^2 = \begin{pmatrix} f_1 & \Pi_1 & 0 \\ \Theta & F_1 & \Pi_1^t \\ 0 & \Theta^t & -f_1 \end{pmatrix}.$$

These are *not* gauge transformations (see B and [26–28]) but $\mathcal{K}_1$-invariant composite fields. Nevertheless, they still transform as $\mathcal{K}_0$-gauge fields. Thus, in ϖ_1, the 1-form A_1 is the genuine spin connection.

In the normal case, that is imposing the condition (4), α_1 is the Schouten 1-form with components given, *mutadis mutandis*, by (5). Since by dressing the gauge invariance of $a_1 = 0$ is guaranteed, the entry $\Pi_1 = d\alpha_1 + A_1\alpha_1$ is the Cotton 2-form. A further consequence is that F_1 is the Weyl 2-form.

By the way, given that $\mathcal{L}_{\mathrm{YM}}(\varpi^{\gamma_1}) = \mathcal{L}_{\mathrm{YM}}(\varpi)$, for $\gamma_1 : \mathcal{U} \to K_1\ \in \mathcal{K}_1$. And using the formal resemblance between gauge transformation and dressing, one has $\mathcal{L}_{\mathrm{YM}}(\varpi) = \mathcal{L}_{\mathrm{YM},1}(\varpi_1)$ with

$$\begin{aligned}\mathcal{L}_{\mathrm{YM},1}(\varpi_1) &= \operatorname{Tr}(\Omega_1 \wedge *\Omega_1) \\ &= \operatorname{Tr}(F_1 \wedge *F_1) + 4\Pi_1 \wedge *\Theta + 2f_1 \wedge *f_1.\end{aligned} \tag{11}$$

This Lagrangian is $\mathcal{K}_1$-invariant because it is constructed with $\mathcal{K}_1$-invariant fields, the only true residual gauge symmetry being $\mathcal{K}_0$ (Lorentz × Weyl). Furthermore, it gives a field equation for the gauge field ϖ_1 which unfolds as three equations only: respectively for the vielbein field θ, the spin connection A_1 and α_1. The Weyl gauge potential of dilation, a in the previous writing of the theory, was a spurious degree of freedom, compensated by an 'artificial' $\mathcal{K}_1$ gauge symmetry.[3]

The analogue of (7) for the dressed variables,

$$\begin{aligned}&\mathcal{L}_{\mathrm{gen},1} = \\ &\quad c_1 \operatorname{Tr}(F_1 \wedge *F_1) + c_3\ \Pi_1 \wedge *\Theta + c_2\ f_1 \wedge *f_1\end{aligned} \tag{12}$$

is invariant under the Lorentz gauge group $\mathcal{SO} \subset \mathcal{K}_0$, but not under the Weyl gauge group $\mathcal{W}$ (see appendix B). Indeed, if δ_W is the infinitesimal Weyl action with parameter $\epsilon \in \mathrm{Lie}\mathcal{W}$ ($z = \exp(\epsilon)$), then

$$\begin{aligned}\delta_W \mathcal{L}_{\mathrm{gen},1} = (4c_1 - c_3) &\operatorname{Tr}\left(\Theta(\partial\epsilon\cdot e^{-1}) \wedge *F_1\right) \\ &+ (c_3 - 2c_2)(\partial\epsilon\cdot e^{-1})\Theta \wedge *f_1.\end{aligned}$$

This vanishes only if $\Theta = 0$, or if $4c_1 = c_3 = 2c_2$, that is $\mathcal{L}_{\mathrm{gen},1} = c_1\mathcal{L}_{\mathrm{YM},1}$, the natural choice for which $\delta_W\mathcal{L}_{\mathrm{YM},1} = 0$ as expected.

But now we have no choice, we cannot freeze the action of the Weyl gauge group $\mathcal{W}$, neither by decree nor by dressing. In order to preserve the $\mathcal{W}$-invariance, one *must* require $\Theta = 0$, the torsionless condition. Implementing the latter in (12) restricts oneself to

$$\mathcal{L}_{\mathrm{W},1}(\varpi_1) = c_1 \operatorname{Tr}(F_1 \wedge *F_1) + c_2\ f_1 \wedge *f_1 \tag{13}$$

as a theory for the gauge potential and field strength

$$\varpi_1 = \begin{pmatrix} 0 & \alpha_1 & 0 \\ \theta & A_1 & \alpha_1^t \\ 0 & \theta^t & 0 \end{pmatrix}, \quad \Omega_1 = \begin{pmatrix} f_1 & \Pi_1 & 0 \\ 0 & F_1 & \Pi_1^t \\ 0 & 0 & -f_1 \end{pmatrix}.$$

