

# Charged-Rotating Black Holes in Higher-dimensional (A)DS-Gravity

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## Abstract

We present numerical evidences for the existence of rotating black hole solutions in  $d$ -dimensional Einstein-Maxwell theory with a cosmological constant and for  $d$  odd. The metric used possesses  $(d+1)/2$  Killing vectors and the solutions have  $(d-1)/2$  equal angular momenta. A Schwarzschild-type coordinate is used for the radial variable and both signs of the cosmological constant are emphasized. Several properties of the solutions are studied, namely their surface gravity, mass and angular momentum as functions of two parameters : the magnetic field and the angular velocity at the horizon. The influence of the electromagnetic field on the domain of existence of the black holes is studied and compared to the vacuum case where analytic solutions are available.

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## I. INTRODUCTION

Black holes constitute one of the most exciting predictions of four-dimensional general relativity (GR) and the astrophysical detections of them constitute a great advance in physics. Black holes further played a major role in establishing several mathematical aspects of GR and in understanding the physical interpretations of its solutions [1, 2, 3]. For numerous reasons, the interest for general relativity and of gravity in higher dimensions has increased considerably in the last years (see e.g. [4],[5], [6],[7]). The question of the existence of black holes in higher dimensions therefore occurred naturally. One striking feature of the Einstein equations in more than four dimensions is that many uniqueness properties holding in four dimensions are lost. For example, black holes presenting an horizon of topology  $S_{d-2}$  can be constructed explicitly in arbitrary dimensions [8]; their rotating generalisations [9], the Myers-Perry (MP) solutions, are characterized by several angular momenta. However several other types of solutions exist in higher dimensional gravity. Black strings are one of them; they denote string-like generalisations of 4-dimensional black holes in d-dimensional Einstein gravity; in particular, they are characterized by an event horizon of topology  $S_{d-3} \times S_1$  [10]. The simplest black strings are independent on the extra coordinate, say  $y$ . Later on it was realized that nonuniform solutions (depending on  $y$ ) exist as well [11]; these can further be done rotating [12].

Among possible solutions in higher dimensional gravity, we should also mention black ring [13], with toroidal horizon topology and black saturn [14], where a black hole is surrounded by a black ring etc. These black object result from a balance between rotation along the torus and gravitational attraction and are known to be connected to the Myers Perry black hole in a limit where the rotation becomes high [13, 14]. However, it has been found recently that black saturns can also exist without rotation in Einstein Maxwell theory [15]. In this solution, it is the charge that balances gravity instead of rotation.

As a consequence, the classification of localized objects (black holes, black strings or other) in higher dimensional gravity poses a serious mathematical problem.

Recently, there has been a lot of interest for d-dimensional GR with a cosmological constant  $\Lambda$ . In the case of a negative cosmological constant, this interest is motivated by the correspondance between the gravitating fields in an AdS space-time and conformal field theory on the boundary of the AdS space-time [16, 17]. The prototype of higher-dimensional black holes with cosmological constant are known in an explicit form [18]. They generalize the solutions of [8, 9]. Further explicit solutions are known in non minimal model e.g. [19, 20] or in the presence of a dilaton [21, 22]. As far as we know, rotating black holes for the Maxwell field minimally coupled to Einstein gravity do not exist in a closed form and one has to rely on perturbative or numerical methods to construct them respectively in asymptotically flat [23, 24], anti-DeSitter [25, 26] and DeSitter [27] space-times. The numerical construction reported in [23, 25] uses the isotropic coordinates, making the comparison with the explicit solution of [9] (usually expressed in Schwarzschild coordinate) not easy. In particular, the deformation of the MP analytic solution due to the electromagnetic field can hardly be estimated as explained later in the text. On the other hand, for  $\Lambda > 0$ , the occurrence of a cosmological horizon leads to technical difficulties which do not allow a direct comparison between the domain of existence of the MP solutions and those of charged black holes for both signs of the cosmological constant.

The purpose of this paper is to reexamine the equations of rotating black holes in d-dimensional Einstein gravity with a cosmological constant and with a minimal coupling

to the electromagnetic field. We can then study the effects both of the sign of  $\Lambda$  and of the electromagnetic field on the MP solutions. The charged rotating solutions exist for angular momenta higher than some critical momenta. When this critical value is approached, the solution becomes extremal. However, we were able to address some extremal solution numerically by implementing a suitable set of boundary conditions. Extremal solution are interesting on their own, in particular, they are known to present symmetry enhancement [28] and play an important role in supergravity theory [29].

