On Carrollian and Galilean higher-spin algebras

Andrea Campoleoni

Physique de l'Univers, Champs et Gravitation

UMONS
Université de Mons

based on A.C., S. Pekar, arXiv:2109.xxxxx

ESI workshop "Geometry for Higher Spin Gravity", *2/9/2021*

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∞-dim Lie algebras & higher spins

- Historical path to Vasiliev theories
  - *1987*: proposal for a **higher-spin algebra** in AdS_4 — Fradkin, Vasiliev
  - *1990*: procedure to implement its **gauging** → Vasiliev's equations — Vasiliev
  - 2003: higher-spin algebras and interacting e.o.m. in AdS_D — Vasiliev
- Other recent (and less recent) developments
  - 3D HS algebras → Chern-Simons gauge theories (& matter couplings)
    Blencowe (1989); Porkushkin, Vasiliev (1999) & many others…
  - (Eastwood-)Fradkin-Vasiliev algebras ⇔ non-Abelian "gauge" algebras associated with the known perturbatively-local cubic vertices in AdS_D
    Maldacena, Zhiboedov (2011); Boulanger, Ponomarev, (Joung), Skvortsov, Taronna (2013)
  - HS algebras for mixed symmetry and partially-massless fields
    Boulanger, Skvortsov (2011); Joung, Mkrtchyan (2016)

Higher spins & (A)dS

- Why (massless) HS fields like (A)dS?
  - Minimal coupling to gravity? OK in (A)dS, obstructed for $\Lambda = 0$ — Aragone, Deser (1979); Fradkin, Vasiliev (1987)
  - Long-range HS interactions:
    - in flat-space → trivial S-matrix — Weinberg (1964)
    - in AdS → free CFT boundary correlators → "soluble" AdS/CFT

      Sezgin, Sundell (2002); Klebanov, Polyakov (2002); Maldacena, Zhiboedov (2011) et al.
- May Minkowski still play a role?
  - Is String Theory a broken phase of a HS gauge theory?
  - Models with trivial S-matrix, but non-trivial interactions (*& symmetries*)?

    see, e.g., the talk by Zhenya Skvortsov & A.C., Francia, Heissenberg (2017)
- Outlook: *flat and "non-AdS" holography with higher spins*

  see the talks by Daniel Grumiller and Stefan Prohazka

Higher-spin algebras

- Key ingredient in building HS theories and studying HS holography
- **What is a HS algebra?** *Lie algebra on traceless Killing tensors*
  - Poincaré & (A)dS algebras: isometries of the vacuum

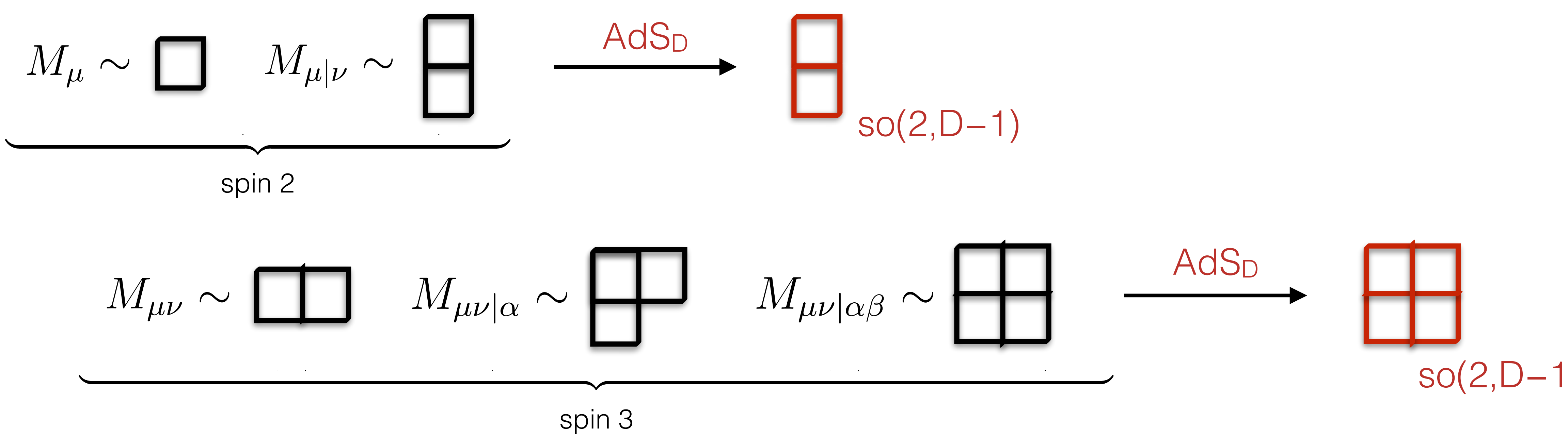
HS "isometries" of the vacuum

- Fronsdal's gauge transf.: $\delta\varphi_{\mu_1\cdots\mu_s} = \bar{\nabla}_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)} + \mathcal{O}(\varphi)$
- Vacuum-preserving symm.: $\bar{\nabla}_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)} = 0$
- Solution (in Minkowski): $\epsilon_{\mu_1\cdots\mu_{s-1}} = \sum_{k=0}^{s-1} M_{\mu_1\cdots\mu_{s-1}|\nu_1\cdots\nu_k} x^{\nu_1}\cdots x^{\nu_k}$

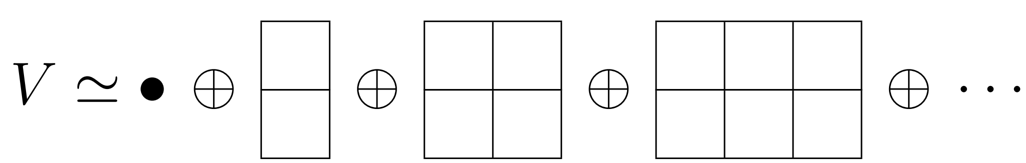
$$\epsilon_{\mu_1\cdots\mu_{s-3}\lambda}{}^{\lambda} = 0$$

Higher-spin algebras

- Vector space of traceless Killing tensors:



Eastwood-Vasiliev algebras in any D: non-Abelian Lie algebras on V including a so(2,D−1) subalgebra



Fradkin, Vasiliev (1987);
Vasiliev (2003);
Eastwood (2002)

Higher-spin algebras

so(2,D−1): isometries of AdS_D &
conformal symmetries (in D−1)

- Vector space of traceless Killing te

$M_\mu \sim \ydiagram{1} \qquad M_{\mu|\nu} \sim \ydiagram{1,1}$ (spin 2) $\xrightarrow{AdS_D}$ $\ydiagram{1,1}$ so(2,D−1)

$M_{\mu\nu} \sim \ydiagram{2} \qquad M_{\mu\nu|\alpha} \sim \ydiagram{2,1} \qquad M_{\mu\nu|\alpha\beta} \sim \ydiagram{2,2}$ (spin 3) $\xrightarrow{AdS_D}$ $\ydiagram{2,2}$ so(2,D−1)

Eastwood-Vasiliev algebras in any D: non-Abelian Lie algebras on V including a so(2,D−1) subalgebra

$$V \simeq \bullet \oplus \ydiagram{1,1} \oplus \ydiagram{2,2} \oplus \ydiagram{3,3} \oplus \cdots$$