Normality and field equations

The field equations deriving from $\mathcal{L}_{\mathrm{YM},1}$ (11) are obtained by varying the corresponding action with respect to (w.r.t.) the dressed Cartan connection ϖ_1 (see (10)). Two contributions must be considered: one is the standard Yang-Mills term, the other comes from variation of the Hodge-$*$ operator, defined w.r.t. the coframe basis $\{\theta\}$ for differential forms:

$$\delta_{\varpi_1} S_{\mathrm{YM},1} = \int \left(\operatorname{Tr}(\delta\varpi_1 \wedge D_1 * \Omega_1) + \delta\theta \wedge T^{\Omega_1}\right) = 0$$

where $D_1 := d + [\varpi_1,\ ]$ and T^{Ω_1} is the energy-momentum 3-form of Ω_1. Thanks to the non-degeneracy of the Killing form and taking into account the various sectors of the Lie algebra, one gets

[3] The dressing field method is shown to be here the inverse of the Stueckelberg procedure, which aims at implementing a gauge symmetry by adding the so-called Stueckelberg field. In the situation at hand, a is such a Stueckelberg field indeed. See appendix in [28] for a discussion.

three equations w.r.t. the respective three gauge fields

$$\begin{aligned}
\delta\alpha_1: &\quad D*\Theta - *F_1\wedge\theta + \theta\wedge *f_1 = 0,\\
\delta A_1: &\quad D*F_1 - *\Theta\wedge\alpha_1 + \alpha_1^t\wedge *\Theta^t\\
&\qquad\qquad + \theta\wedge *\Pi_1 - *\Pi_1^t\wedge\theta = 0,\\
\delta\theta: &\quad D*\Pi_1 - *f_1\wedge\alpha_1 + \alpha_1\wedge *F_1 = -\tfrac{1}{2}T^{\Omega_1},
\end{aligned}$$

where $D := d + [A_1,\ ]$.

The field equations for $\mathcal{L}_{\mathrm{W},1}$ (13) are a special case of those of $\mathcal{L}_{\mathrm{gen},1}$ (12) (see appendix C). They read

$$\begin{aligned}
\delta\alpha_1: &\qquad 2c_1 * F_1\wedge\theta - c_2\theta\wedge *f_1 = 0,\\
\delta A_1: &\qquad\qquad\qquad D*F_1 = 0,\\
\delta\theta: &\quad c_2 * f_1\wedge\alpha_1 - 2c_1\alpha_1\wedge *F_1 = c_1 T^{F_1} + c_2 T^{f_1}.
\end{aligned}$$

Dropping out the subscript "1" for convenience, one has in components,

$$2c_1\ {F^c}_{b,ca} - c_2\ f_{ab} = 0, \tag{14}$$

$$D^c {F^d}_{a,cb} = 0, \tag{15}$$

$$c_2\ \alpha_{a,c}{f_b}^c - 2c_1\ \alpha_{c,d}{{F^c}_{a,b}}^d = c_1 T^{F_1}_{ab} + c_2 T^{f_1}_{ab} \tag{16}$$

with the two energy-momentum tensors,

$$\begin{aligned}
T^{F_1}_{ab} &= \tfrac{1}{4}{F^i}_{j,cd}{{F^j}_{i,}}^{cd}\eta_{ab} + {F^i}_{j,bc}{{F^j}_{i,}}^{cd}\eta_{da},\\
T^{f_1}_{ab} &= \tfrac{1}{4}f_{cd}f^{cd}\eta_{ab} + f_{bc}f^{cd}\eta_{da}.
\end{aligned}$$

A remarkable fact is that the field equations (14) select the (dressed) normal conformal Cartan connection as gauge field, *provided* that $c_2 \neq 2c_1$. Let us prove this.