We consider odd values  $d$ , the ansatz for the metric is inspired from [9, 23]. It contains  $(d + 1)/2$  killing vectors, namely time-translation and  $(d - 1)/2$  related to rotations in suitable two-planes of space. Assuming all angular momenta in the two-planes to be equal and completing the ansatz by an appropriate form for the electromagnetic fields, we can transform the Einstein-Maxwell into a system of six differential equations. The remaining arbitrariness in the coordinate system is fixed by using a Schwarzschild-like radial variable. Charged, rotating black holes can then be constructed numerically by fixing three parameters by hand, namely the values of the horizon, the angular velocity and the magnetic field at the horizon. The challenging question is to determine the domain of existence of the solutions in this three-parameter space.

Although the equations are basically the same for both signs of the cosmological constant, the pattern of the solutions can be considerably different.

The paper is organized as follow : in Sect. 2 we present the model, the ansatz and the relevant one-dimensional reduced effective action (the equations are written in an Appendix). In Sect 3 we discuss the explicit solutions known in some limits, namely the non rotating case and the vacuum case. The boundary conditions are presented in Sect. 4, together with some physical quantities characterizing the solutions. The numerical results are described in Sect. 5 and illustrated by several figures. In particular, the domain of existence of the black holes is emphasized for both  $\Lambda > 0$  and  $\Lambda < 0$  and compared to the vacuum case. Finally, some conclusions are drawn in Sect. 6.

## II. GENERAL FORMALISM

### A. The Action

We consider the Einstein-Maxwell equations with a cosmological constant  $\Lambda$  in a  $d$ -dimensional spacetime. The equations follow from the variations of the action

$$I = \frac{1}{16\pi G_d} \int_M d^d x \sqrt{-g} (R - 2\Lambda - F_{\mu\nu} F^{\mu\nu}) - \frac{1}{8\pi G_d} \int_{\partial M} d^{d-1} x \sqrt{-h} K, \quad (1)$$

with respect to the metric and the electromagnetic field. Here  $G_d$  denotes the  $d$ -dimensional Newton constant and the units are chosen in such a way that  $G_d$  appears as an overall factor. The last term is the Gibbons-Hawking surface term [30]. It is required for the variational principle to be well-defined. The factor  $K$  represents the trace of the extrinsic curvature for the boundary  $\partial M$  and  $h$  is the induced metric on the boundary. It is standard to parametrize the cosmological constant by means of an (anti)-DeSitter radius  $\ell$  according to  $\Lambda = \epsilon(d - 2)(d - 1)/(2\ell^2)$ ;  $\epsilon = +1$  and  $\epsilon = -1$  corresponds respectively to an asymptotic DeSitter and Anti-DeSitter space-time.

## B. The ansatz

To obtain rotating black hole solutions, representing charged  $U(1)$  generalizations of the MP solutions we consider space-times with odd dimensions:  $d = 2N + 1$  and we parametrize the metric by means of an Ansatz previously used for asymptotically flat solutions [9, 23]

$$\begin{aligned}
ds^2 = & -b(r)dt^2 + \frac{dr^2}{f(r)} + g(r) \sum_{i=1}^{N-1} \left( \prod_{j=0}^{i-1} \cos^2 \theta_j \right) d\theta_i^2 \\
& + h(r) \sum_{k=1}^N \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k (d\varphi_k - w(r)dt)^2 \\
& + p(r) \left\{ \sum_{k=1}^N \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k d\varphi_k^2 - \left[ \sum_{k=1}^N \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k d\varphi_k \right]^2 \right\}, \quad (2)
\end{aligned}$$

In the above formula  $\theta_0 \equiv 0$  and  $\theta_N \equiv \pi/2$  are assumed; the non trivial angles have  $\theta_i \in [0, \pi/2]$  for  $i = 1, \dots, N-1$ , while  $\varphi_k \in [0, 2\pi]$  for  $k = 1, \dots, N$ . The corresponding space-time possesses  $N$  commuting Killing-vectors  $\eta_k = \partial_{\varphi_k}$  supplementing the standard invariance of time translations  $\partial_t$ . The functions  $b, f, h, g, w$  depend on the variable  $r$ , the consistency of the ansatz imposes  $p(r) = g(r) - h(r)$ .

The most general Maxwell potential consistent with the symmetries of the line element (2) reads

$$A_\mu dx^\mu = V(r)dt + a_\varphi(r) \sum_{k=1}^N \left( \prod_{l=0}^{k-1} \cos^2 \theta_l \right) \sin^2 \theta_k d\varphi_k \quad (3)$$

Where the electric and magnetic potentials  $V(r)$  and  $a_\varphi(r)$  depend on  $r$ .