Fradkin, Vasiliev (1987);
Vasiliev (2003);
Eastwood (2002)

Notable so(2,D−1) Inönü-Wigner contractions

AdS isometries — Minkowski isometries

$$so(2,D-1) \xrightarrow{\Lambda \to 0} iso(1,D-1)$$

Notable so(2,D–1) Inönü-Wigner contractions

AdS isometries

so(2,D–1)

conformal algebra

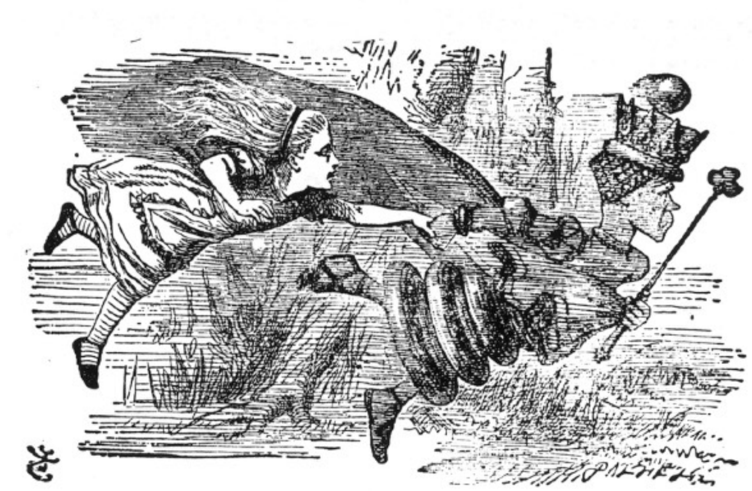
$\Lambda \to 0$

$c \to 0$

Minkowski isometries

iso(1,D–1)

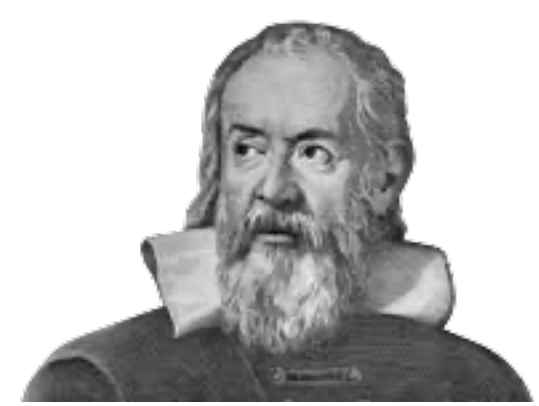
conformal Carroll algebra



$c \to \infty$

gca_{D-1}

conformal Galilei algebra



What about higher-spin algebras?

Goals & strategy/hypotheses

- **Goal:** classify Lie algebras defined on the vector space V (traceless Killing tensors) that

  1. contain a Poincaré subalgebra, **iso(1,D−1)**
  2. contain a conformal Galilei subalgebra, $\mathbf{gca}_{\mathbf{D-1}}$

  ...and discuss their properties

- **Strategy:** look for *coset algebras*, obtained by quotienting out an ideal from the universal enveloping algebras of iso(1,D−1) or gca_{D-1} (bonus: "good" Lorentz transf. for free)

Eastwood (2002)

partial classification, still with interesting examples

HS algebras in AdS_D

Conformal HS algebras in $D-1$ dimensions

Coset construction of HS algebras

- so(2,D−1) algebra: $[J_{AB}\,,\,J_{CD}] = \tilde{\eta}_{AC}\,J_{BD} - \tilde{\eta}_{BC}\,J_{AD} - \tilde{\eta}_{AD}\,J_{BC} + \tilde{\eta}_{BD}\,J_{AC}$

- Quadratic products of the generators

$$J_{A(B} \odot J_{C)D} - \text{traces} \sim \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \qquad\qquad C_2 \equiv \frac{1}{2}\,J_{AB} \odot J^{BA} \sim \bullet$$

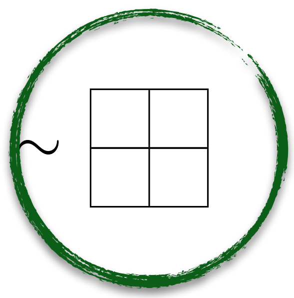
$$\mathcal{I}_{AB} \equiv J_{C(A} \odot J_{B)}{}^{C} - \frac{2}{D+1}\,\tilde{\eta}_{AB}\,C_2 \sim \begin{array}{|c|c|}\hline & \\ \hline \end{array} \qquad\qquad \mathcal{I}_{ABCD} \equiv J_{[AB} \odot J_{CD]} \sim \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

Coset construction of HS algebras

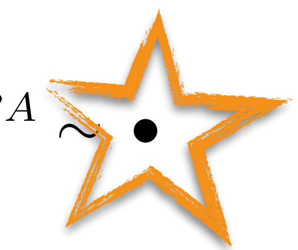
- so(2,D−1) algebra: $[J_{AB}\,,\,J_{CD}] = \tilde{\eta}_{AC}\,J_{BD} - \tilde{\eta}_{BC}\,J_{AD} - \tilde{\eta}_{AD}\,J_{BC} + \tilde{\eta}_{BD}\,J_{AC}$

- Quadratic products of the generators

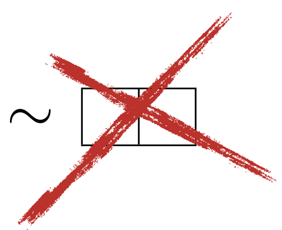
$$J_{A(B} \odot J_{C)D} - \text{traces} \sim$$



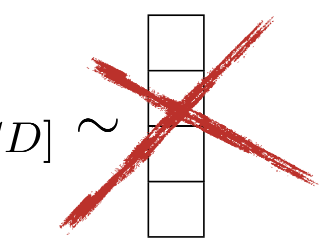
$$C_2 \equiv \frac{1}{2}\,J_{AB} \odot J^{BA} \sim$$



$$\mathcal{I}_{AB} \equiv J_{C(A} \odot J_{B)}{}^{C} - \frac{2}{D+1}\,\tilde{\eta}_{AB}\,C_2 \sim$$



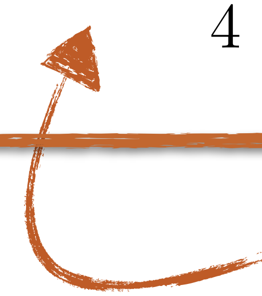
$$\mathcal{I}_{ABCD} \equiv J_{[AB} \odot J_{CD]} \sim$$



- Eastwood-Vasiliev algebras:

$$\mathfrak{hs}_D = \frac{\mathcal{U}(\mathfrak{so}(2,D-1))}{\langle \mathcal{I}_{AB} \oplus \mathcal{I}_{ABCD} \rangle} \;\Rightarrow\; C_2 \sim -\frac{(D+1)(D-3)}{4}\,id$$

see the lectures by Misha Vasiliev



scalar singleton module

Special cases: D=3

- so(2,2) algebra: $[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}\,, \quad [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n}\,, \quad [\mathcal{L}_m, \bar{\mathcal{L}}_n] = 0$