From the Bianchi identity $D\Omega_1 = [\Omega_1, \varpi_1]$ which is easily written in matrix form, the $\mathfrak{g}_{-1}$-sector reads $d\Theta = (F_1 - f_1\mathbb{1})\theta - A_1\Theta$. Since $\Theta = 0$ this reduces to $(F_1 - f_1\mathbb{1})\theta = 0$, or in components ${F^a}_{[b,cd]} = f_{[cd}{\delta^a}_{b]}$. By contracting over a and b and remembering that ${F^a}_{a,cd} = 0$ since $F \in \mathfrak{so}(1,3)$, one has

$${F^a}_{c,ad} - {F^a}_{d,ac} = -2f_{cd}.$$

Now the antisymmetric part of (14) is

$$c_1({F^c}_{a,cb} - {F^c}_{b,ca}) + c_2 f_{ab} = 0.$$

Combining these two equations, we end up with

$$(c_2 - 2c_1)f_{ab} = 0.$$

Now the point in writing the linear combination (12), thus (13), was to depart from the natural (and rigid) geometric case $c_2 = 2c_1$. So the above equation implies $f_{ab} = 0$, which in turn implies that (14) reduces to

$${F^c}_{acb} = 0. \tag{17}$$

In other words, the field equations of $\mathcal{L}_{W,1}$ single out the dressed normal Cartan connection as gauge field.

Since in this case α_1 is the Schouten 1-form (a function of A_1 through solving (17)), it is not an independent field variable. Furthermore, since $A_1 \in \mathfrak{so}(1,3)$ and $\Theta = 0$, the spin connection A_1 is a function of the vielbein field $e = {e^a}_\mu$. Thus, the only independent gauge field in ϖ_1 is the vielbein 1-form $\theta = e\cdot dx = {e^a}_\mu dx^\mu$.

It is quite easy to see that it induces a conformal class of metrics $\{g\}$. Indeed from (29) in appendix B, one has that the gauge BRST variation of ϖ_1 provides

$$s_L\theta = -v_L\theta, \quad \text{and} \quad s_W\theta = \epsilon\theta,$$

where $v_L \in \mathfrak{so}(1,3)$ is the Lorentz ghost, and ϵ is the Weyl ghost. So that defining a metric by $g := e^T\eta e$, one has the infinitesimal gauge transformations,

$$\begin{aligned}
s_L g &= (s_L e)^T\eta e + e^T\eta s_L e = -e^T(v_L^T\eta + \eta v_L)e = 0,\\
s_W g &= (s_W e)^T\eta e + e^T\eta s_W e = 2\epsilon(e^t\eta e) = 2\epsilon g.
\end{aligned}$$

In other words, at the finite level, one has $g^{\gamma_0} = z^2 g$. This means that the true degrees of freedom of the theory described by $\mathcal{L}_{\mathrm{W},1}$ (13) are those of a conformal class of metric $\{g\}$ ($\frac{m(m+1)}{2} - 1 = 9$ in dimension $m = 4$).

Moreover, in dimension $m = 4$, the tensor $T^{F_1}_{ab}$ vanishes identically, see [29; 30]. It is then easily seen that while (14) enforces the normality, combining (15) and (16) provides particular solutions of the Bach equation,

$$2D_d D^c {F^d}_{a,bc} + \alpha_{c,d}{{F^c}_{a,b}}^d = 0 \tag{18}$$

but do not exhaust them.

3 Discussion

Aiming at finding the vacuum Einstein equations from conformal gravity, the author of [1] (see also [31]) *starts* with the Lagrangian $\mathcal{L}_{\mathrm{W}}$ (9), that is setting $c_3 = 0$ in (7). With this choice of Lagrangian he needs to *assume*, first that $\mathcal{K}_1$ does not act (breaking of the gauge symmetry by hand), and second that $\Theta = 0$ for the field equations to enforce normality. Subsequently, he also requires the gauge fixing condition $a = 0$ (there referred to as the 'Riemann gauge') for the Cartan connection ϖ.

Obtaining the Lagrangian $\mathcal{L}_{\mathrm{W},1}$ (13) by redefining the fields through the dressing field method has several advantages. Indeed, the vanishing of the (dressed) Weyl potential a_1 and the $\mathcal{K}_1$-invariance are simultaneously guaranteed by the dressing construction. Furthermore, $\mathcal{L}_{\mathrm{W},1}$ is $\mathcal{SO}$-invariant, and requiring the invariance under $\mathcal{W}$ *imposes* $\Theta = 0$ right away. Then, the field equations for $\mathcal{L}_{\mathrm{W},1}$, directly select the normal conformal Cartan connection as gauge field.

Suppose that the choice of the constants in $\mathcal{L}_{\mathrm{W},1}$ is taken to be the natural one with respect to the underlying geometry of the second-order conformal structure, $c_2 = 2c_1$. Then the field equations fail to select the (dressed) normal conformal Cartan connection.

The authors of [2] made the mathematical observation that, in dimension 4, the Bach tensor can be identified with the Yang-Mills current of the normal conformal Cartan connection in what they refer to as the 'natural gauge', that is with $a = 0$ (in our notation). This observation receives a clear meaning in the dressing field scheme and in a Lagrangian field theory approach.

Indeed, starting with the normal subgeometry of the second-order conformal structure $\mathcal{P}(\mathcal{M}, H = K_0K_1)$, and after dressing (w.r.t. the K_1 direction), the normal conformal Cartan connection associated to $\mathcal{P}(\mathcal{M}, K_0)$ and its curvature read

$$\varpi_1 = \begin{pmatrix} 0 & \alpha_1 & 0 \\ \theta & A_1 & \alpha_1^t \\ 0 & \theta^t & 0 \end{pmatrix}, \quad \Omega_1 = \begin{pmatrix} 0 & \Pi_1 & 0 \\ 0 & F_1 & \Pi_1^t \\ 0 & 0 & 0 \end{pmatrix},$$

with α_1 the Schouten 1-form, A_1 the spin connection, $\Pi_1 = D\alpha_1$ the Cotton 2-form and F_1 the Weyl 2-form. The natural Yang-Mills Lagrangian then reduces to

$$\mathcal{L}_{\mathrm{YM},1}(\varpi_1) = \mathrm{Tr}(\Omega_1 \wedge *\Omega_1) = \mathrm{Tr}(F_1 \wedge *F_1). \quad (19)$$

Varying of the action w.r.t. ϖ_1 gives

$$\delta_{\varpi_1} S_{\mathrm{YM},1} = \int \mathrm{Tr}(\delta\varpi_1 \wedge D_1 * \Omega_1) + \delta\theta \wedge T^{\Omega_1} = 0,$$

where the energy-momentum T^{Ω_1} reduces to T^{F_1}, which vanishes identically ($m = 4$). Then, the field equation is just the Yang-Mills equation

$$D_1 * \Omega_1 = 0,$$

the Yang-Mills current of [2]. Unfolding it we get,

$$\begin{aligned} \delta\alpha_1 : &\quad *F_1 \wedge \theta = 0, \\ \delta A_1 : &\quad D * F_1 + \theta \wedge *\Pi_1 - *\Pi_1^t \wedge \theta^t = 0, \\ \delta\theta : &\quad D * \Pi_1 + \alpha_1 \wedge *F_1 = 0. \end{aligned}$$

After dualizing through the Hodge $*$ and dropping out once more the subscript 1 for convenience, one has

$$\begin{aligned} \delta\alpha_1 : &\quad F^c{}_{a,cb} = 0, \\ \delta A_1 : &\quad D^j F^a{}_{b,rj} + \Pi_{b,rj}\eta^{aj} + \eta^{aj}\Pi_{j,br} = 0, \\ \delta\theta : &\quad D^c \Pi_{a,bc} + \alpha_{cd} F^c{}_{a,b}{}^d = 0. \end{aligned}$$

The first equation above is identically satisfied because it gives back one of the two conditions of normality assumed from the very beginning. Using the $\mathfrak{g}_0$-sector of the Bianchi identity $D_1\Omega_1 = 0$, which is the well-known result $D_d F^d{}_{a,bc} + \Pi_{a,bc} = 0$, one shows that the second equation above is also identically satisfied. Thus, the only equation giving information is that stemming from the variation of the tetrad field,

$$D^c D_{[b}\alpha_{c],a} + \alpha_{cd} F^c{}_{a,b}{}^d = 0. \quad (20)$$

This is nothing but the Bach equation (in an alternative form equivalent to (18) in dimension 4).

In other words, in dimension 4, the field equation for $\mathcal{L}_{\mathrm{YM},1}$ (19) is the Yang-Mills equation,

$$D_1 * \Omega_1 = \begin{pmatrix} 0 & D * D\alpha_1 + \alpha_1 \wedge *F_1 & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} = 0 \quad (21)$$

and is equivalent to the Bach equation (20).

This was naturally expected since $\mathcal{L}_{\mathrm{YM},1}$ (19) is nothing but the Lagrangian $\mathcal{L}_{\mathrm{Weyl}}$ (1) of Weyl gravity, and as noted above, the vielbein θ is the only independent field in the dressed normal conformal Cartan connection ϖ_1. Thus, variation of $\mathcal{L}_{\mathrm{YM},1}$ under ϖ_1 giving $D_1 * \Omega_1 = 0$ is the same as variation of $\mathcal{L}_{\mathrm{Weyl}}$ under θ giving the Bach equation as usual.

Conclusion

In this paper we highlighted the second-order conformal structure as the global geometrical framework underlying gauge conformal theories of gravity, and the conformal Cartan connection as the natural gauge potential.