- Ideal to be factored out: $\mathcal{I}_{AB} \sim 0 \quad \Rightarrow \quad \mathcal{L}_m \bar{\mathcal{L}}_n \sim 0$

- No need to factor out $W \equiv \frac{1}{8}\,\varepsilon^{ABCD}\mathcal{I}_{ABCD}$ but $W^2 \sim \frac{1}{4}\,(C_2)^2$

- Still, better to get rid of C_2: $C_2 = 2\left(\mathcal{L}^2 + \bar{\mathcal{L}}^2\right) \sim \frac{\lambda^2-1}{2}\,id$

- Not factorising $\mathcal{I}_{ABCD}$ gives a one-parameter family of HS algebras

$$\mathfrak{hs}_3[\lambda] = id \oplus W \oplus \mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda] \qquad \text{with} \quad \mathbb{1} \oplus \mathfrak{hs}[\lambda] = \frac{\mathcal{U}(\mathfrak{sl}\,(2,\mathbb{R}))}{\left\langle \mathcal{C}_2 - \frac{\lambda^2-1}{4}\,\mathbb{1} \right\rangle}$$

Special cases: D=3

- Ideal to be factored out:

$$\mathcal{I}_{AB} \sim 0 \quad \Rightarrow \quad \mathcal{L}_m \bar{\mathcal{L}}_n \sim 0 \qquad C_2 \sim \frac{\lambda^2 - 1}{2}\, id \qquad W^2 \sim \frac{1}{4}(C_2)^2$$

- Are we evaluating *U*(so(2,2)) on which module?

- Simple answer for $\lambda \in \mathbb{N}$:

$$\mathcal{L}_m = \begin{pmatrix} l_m & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\mathcal{L}}_m = \begin{pmatrix} 0 & 0 \\ 0 & \bar{l}_m \end{pmatrix}$$ with l_m *N*× *N irrep of* $\mathfrak{sl}(2,\mathbb{R})$

$$\Rightarrow \quad C_2 = \frac{N^2-1}{2}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad W = \frac{N^2-1}{4}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

Casimir operator not proportional to the identity

Special cases: D=5

- Again, no need to factor out $\mathcal{I}_{ABCD}$

- Ideal:

$$\mathcal{I}_{AB} \equiv J_{C(A}J_{B)}{}^{C} - \frac{1}{3}\,\tilde{\eta}_{AB}\,C_2\,,$$

$$\mathcal{I}^{\lambda}_{ABCD} \equiv J_{[AB}J_{CD]} - i\,\frac{\lambda}{6}\,\varepsilon_{ABCDEF}J^{EF}$$

mixing terms with different # of J_{AB}

- One parameter family of HS algebras

Boulanger, Skvortsov (2011)

$$\mathfrak{hs}_5[\lambda] = \frac{\mathcal{U}(\mathfrak{so}(2,4))}{\langle \mathcal{I}_{AB} \oplus \mathcal{I}^{\lambda}_{ABDC} \rangle} \quad \Rightarrow \quad C_2 \sim 3\left(\lambda^2 - 1\right) id$$

Warming up in 3D

Carrollian and Galilean 3D HS algebras

- iso(1,2) and gca₂ are isomorphic

Bagchi, Gopakumar, Mandal, Miwa (2009)

- $\mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda]$ algebra:

$$P_m^{(s)} \equiv \epsilon \left(\mathcal{L}_m^{(s)} - \bar{\mathcal{L}}_m^{(s)} \right) , \qquad L_m^{(s)} \equiv \mathcal{L}_m^{(s)} + \bar{\mathcal{L}}_m^{(s)}$$

$$\left[P_m^{(s)}, P_n^{(t)} \right] = \epsilon^2 \sum_{\substack{u=|s-t|+2 \\ s+t+u \text{ even}}}^{s+t-2} g_{s+t-u}^{st}(m,n;\lambda) L_{m+n}^{(u)} \, ,$$

$$\left[L_m^{(s)}, P_n^{(t)} \right] = \sum_{\substack{u=|s-t|+2 \\ s+t+u \text{ even}}}^{s+t-2} g_{s+t-u}^{st}(m,n;\lambda) P_{m+n}^{(u)} \, ,$$

$$\left[L_m^{(s)}, L_n^{(t)} \right] = \sum_{\substack{u=|s-t|+2 \\ s+t+u \text{ even}}}^{s+t-2} g_{s+t-u}^{st}(m,n;\lambda) L_{m+n}^{(u)} \, ,$$

Carrollian and Galilean 3D HS algebras

- iso(1,2) and gca_2 are isomorphic

Bagchi, Gopakumar, Mandal, Miwa (2009)

- Carrollian (aka flat!) & Galilean limits defined by $\epsilon \to 0$

$$P_m^{(s)} \equiv \epsilon \left(\mathcal{L}_m^{(s)} - \bar{\mathcal{L}}_m^{(s)} \right) , \qquad L_m^{(s)} \equiv \mathcal{L}_m^{(s)} + \bar{\mathcal{L}}_m^{(s)}$$

Blencowe (1989); Afshar, Bagchi, Fareghbal, Grumiller, Rosseel (2013); Gonzalez, Matulich, Pino, Troncoso (2013); Ammon, Grumiller, Prohazka, Riegler, Wutte (2017)

$$\left[P_m^{(s)}, P_n^{(t)} \right] = 0$$

ihs[λ] algebra

$$\left[L_m^{(s)}, P_n^{(t)} \right] = \sum_{\substack{u=|s-t|+2 \\ s+t+u \text{ even}}}^{s+t-2} g_{s+t-u}^{st}(m,n;\lambda) P_{m+n}^{(u)} \,,$$

$$\left[L_m^{(s)}, L_n^{(t)} \right] = \sum_{\substack{u=|s-t|+2 \\ s+t+u \text{ even}}}^{s+t-2} g_{s+t-u}^{st}(m,n;\lambda) L_{m+n}^{(u)} \,,$$

[P,P] ≃ ~~L~~

[L,P] ≃ P

[L,L] ≃ L

see the talk by Stefan Prohazka

Coset construction from *U*(iso(1,2))

- $\mathfrak{hs}[\lambda]$ generators: $\mathcal{L}^{(s)}_{\pm(s-1)} \equiv (\mathcal{L}_\pm)^{s-1}$ & $\mathcal{L}^{(s)}_{m\mp1} \equiv \dfrac{\mp1}{s\pm m-1}\,[\,\mathcal{L}_\mp, \mathcal{L}^{(s)}_m\,]$
- We wish to get [P,P] ≃ 0, [L,P] ≃ P and [L,L] ≃ L
- ***Which option do you choose?***

A $$P^{(s)}_{\pm(s-1)} \equiv (P_\pm)^{s-1} \quad \& \quad L^{(s)}_{\pm(s-1)} \equiv (s-1)\,(P_\pm)^{s-2}\,L_\pm$$

B $$L^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-1} \quad \& \quad P^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-2}\,P_\pm$$

(other components fixed by [$L_{\pm}$, .])