We have shown that the Weyl potential a for dilation is a Stueckelberg-like field whose spurious degrees of freedom can be absorbed through the dressing field method. This provides an advantageous substitute to the gauge fixing a=0 imposed in [1], and results in the effective local reduction of the second-order conformal structure to the first-order conformal structure.

We have discussed two choices of Lagrangians, a Yang-Mills type Lagrangian dictated by the conformal geometry and a more generalized one, inspired by [1], which relaxes the conformal geometry. In the latter case, we have stressed that the field equations select the unique (dressed) normal conformal Cartan connection as gauge potential.

Furthermore, in this geometrical setup, we have provided a Yang-Mills theory which justifies (see Lagrangian (19) and eq.(21)) the identification, in dimension 4, [2, see there section 3] of the Bach tensor with the Yang-Mills current of the normal conformal Cartan connection.

A Symmetries of the Lagrangians

Under the gauge group $\mathcal{H} := \{\gamma : \mathcal{U} \subset \mathcal{M} \to H\}$, the curvature Ω transforms by the adjoint: $\Omega^\gamma = \gamma^{-1}\Omega\gamma$. This is why the choice $\mathcal{L}_{\text{YM}}(\varpi) = \text{Tr}(\Omega \wedge *\Omega)$ as $\mathcal{H}$-invariant Lagrangian is natural. To consider other possibilities, it is interesting to pay attention to the action of the subgroups of $\mathcal{H}$.

Consider the gauge transformations

$$\gamma_0 : \mathcal{U} \to K_0 \quad \text{and} \quad \gamma_1 : \mathcal{U} \to K_1$$

elements of the subgroup $\mathcal{K}_0$ and $\mathcal{K}_1$ respectively. Given the matrix representation (2), one has

$$\Omega^{\gamma_0} = \begin{pmatrix} f & z^{-1}\Pi S & 0 \\ S^{-1}\Theta z & S^{-1}FS & S^{-1}\Pi^t z^{-1} \\ 0 & z\Theta^t S & -f \end{pmatrix} \quad \text{and}$$

$$\Omega^{\gamma_1} = \begin{pmatrix} f - r\Theta & \Pi - r(F - f\mathbb{1}) - r\Theta r + \frac{1}{2}rr^t\Theta^t & 0 \\ \Theta & \Theta r + F - r^t\Theta^t & * \\ 0 & \Theta^t & * \end{pmatrix}.$$

By inspection one sees that each term in the natural Lagrangian (6)

$$\mathcal{L}_{\text{YM}}(\varpi) = \text{Tr}(F \wedge *F) + 4\Pi \wedge *\Theta + 2f \wedge *f$$

are separetely $\mathcal{K}_0$-invariant. This means that even the more general Lagrangian

$$\mathcal{L}_{\text{gen}} = c_1 \,\text{Tr}(F \wedge *F) + c_3 \;\Pi \wedge *\Theta + c_2 \; f \wedge *f \tag{22}$$

with c_1, c_2 and c_3 arbitrary constants, is $\mathcal{K}_0$-invariant. Thus, so is the Lagrangian (9) considered in [1; 31].

The $\mathcal{K}_1$-invariance imposes more restrictions. For simplicity, consider an infinitesimal conformal boost $r = \kappa$ (an inversion). The linear variation of Ω is [4]

$$\delta_1\Omega = \begin{pmatrix} -\kappa\Theta & -\kappa(F - f\mathbb{1}) & 0 \\ 0 & \Theta\kappa - \kappa^t\Theta^t & * \\ 0 & 0 & * \end{pmatrix}.$$

It is then easy to show that

$$\delta_1\mathcal{L}_{\text{YM}} = 4\,\text{Tr}\left(\Theta\kappa \wedge *F\right) - 4\kappa F \wedge *\Theta = 0,$$

as expected. But the general Lagrangian transforms as

$$\delta_1\mathcal{L}_{\text{gen}} = (4c_1 - c_3)\,\text{Tr}\left(\Theta\,\kappa \wedge *F\right) + (c_3 - 2c_2)\kappa\,\Theta \wedge *f. \tag{23}$$

This vanishes only if $\Theta = 0$, or if $4c_1 = c_3 = 2c_2$.[5] The latter case is $\mathcal{L}_{\text{gen}} = c_1\mathcal{L}_{\text{YM}}$, the natural choice dictated by the geometry.

If one doesn't want to be restricted to a torsion free geometry, and nevertheless wants to restore full gauge-invariance, then the so-called dressing field method is the way forward. See [26–28] for details, and the following for a brief recap.

[4] Along with the linear variation of the Cartan connection ϖ, they can be both obtained by writing the $\mathcal{K}_1$ sector of the BRST algebra of the theory (the subscript i stands for inversion)

$$s_i\varpi = -dv_i - [\varpi, v_i], \quad s_i\Omega = [\Omega, v_i] \quad \text{and} \quad s_iv_i = -\tfrac{1}{2}[v_i, v_i] = -v_i^2 = 0, \quad \text{with} \quad v_i = \begin{pmatrix} 0 & \kappa & 0 \\ 0 & 0 & \kappa^t \\ 0 & 0 & 0 \end{pmatrix},$$

where v_i is the anticommuting ghost field associated with infinitesimal conformal boosts. See [26] for an extensive treatment of the BRST algebras associated with the second-order conformal structure $\mathcal{P}(\mathcal{M}, H)$.

[5] These relations can also be found by requiring the nilpotency of the BRST operator, $s_i^2\mathcal{L}_{\text{gen}} = 0$.

B The dressing field method

The gauge group of a gauge field theory is defined as $\mathcal{H} := \{\gamma : \mathcal{U} \to H\}$ and acts on itself by $\gamma_1^{\gamma_2} = \gamma_2^{-1}\gamma_1\gamma_2$ for any $\gamma_1, \gamma_2 \in \mathcal{H}$. It acts on the gauge potential and the field strength according to,

$$A^\gamma = \gamma^{-1}A\gamma + \gamma d\gamma, \qquad F^\gamma = \gamma^{-1}F\gamma. \tag{24}$$

Suppose the theory also contains a (Lie) group-valued field $u : \mathcal{U} \to G'$ defined by its transformation under $\mathcal{H}' = \{\gamma' : \mathcal{U} \to H'\}$, where $H' \subseteq H$ is a subgroup, given by $u^{\gamma'} := \gamma'^{-1}u$. One can then define the following *composite fields*,

$$\widehat{A} := u^{-1}Au + u^{-1}du, \qquad \widehat{F} := u^{-1}Fu. \tag{25}$$

The Cartan structure equation holds for the dressed curvature $\widehat{F} = d\widehat{A} + \widehat{A}^2$.

Despite the formal similarity with (24), the composite fields (25) are not mere gauge transformations since $u \notin \mathcal{H}$, as witnessed by its transformation property under $\mathcal{H}'$ and the fact that in general G' can be different from H. This implies that the composite field $\widehat{A}$ does no longer belong to the space of local connections.

As is easily checked, the composite fields (25) are $\mathcal{H}'$-invariant and are only subject to residual gauge transformation laws in $\mathcal{H} \setminus \mathcal{H}'$. In the case $H' = H$, these composite fields are $\mathcal{H}$-gauge invariants.

It is easy to show that the BRST gauge algebra pertaining to a pure gauge theory is modified by the dressing as

$$s\widehat{A} = -\widehat{D}\widehat{v} = -d\widehat{v} - [\widehat{A}, \widehat{v}], \quad s\widehat{F} = [\widehat{F}, \widehat{v}],$$
$$\text{and} \quad s\widehat{v} = -\tfrac{1}{2}[\widehat{v}, \widehat{v}] = -\widehat{v}^{\,2}, \tag{26}$$

upon defining the *composite ghost*

$$\widehat{v} := u^{-1}vu + u^{-1}su\ . \tag{27}$$

It encodes the infinitesimal residual gauge symmetry, if any. If $\widehat{v} = 0$, the BRST algebra (26) becomes trivial, thus expressing the gauge invariance of the composite fields.

As for the case of the second-order conformal structure, the gauge group is $\mathcal{H} = \mathcal{K}_0\mathcal{K}_1$, and it is possible to reduce $\mathcal{H}$ down to $\mathcal{K}_0$ by dressing in the $\mathcal{K}_1$-direction. Consider the field $u_1 : \mathcal{U} \to K_1$ with

$$u_1 = \begin{pmatrix} 1 & q & \frac{qq^t}{2} \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix}.$$

Imposing on the Cartan connection ϖ the gauge-like condition $\chi(\varpi^{u_1}) = a^{u_1} = a - q\theta = 0$ and solving for q, one can check that $u_1^{\gamma_1} = \gamma_1^{-1}u_1$ for $\gamma_1 \in \mathcal{K}_1$. Then u_1 is indeed a $\mathcal{K}_1$-dressing field which can be used to form the $\mathcal{K}_1$-invariant composite fields [6]

$$\varpi_1 := u_1^{-1}\varpi u_1 + u_1^{-1}du_1, \quad \text{and} \quad \Omega_1 := u_1^{-1}\Omega u_1$$

whose matrix forms are displayed in (10). These fields are well behaved as $\mathcal{K}_0$-gauge fields, so that the dressing amount to a (local) reduction of the second-order conformal structure $\mathcal{P}(\mathcal{M}, H)$ to the first-order conformal structure $\mathcal{P}(\mathcal{M}, K_0)$. See [26] for details.