Coset construction from $U(iso(1,2))$

- $\mathfrak{hs}[\lambda]$ generators: $\mathcal{L}^{(s)}_{\pm(s-1)} \equiv (\mathcal{L}_\pm)^{s-1}$ & $\mathcal{L}^{(s)}_{m\mp1} \equiv \frac{\mp1}{s \pm m - 1}[\mathcal{L}_\mp, \mathcal{L}^{(s)}_m]$

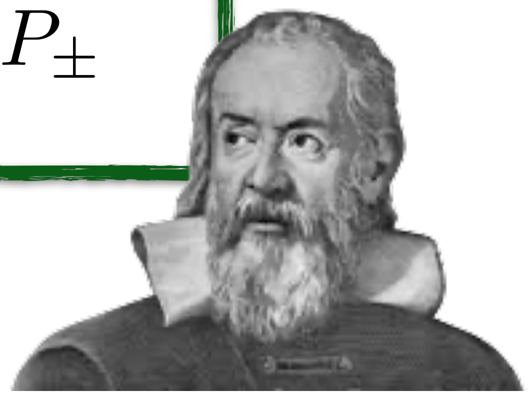
- We wish to get [P,P] ≃ 0, [L,P] ≃ P and [L,L] ≃ L

- ***Which option do you choose?***

commutators close, but we can only get ihs[∞]

A $$P^{(s)}_{\pm(s-1)} \equiv (P_\pm)^{s-1} \quad \& \quad L^{(s)}_{\pm(s-1)} \equiv (s-1)\,(P_\pm)^{s-2}\,L_\pm$$

B $$L^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-1} \quad \& \quad P^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-2}\,P_\pm$$



(other components fixed by [L±, .])

Coset construction from U(iso(1,2))

see also Ammon, Pannier, Riegler (2009)

- HS generators: $L^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-1}$ & $P^{(s)}_{\pm(s-1)} \equiv (L_\pm)^{s-2} P_\pm$ etc.

- Consistency conditions to recover the ihs[λ] commutators:

$$P_m P_n \sim 0 \qquad L_m P_n \sim P_m L_n \qquad L^2 - \frac{\lambda^2-1}{4}\, id \sim 0$$

Poincaré ideal

- On which representation are we evaluating U(iso(1,2))?

$$L_m = \begin{pmatrix} l_m & 0 \\ 0 & l_m \end{pmatrix}, \quad P_m = \begin{pmatrix} 0 & l_m \\ 0 & 0 \end{pmatrix}$$ with l_m $N \times N$ *irrep of* $\mathfrak{so}(1,2) \simeq \mathfrak{sl}(2,\mathbb{R})$

$$\Rightarrow \quad L^2 = \begin{pmatrix} l^2 & 0 \\ 0 & l^2 \end{pmatrix} = \frac{N^2-1}{4} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad W = \begin{pmatrix} 0 & l^2 \\ 0 & 0 \end{pmatrix} = \frac{N^2-1}{4} \begin{pmatrix} 0 & \mathbb{1} \\ 0 & 0 \end{pmatrix}$$

From U(so(2,D)) to U(iso(1,2))

- so(2,2) ideal: $\mathcal{I}_{AB} \sim 0 \;\Rightarrow\; \mathcal{L}_m \bar{\mathcal{L}}_n \sim 0 \;\Rightarrow\; \begin{cases} P_m P_n - L_m L_n \sim 0 \\ L_m P_n - P_m L_n \sim 0 \end{cases}$

$$C_2 = L^2 + P^2 \sim 2\,L^2 \sim \frac{\lambda^2 - 1}{2}\, id$$

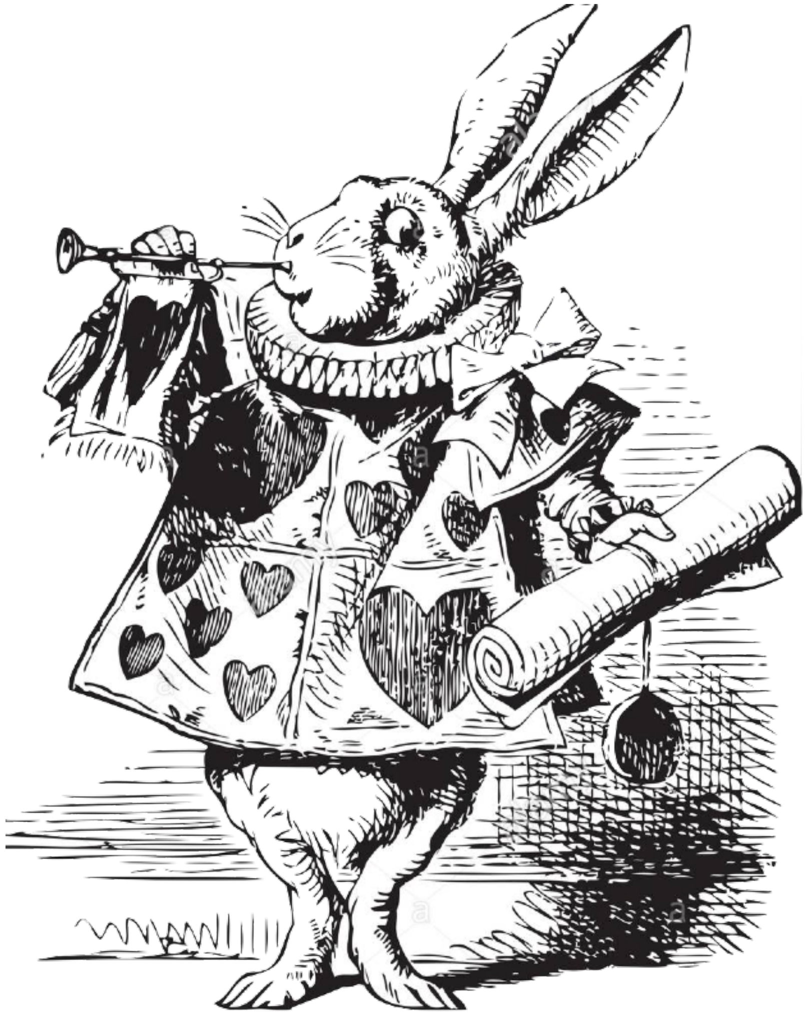
- Introducing the contraction parameter via $P_m \to \epsilon^{-1} P_m$

$$\begin{aligned} \epsilon^{-2} P_m P_n - L_m L_n &\sim 0 \\ \epsilon^{-1}\left(L_m P_n - P_m L_n\right) &\sim 0 \\ L^2 - \frac{\lambda^2 - 1}{4}\, id &\sim 0 \end{aligned} \quad \underset{\epsilon \to 0}{\Longrightarrow} \quad \boxed{\begin{aligned} P_m P_n &\sim 0 \\ L_m P_n - P_m L_n &\sim 0 \\ L^2 - \frac{\lambda^2 - 1}{4}\, id &\sim 0 \end{aligned}}$$

Poincaré ideal

Carrollian HS algebras

(in any dimensions)



From *U*(so(2,D–1)) to *U*(iso(1,D–1))