Furtermore, one can also check that not only $\chi\big((\varpi^{\gamma_1})^{u_1^{\gamma_1}}\big) = \chi(\varpi^{u_1})$, which is the gauge-like condition's $\mathcal{K}_1$-invariance that enforces the dressing transformation law for u_1, but also that $\chi\big((\varpi^{\gamma})^{u_1^{\gamma}}\big) = \chi(\varpi^{u_1})$ for $\gamma \in \mathcal{H}$. Which means that the condition $a_1 := a^{u_1} = 0$ in the dressed field ϖ_1 displayed in (10), is fully $\mathcal{H}$-invariant.

The BRST algebra of $\mathcal{P}(\mathcal{M}, H)$ is modified. The initial full ghost is,

$$v = v_0 + v_i = v_W + v_L + v_i = \begin{pmatrix} \epsilon & \kappa & 0 \\ 0 & v_L & \kappa^t \\ 0 & 0 & -\epsilon \end{pmatrix} \in \text{Lie}\mathcal{H}$$

with $v_0 = v_W + v_L \in \text{Lie}\mathcal{K}_0$ the decomposition in the Weyl and Lorentz sector, and $v_i \in \text{Lie}\mathcal{K}_1$ the ghost of conformal boost.

After dressing the composite ghost is

$$v_1 := u_1^{-1}vu_1 + u_1^{-1}su_1 = \begin{pmatrix} \epsilon & \partial\epsilon{\cdot}e^{-1} & 0 \\ 0 & v_L & (\partial\epsilon{\cdot}e^{-1})^t \\ 0 & 0 & -\epsilon \end{pmatrix} \tag{28}$$

where $\partial\epsilon \cdot e^{-1} = \partial_\mu\epsilon\, e^\mu{}_a$ replaces the ghost of conformal boost κ. The associated modified BRST algebra is

$$s_1\varpi_1 = -D_1v_1, \qquad s_1v_1 = -v_1^2 \tag{29}$$
$$s_1\Omega_1 = [\Omega_1, v_1]$$

with $s_1^2 = 0$. Now, since the composite ghost (28) admits the decomposition,

$$v_1 = v_L + v_W' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & v_L & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \epsilon & \partial\epsilon{\cdot}e^{-1} & 0 \\ 0 & 0 & (\partial\epsilon{\cdot}e^{-1})^t \\ 0 & 0 & -\epsilon \end{pmatrix}.$$

[6] In order to stick to [26] the $\widehat{\ }$ has been dropped out as in the main text.

The algebra (29) splits into two subalgebras,

$$\begin{aligned} s_L\varpi_1 &= -D_1 v_L, & s_1 v_L &= -v_L^2, \\ s_W\varpi_1 &= -D_1 v'_W, & s_1 v_1 &= -{v'_W}^2 \\ (s_L\Omega_1 &= [\Omega_1, v_L], & s_W\Omega_1 &= [\Omega_1, v'_W]) \end{aligned}$$

with $s_L^2 = 0$ and $s_W^2 = 0$.[7]

In the Lorentz sector let us write explicitly,

$$s_L\Omega_1 = \begin{pmatrix} 0 & \Pi_1 v_L & 0 \\ -v_L\Theta & [F_1, v_1] & -v_L\Pi_1^t \\ 0 & \Theta^t v_L & 0 \end{pmatrix}.$$

This readily gives $s_L\mathcal{L}_{\text{YM},1} = 0$ since each piece in (11) is inert under s_L. This also means that the more general Lagrangian

$$\mathcal{L}_{\text{gen},1} = \\ c_1 \operatorname{Tr}(F_1 \wedge *F_1) + c_3\ \Pi_1 \wedge *\Theta + c_2\ f_1 \wedge *f_1 \tag{30}$$

enjoys Lorentz invariance, $s_L\mathcal{L}_{\text{gen},1} = 0$.