- We now reverse the logic: we look at how the contraction affects the so(2,D–1) ideal and then we define the iso(1,D–1) coset

$$[J_{AB}\,,\,J_{CD}] = \tilde{\eta}_{AC}\,J_{BD} - \tilde{\eta}_{AD}\,J_{BC} - \tilde{\eta}_{BC}\,J_{AD} + \tilde{\eta}_{BD}\,J_{AC}$$

$$\mathcal{P}_\mu \equiv \epsilon\,J_{\mu D}\,, \qquad \mathcal{J}_{\mu\nu} \equiv J_{\mu\nu}$$

$$[\mathcal{J}_{\mu\nu}\,,\,\mathcal{J}_{\rho\sigma}] = \eta_{\mu\rho}\,\mathcal{J}_{\nu\sigma} - \eta_{\mu\sigma}\,\mathcal{J}_{\nu\rho} - \eta_{\nu\rho}\,\mathcal{J}_{\mu\sigma} + \eta_{\nu\sigma}\,\mathcal{J}_{\mu\rho}$$

$$[\mathcal{J}_{\mu\nu}\,,\,\mathcal{P}_\rho] = \eta_{\mu\rho}\,\mathcal{P}_\nu - \eta_{\nu\rho}\,\mathcal{P}_\mu\,,$$

$$[\mathcal{P}_\mu\,,\,\mathcal{P}_\nu] = -\epsilon^2\,\mathcal{J}_{\mu\nu}\,,$$

- Next step: branching so(2,D–1) → so(1,D–1)

From U(so(2,D−1)) to U(iso(1,D−1))

- Branching so(2,D−1) → so(1,D−1) of the ideal

$$\mathcal{I}_{AB} \sim 0 \quad \Rightarrow \quad \begin{aligned} \mathcal{J}^2 - \frac{D-1}{2}\epsilon^{-2}\,\mathcal{P}^2 \sim 0 \\ \epsilon^{-1}\left\{\mathcal{P}^\lambda\,,\,\mathcal{J}_{\lambda\mu}\right\} \sim 0 \\ \{\mathcal{J}^\rho{}_\mu, \mathcal{J}_{\nu\rho}\} - \frac{4}{D}\eta_{\mu\nu}\mathcal{J}^2 + \epsilon^{-2}\{\mathcal{P}_\mu, \mathcal{P}_\nu\} - \frac{2}{D}\eta_{\mu\nu}\epsilon^{-2}\,\mathcal{P}^2 \sim 0 \end{aligned}$$

$$\mathcal{I}_{ABCD} \sim 0 \quad \Rightarrow \quad \begin{aligned} \{\mathcal{J}_{[\mu\nu}, \mathcal{J}_{\rho\sigma]}\} \sim 0 \\ \epsilon^{-1}\{\mathcal{J}_{[\mu\nu}, \mathcal{P}_{\rho]}\} \sim 0 \end{aligned}$$

$$C_2 \equiv \mathcal{J}^2 + \epsilon^{-2}\,\mathcal{P}^2 \sim -\frac{(D+1)(D-3)}{4}id \quad \Rightarrow \quad \begin{aligned} \mathcal{J}^2 \sim \frac{D-1}{D+1}C_2 \sim -\frac{(D-1)(D-3)}{4}\,id \\ \epsilon^{-2}\,\mathcal{P}^2 \sim \frac{2}{D+1}C_2 \sim -\frac{D-3}{2}\,id\,, \end{aligned}$$

Coset construction from *U*(iso(1,D−1))

- iso(1,D−1) ideal

$$\mathcal{P}_\mu \mathcal{P}_\nu \sim 0$$

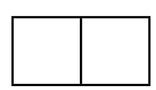
$$\{\mathcal{P}^\lambda\,,\,\mathcal{J}_{\lambda\mu}\} \sim 0$$

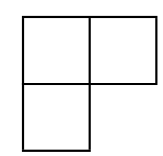
$$\{\mathcal{J}_{[\mu\nu}, \mathcal{P}_{\rho]}\} \sim 0$$

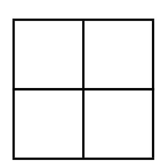
$$\{\mathcal{J}_{[\mu\nu}, \mathcal{J}_{\rho\sigma]}\} \sim 0$$

$$\mathcal{J}^2 \sim -\frac{(D-1)(D-3)}{4}\, id$$

- Leftover quadratic combinations, i.e. spin-3 generators:

$$\{\mathcal{J}^\rho{}_{(\mu}, \mathcal{J}_{\nu)\rho}\} - \text{tr.} \;\simeq$$ 

$$\{\mathcal{P}_{(\mu}, \mathcal{J}_{\nu)\rho}\} - \text{tr.} \;\simeq$$ 

$$\{\mathcal{J}_{(\mu\langle\rho}, \mathcal{J}_{\nu)\sigma\rangle}\} - \text{tr.} \;\simeq$$ 

Coset construction from $U(\mathrm{iso}(1,D-1))$

- iso(1,D−1) ideal

$$\mathcal{P}_\mu \mathcal{P}_\nu \sim 0$$

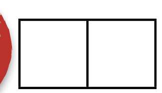
$$\{\mathcal{P}^\lambda\,,\,\mathcal{J}_{\lambda\mu}\} \sim 0$$

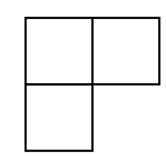
$$\{\mathcal{J}_{[\mu\nu},\mathcal{P}_{\rho]}\} \sim 0$$

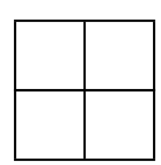
$$\{\mathcal{J}_{[\mu\nu},\mathcal{J}_{\rho\sigma]}\} \sim 0$$

$$\mathcal{J}^2 \sim -\frac{(D-1)(D-3)}{4}\,id$$

- Leftover quadratic combinations, i.e. spin-3 generators:

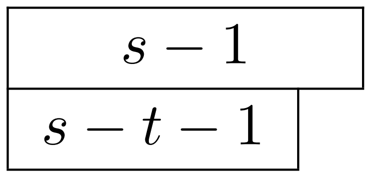
$$\{\mathcal{J}^\rho{}_{(\mu},\mathcal{J}_{\nu)\rho}\} - \mathrm{tr.} \simeq$$ 

$$\{\mathcal{P}_{(\mu},\mathcal{J}_{\nu)\rho}\} - \mathrm{tr.} \simeq$$ 

$$\{\mathcal{J}_{(\mu\langle\rho},\mathcal{J}_{\nu)\sigma\rangle}\} - \mathrm{tr.} \simeq$$ 

Structure of the result

- Higher-spin generators

$\mathcal{Z}^{s,t} \equiv$ 

with $t \in \{0, \dots, s-1\}$

  - t even: no P's
  - t odd: one P

- Structure of the algebra:

For D=4 see also
Fradkin, Vasiliev (1987)

$$[\mathcal{P}, \mathcal{Z}^{s,t}] \propto \mathcal{Z}^{s,t+1} \quad \text{for } t \text{ even}$$

$$[\mathcal{P}, \mathcal{Z}^{s,t}] \propto 0 \quad \text{for } t \text{ odd}$$

- Link with HS algebras for PM fields in D−1 dimensions Joung, Mkrtchyan (2016)
  - The generators with t even form a subalgebra (product of J's only)
  - Within this subalgebra $\mathcal{I}_{\mu\nu\rho\sigma} \sim 0$ and $\mathcal{J}^2 \sim -\frac{(D-1)(D-3)}{4} id$

One-parameter family of algebras

- PM fields admit a one-parameter family of HS algebras   Joung, Mkrtchyan (2016)

- The same "improvement" is consistent also in our setup

$$\mathcal{J}^2 \sim -\tfrac{(D-1)(D-3)}{4}\, id \longrightarrow \boxed{\mathcal{J}^2 \sim \nu\, id}$$

with $\nu_\mu = -\dfrac{(D-2+2\mu)(D-2-2\mu)}{4}$

- <u>One-parameter family of Carrollian conformal HS algebras:</u>

$$\boxed{\mathfrak{ihs}_D[\mu] \equiv \mathcal{U}(\mathfrak{iso}(1,D-1))/\langle \mathcal{I}_\mathfrak{c}[\mu]\rangle \,, \simeq \mathcal{A}^{\rm dS}_{D-1}[\mu] \;\inplus\; \mathcal{B}[\mu]_{\rm Ab}}$$

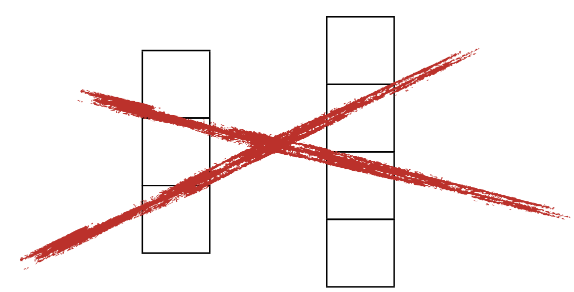
- Finite-dim truncations are possible

Classification of consistent ideals

- Can one build other conformal Carrollian HS algebras from U(iso(1,D−1))?

- Portion of the ideal we need to quotient out:

$$\mathcal{I}_{ABCD} \sim 0 \;\; \Rightarrow \;\; \begin{array}{r} \{\mathcal{J}_{[\mu\nu},\mathcal{J}_{\rho\sigma]}\} \sim 0 \\ \epsilon^{-1}\,\{\mathcal{J}_{[\mu\nu},\mathcal{P}_{\rho]}\} \sim 0 \end{array}$$



recall the 3D poll!

- Candidate spin-3 generators:

$$\{\mathcal{P}_\mu,\mathcal{P}_\nu\} - \text{tr.} \;\simeq\; \Box\Box \qquad \{\mathcal{J}^\rho{}_{(\mu},\mathcal{J}_{\nu)\rho}\} - \text{tr.} \;\simeq\; \Box\Box$$

- Can one use $P_\mu P_\nu$ as spin-3 generator?

$$[\mathcal{P}_\alpha\,,\,\mathcal{J}^\rho{}_{(\mu}\mathcal{J}_{\nu)\rho} - \frac{2}{D}\eta_{\mu\nu}\,\mathcal{J}^2] = \{\mathcal{J}_{\alpha(\mu},\mathcal{P}_{\nu)}\} + \cdots$$

Classification of consistent ideals

- Can one build other conformal Carrollian HS algebras from *U*(iso(1,D−1))?

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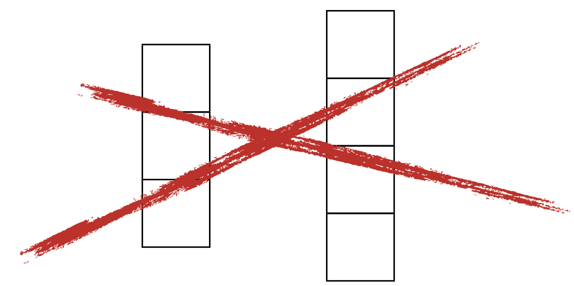

- Candidate spin-3 generators:

recall the 3D poll!

$$\{\mathcal{P}_\mu,\mathcal{P}_\nu\} - \text{tr.} \;\simeq\; \square\square \qquad\qquad \{\mathcal{J}^\rho{}_{(\mu},\mathcal{J}_{\nu)\rho}\} - \text{tr.} \;\simeq\; \square\square$$

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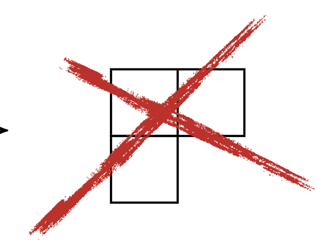
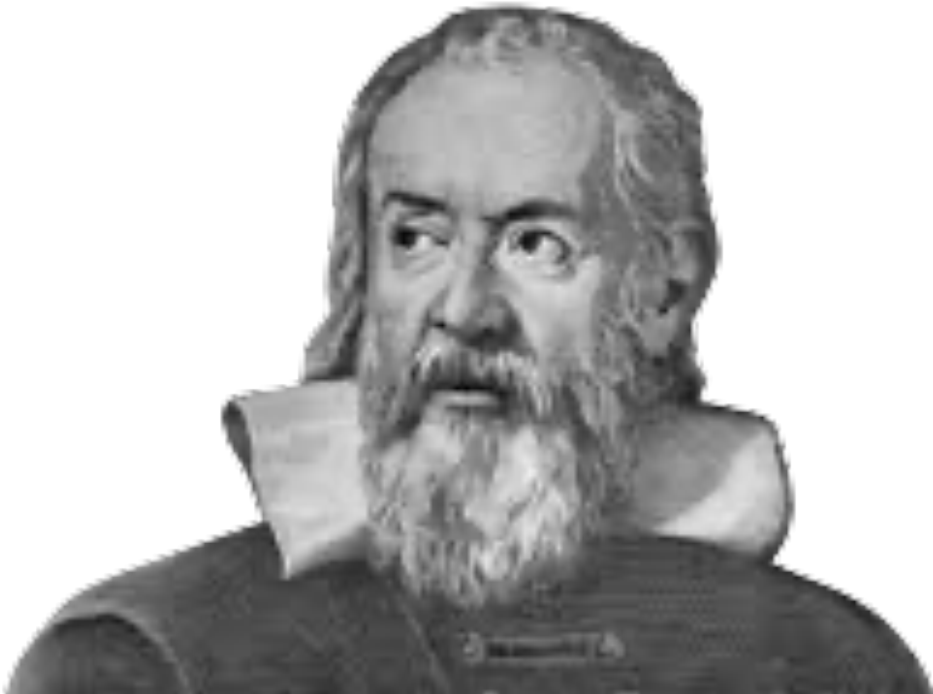


Galilean HS algebras

(in any dimensions)



From U(so(2,D−1)) to U(iso(1,D−1))