In the Weyl subalgebra, let us write explicitly

$$s_W\Omega_1 = \begin{pmatrix} -(\partial\epsilon\cdot e^{-1})\Theta & -\epsilon\Pi_1 - (\partial\epsilon\cdot e^{-1})\big(F_1 - f_1\mathbb{1}\big) & 0 \\ \Theta\epsilon & \Theta(\partial\epsilon\cdot e^{-1}) - (\partial\epsilon\cdot e^{-1})^t\Theta^t & * \\ 0 & \epsilon\Theta^t & * \end{pmatrix}.$$

One can easily show that

$$\begin{aligned} s_W\mathcal{L}_{\text{gen},1} = (4c_1 - c_3) \operatorname{Tr}\big(\Theta(\partial\epsilon\cdot e^{-1}) \wedge *F_1\big) \\ + (c_3 - 2c_2)(\partial\epsilon\cdot e^{-1})\Theta \wedge *f_1 \end{aligned}$$

which is the analogue of (8) but where the infinitesimal conformal boost κ has been replaced by $\partial\epsilon\cdot e^{-1}$. This vanishes only if $\Theta = 0$, or if $4c_1 = c_3 = 2c_2$.[8] The latter case is $\mathcal{L}_{\text{gen},1} = c_1\mathcal{L}_{\text{YM},1}$, the natural choice for which $s_W\mathcal{L}_{\text{YM},1} = 0$ is expected.

C General field equations

For the sake of completeness, we here provide the field equations for $\mathcal{L}_{\text{gen},1}$ stemming from the variations $\delta\alpha_1$, δA_1 and $\delta\theta$ respectively,

$$\begin{aligned} c_3 D * \Theta - 4c_1 * F_1 \wedge \theta + 2c_2\theta \wedge *f_1 &= 0, \\ 2c_2 D * F_1 - \tfrac{c_3}{2}\big(* \Theta \wedge \alpha_1 - \alpha_1^t \wedge *\Theta^t\big) & \\ + \tfrac{c_3}{2}\big(\theta \wedge *\Pi_1 - *\Pi_1 \wedge \theta^t\big) &= 0, \\ c_3 D * \Pi_1 - 2c_2 * f_1 \wedge \alpha_1 + 4c_1\alpha_1 \wedge *F_1 &= -\ 2T^{EM} \end{aligned}$$

where $D := d + [A_1,\ ]$. Applying the Hodge star operator to get equations for 1-forms, and dropping the subscript "1" for convenience, one has in components,

$$\begin{aligned} & c_3\ D^c\Theta^d{}_{,ac}\eta_{db} - 4c_1\ F^c{}_{b,ca} + 2c_2\ f_{ab} = 0, \\ & 2c_1\ D^cF^d{}_{a,bc} + \tfrac{c_3}{2}\big(\Pi_{a,bc}\eta^{cd} - \eta^{dc}\Pi_{c,ba}\big) \\ & \quad -\tfrac{c_3}{2}\big(\Theta^d{}_{,bc}\alpha_{a,e}\eta^{ec} - \eta^{dc}\alpha_{c,e}\Theta^n{}_{bm}\eta_{na}\eta^{em}\big) = 0, \\ & 2c_2\ \alpha_{a,c}f_b{}^c - c_3\ D^c\Pi_{a,bc} - 4c_1\ \alpha_{c,d}F^c{}_{a,b}{}^d = 2T^{EM}_{ab} \end{aligned}$$

with the symmetric energy-momentum tensor,

$$\begin{aligned} T^{EM}_{ab} = {} & c_1\big(\tfrac{1}{4}F^i{}_{j,cd}F^j{}_{i,}{}^{cd}\eta_{ab} + F^i{}_{j,bc}F^j{}_{i,}{}^{cd}\eta_{da}\big) \\ & + c_3\big(\tfrac{1}{4}\Pi_{j,cd}\Theta^{j,cd}\eta_{ab} + \Pi_{j,bc}\Theta^{j,cd}\eta_{da}\big) \\ & + c_2\big(\tfrac{1}{4}f_{cd}f^{cd}\eta_{ab} + f_{bc}f^{cd}\eta_{da}\big). \end{aligned}$$

Notice that the last term (which is similar to the energy-momentum tensor of Electromagnetism), exists even if the gauge field of Weyl dilation a_1 vanishes.

Obviously, with the natural values $4c_1 = c_3 = 2c_2$ the above equations reduce to those of $\mathcal{L}_{\text{YM},1}$. For $c_3 = 0$ they provide the equations for $\mathcal{L}_{\text{W},1}$.

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[7] And $s_Ls_W + s_Ws_L = 0$ since $s_Lv_W = -v_Lv_W$ and $s_Wv_L = -v_Wv_L$.

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