- Same approach as for Carroll, but with a new splitting of so(2,D−1)

$$[J_{AB}\,,\,J_{CD}] = \tilde\eta_{AC}\,J_{BD} - \tilde\eta_{AD}\,J_{BC} - \tilde\eta_{BC}\,J_{AD} + \tilde\eta_{BD}\,J_{AC}$$

$$J_{ij} \longrightarrow \mathfrak{so}(D-2)$$

$$\bar L_- = H\,,\quad \bar L_0 = D\,,\quad \bar L_+ = K\,, \longrightarrow \mathfrak{sl}(2,\mathbb{R})$$

$$T_{i,-} = P_i\,,\quad T_{i,0} = B_i\,,\quad T_{i,+} = K_i$$

$$[J_{ij}, \bar L_m] = 0 \qquad [J_{ij}, T_{k,m}] = \delta_{ik}T_{j,m} - \delta_{jk}T_{i,m} \qquad [\bar L_m, T_{i,n}] = (m-n)T_{i,m+n}$$

$$[T_{i,m}, T_{j,n}] = \delta_{ij}(m-n)\bar L_{m+n} + \gamma_{mn}J_{ij}$$

Contraction: $T_{i,m} \to \epsilon^{-1}T_{i,m}$ with $\epsilon \to 0$

Bagchi, Gopakumar (2009)

The so(2,D–1) ideal

$\mathcal{I}_{AB} \sim 0$ $\qquad$ $\mathcal{I}_{ABCD} \sim 0$ $\qquad$ $C_2 \sim -\frac{(D+1)(D-3)}{4}\, id$ $\qquad$ or…

$$\gamma^{mn} \{T_{i,m}, T_{j,n}\} - J_{k(i}J_{j)}{}^{k} - \frac{2}{D-2}\delta_{ij}\left(T^2 - J^2\right) \sim 0\,,$$

$$\delta^{ij} \{T_{i,m}, T_{j,n}\} - \{\bar{L}_m, \bar{L}_n\} - \frac{2}{3}\gamma_{mn}\left(T^2 - \bar{L}^2\right) \sim 0\,,$$

$$6J^2 - 2(D-2)\bar{L}^2 - (D-5)T^2 \sim 0\,,$$

$$\left\{J_i{}^j, T_{j,m}\right\} + \gamma^{kn}(m-n)\left\{\bar{L}_k, T_{i,m+n}\right\} \sim 0\,,$$

$$\left\{J_{[ij}, T_{k],m}\right\} \sim 0\,,$$

$$\gamma^{mn}\left\{\bar{L}_m, T_{i,n}\right\} \sim 0\,,$$

$$2\left\{T_{[i,m}, T_{j],n}\right\} + (m-n)\left\{J_{ij}, \bar{L}_{m+n}\right\} \sim 0\,,$$

$$J_{[ij}J_{kl]} \sim 0\,,$$

$$C_2 \equiv J^2 + \bar{L}^2 + T^2 \sim -\frac{(D+1)(D-3)}{2}\, id$$

The gca$_{D-1}$ ideal and Galilean HS algebras

$$
\begin{aligned}
\gamma^{mn}\left\{T_{i,m},T_{j,n}\right\} - \frac{2}{D-2}\delta_{ij}T^2 &\sim 0\,,\\
\delta^{ij}\left\{T_{i,m},T_{j,n}\right\} - \frac{2}{3}\gamma_{mn}T^2 &\sim 0\,,\\
J^2 - \bar{L}^2 &\sim -\frac{(D-3)(D-5)}{4}\,id\,,\\
\left\{J_i{}^j,T_{j,m}\right\} + \gamma^{kn}(m-n)\left\{\bar{L}_k,T_{i,m+n}\right\} &\sim 0\,,\\
\left\{J_{[ij},T_{k],m}\right\} &\sim 0\,,\\
\gamma^{mn}\left\{\bar{L}_m,T_{i,n}\right\} &\sim 0\,,\\
\left\{T_{[i,m},T_{j],n}\right\} &\sim 0\,,\\
J_{[ij}J_{kl]} &\sim 0\,,\\
T^2 &\sim 0\,.
\end{aligned}
$$

- Galilean conformal HS algebra: $\mathfrak{ghs}_D \equiv \mathcal{U}(\mathfrak{gca}_{D-1})/\langle \mathcal{I}_{\mathfrak{g}}\rangle$

One-parameter family of algebras

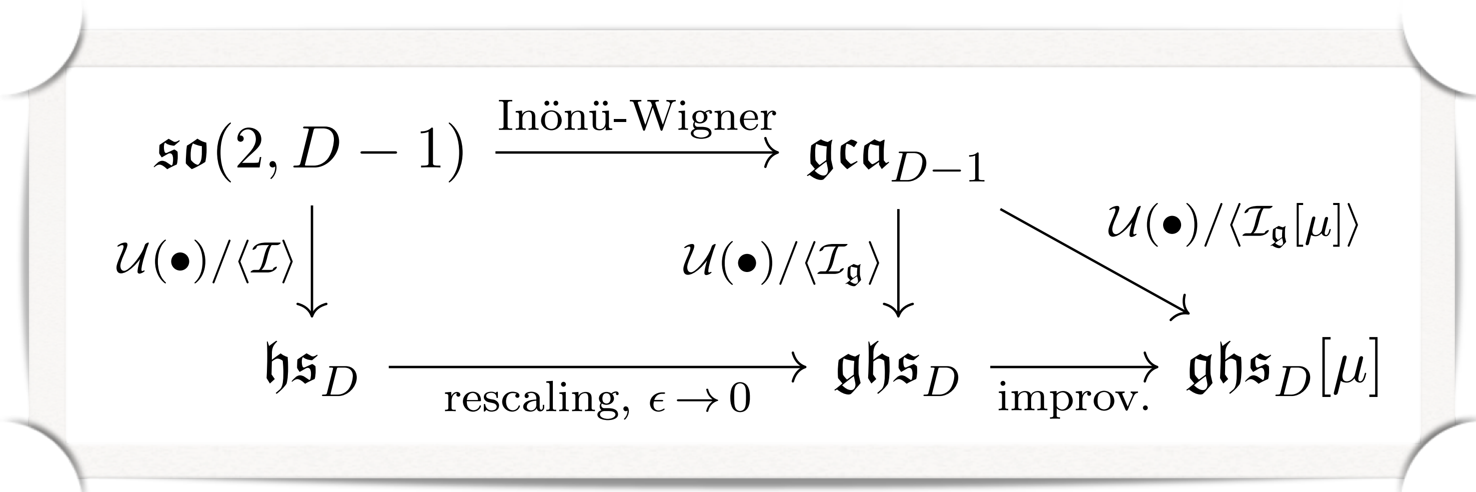
- Same "trick" as for Carroll

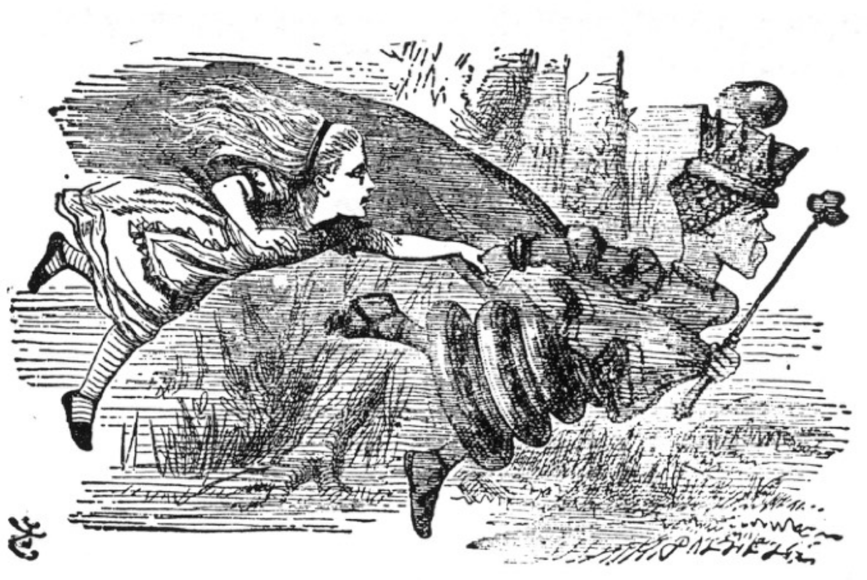
$$J^2 - \bar{L}^2 \sim -\frac{(D-3)(D-5)}{4} id \longrightarrow \boxed{J^2 - \bar{L}^2 \sim \nu \, id}$$

with $\nu_\mu = -\dfrac{(D-4+2\mu)(D-4-2\mu)}{4}$

- <u>One-parameter family of Galilean conformal HS algebras:</u>

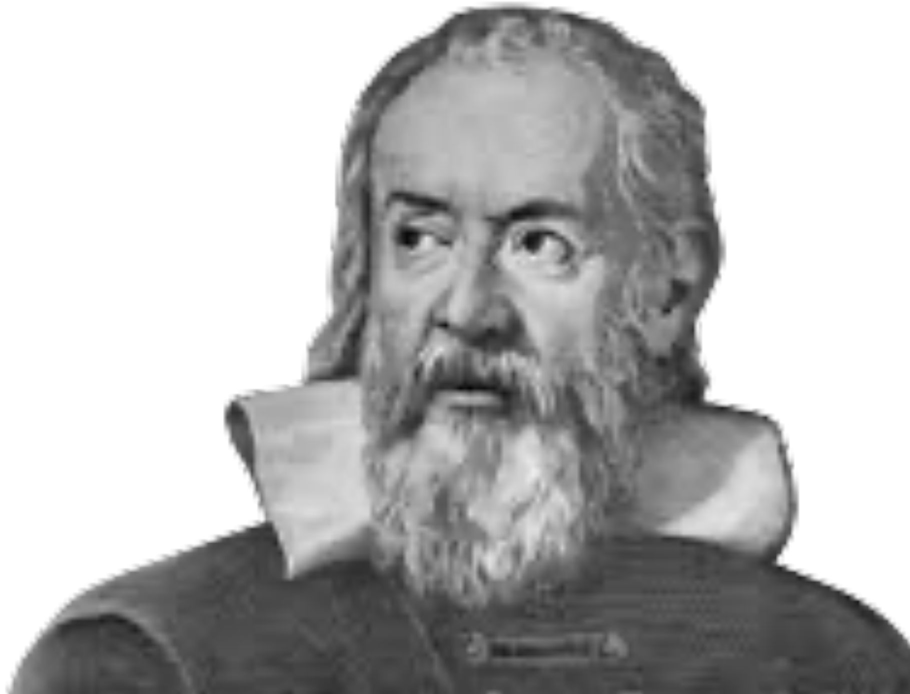
$$\boxed{\mathfrak{ghs}_D[\mu] \equiv \mathcal{U}(\mathfrak{gca}_{D-1})/\langle \mathcal{I}_{\mathfrak{g}}[\mu] \rangle}$$





Miscellaneous additional results

Stay tuned!



Carrollian and Galilean HS algebras in D=5

- In D=5 we start from a one-parameter family of algebras
  - Carrollian contraction: one extra non-isomorphic algebra obtained in the limit λ → 0
  - Galilean contraction: a 3D like structure emerge

$$L_m = \{J_{31} + iJ_{12}, iJ_{23}, J_{31} - iJ_{12}\}$$
$$\bar{L}_n = \{H, D, K\}$$

$$T_{m,n} = \left(\begin{array}{c|c|c} P_2 + iP_3 & iP_1 & P_2 - iP_3 \\ \hline B_2 + iB_3 & iB_1 & B_2 - iB_3 \\ \hline K_2 + iK_3 & iK_1 & K_2 - iK_3 \end{array}\right)$$

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} \,, \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} \,, \\
[\bar{L}_m, L_m] &= 0 \,, \\
[L_m, T_{n,k}] &= (m-n)T_{m+n,k} \,, \\
[\bar{L}_m, T_{k,n}] &= (m-n)T_{k,m+n} \,, \\
[T_{m,k}, T_{n,l}] &= (m-n)\gamma_{kl}L_{m+n} + (k-l)\gamma_{mn}\bar{L}_{k+l}
\end{aligned}$$

improvements of the limiting ideal are possible and one obtains algebras admitting a $\mathfrak{hs}^{(+)}(\lambda, \bar{\lambda})$ subalgebra

Ammon, Pannier, Riegler (2009)

“Geometric” algebras for Killing tensors?

- Why cannot we use the following bracket? Schouten (1940)
  - $$[v, w]^{\mu_1 \cdots \mu_{p+q-1}} \equiv \frac{(p+q-1)!}{p!q!} \left(p\, v^{\alpha(\mu_1 \cdots} \partial_\alpha w^{\cdots \mu_{p+q-1})} - q\, w^{\alpha(\mu_1 \cdots} \partial_\alpha v^{\cdots \mu_{p+q-1})} \right)$$
  - for p=1 and q=1 it coincides with the Lie bracket
  - the bracket of two Killing tensors is a Killing tensor
  - *the bracket of two traceless tensors isn't traceless*

"Geometric" algebras for Killing tensors?

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  - for *p*=1 and *q*=1 it coincides with the Lie bracket
  - the bracket of two Killing tensors is a Killing tensor
  - *the bracket of two traceless tensors isn't traceless*
- Exception in D=3: $(P^{(s)}_{\pm(s-1)})^{\mu_1\cdots\mu_{s-1}} \equiv \frac{(s-1)!}{(2\sqrt{2})^{s-2}}\,(P_{\pm1})^{\mu_1}\cdots(P_{\pm1})^{\mu_{s-1}},$

AC, Henneaux (2014)

$$(L^{(s)}_{\pm(s-1)})^{\mu_1\cdots\mu_{s-1}} \equiv (s-1)\frac{(s-1)!}{(2\sqrt{2})^{s-2}}\,(P_{\pm1})^{(\mu_1}\cdots(P_{\pm1})^{\mu_{s-2}}(L_{\pm1})^{\mu_{s-1})}$$

$$[L^{(3)}_m, P^{(3)}_n]^{\mu\nu\rho} = (m-n)\left(2\,(P^{(4)}_{m+n})^{\mu\nu\rho} - \frac{2m^2+2n^2-mn-8}{20}\,\eta^{(\mu\nu}(P_{m+n})^{\rho)}\right)$$

ihs[∞] !

Summary & overview

- One can build non-Abelian HS algebras including subalgebras $h = iso(1,D-1)$ or $h = gca_{D-1}$
- There exists a one-parameter family of coset algebras (built out of U(h)) in both cases
- “Good” Lorentz commutators guaranteed in UEA constructions
- Atypical commutators with translations (counterpart of the absence of minimal coupling?)

What's next?

- Asymptotic symmetries?
- Linearised curvatures?
- Recovering the algebras in interacting theories?