Substituting the ansatz (2),(3) into the action (1) leads to an effective one-dimensional reduced Lagrangian

$$S_{red} \int dr (\mathcal{L}_E^{eff} + \mathcal{L}_M^{eff}) \quad (4)$$

where the Einstein part  $\mathcal{L}_E^{eff}$  and the Maxwell part  $\mathcal{L}_M^{eff}$  read respectively :

$$\begin{aligned}
\mathcal{L}_E^{eff} = & (d-3)g^{\frac{(d-7)}{2}} \sqrt{\frac{bh}{f}} ((d-1)g - h) + \frac{1}{2} \sqrt{f h b g}^{\frac{(d-3)}{2}} \left( \frac{b'}{b} + (d-3) \frac{g'}{g} \right) \left( \frac{h'}{h} + (d-3) \frac{g'}{g} \right) \quad (5) \\
& - \frac{1}{4} (d-2)(d-3) \sqrt{f h b g}^{\frac{(d-7)}{2}} g'^2 + \frac{1}{2} g^{\frac{(d-3)}{2}} h \sqrt{\frac{f h}{b}} w'^2 - \epsilon \frac{(d-2)(d-1)}{\ell^2} g^{\frac{(d-3)}{2}} \sqrt{\frac{bh}{f}}
\end{aligned}$$

$$\mathcal{L}_M^{eff} = \frac{g^{\frac{(d-7)}{2}}}{\sqrt{b f h}} \left( 2b \left( 2(d-3)a_\varphi'^2 h + f g^2 a_\varphi'^2 \right) - 2f g^2 h (w a_\varphi' + V')^2 \right). \quad (6)$$

The Einstein-Maxwell equations obtained from the ansatz (2),(3) can be obtained in a standard way. For the numerical construction of the solutions, the radial variable  $r$  has to be

specified by fixing the "metric gauge"; we find it convenient to fix this arbitrariness by choosing  $g(r) = r^2$ , defining a radial coordinate  $r$  of the Schwarzschild-type. The corresponding equations are written explicitly in an Appendix. Similarly to the asymptotically flat case [23], the electric potential can be eliminated from the equations by making use of the first integral (37). The cosmological constant can be arbitrarily redefined by means of a rescaling of the radial variable  $r$  and of the fields  $h, V$  and  $w$ . For the numerical analysis, we use this arbitrariness to choose  $r_h = 1$  without loosing generality.

### III. KNOWN SOLUTIONS

Solutions of the system of equations (31)-(36) are known in two particular cases.

(i) The de Sitter-Reissner-Nordstrom black holes [31, 32] are recovered in the limit  $w(r) = a_\varphi(r) = 0$  :

$$\begin{aligned} f(r) &= b(r) = 1 - \epsilon \frac{r^2}{\ell^2} - \frac{2M}{r^{d-3}} + \frac{q^2}{2(d-2)(d-3)r^{2(d-3)}} , \\ h(r) &= g(r) = r^2 , \quad V(r) = \frac{q}{(d-3)r^{d-3}} \end{aligned} \quad (7)$$

where  $M$  and  $q$  are the mass and electric charge of the solution.

(ii) The vacuum black holes discussed in [18] are recovered for a vanishing gauge fields ( $V = a_\varphi = 0$ ) and

$$\begin{aligned} f(r) &= 1 - \epsilon \frac{r^2}{\ell^2} - \frac{2M\Xi}{r^{d-3}} + \frac{2Ma^2}{r^{d-1}}, \quad h(r) = r^2 \left(1 + \frac{2Ma^2}{r^{d-1}}\right), \\ w(r) &= \frac{2Ma}{r^{d-3}h(r)}, \quad g(r) = r^2, \quad b(r) = \frac{r^2 f(r)}{h(r)}, \end{aligned} \quad (8)$$

where  $M$  and  $a$  are two constants related to the solution's mass and angular momentum and  $\Xi = 1 + a^2/\ell^2$ . These are generalisations of the Myers-Perry (MP) solutions [9] to the case of non-vanishing cosmological constant. Here and in the following,  $\Omega \equiv w(r_h)$  will denote the angular velocity at the event horizon.

Some properties of the MP solutions are useful for the discussion of charged rotating black holes in the case  $\Lambda > 0$ . If we consider uncharged black holes solutions with fixed  $r_h$  and  $r_c$ , the explicit form provided by the Myers-Perry solutions reveals that rotating solutions exist up to a maximal value of  $\Omega$ . For  $d = 5$ , we find

$$\Omega_{max} = \sqrt{\frac{2}{r_c^2 + 2r_h^2}} \frac{r_c r_h^3 (r_c^2 + r_h^2)}{r_c^2 r_h^4 + r_c^2 + 2r_h^2} , \quad \frac{1}{\ell^2} = \frac{1}{r_c^2 + 2r_h^2} \quad (9)$$

For  $\Omega \rightarrow \Omega_{max}$  the event horizon  $r_h$  becomes a double root of the function  $f$ , i.e. the horizon becomes extremal. In fact for  $\Omega > \Omega_{max}$ , a third horizon occurs in the interval  $[r_h, r_c]$  in the neighbourhood of  $r_h$ . The values  $\Omega_{max}$  and the corresponding value of the DeSitter radius  $1/\ell^2$  are reported as functions of the cosmological horizon  $r_c$  for three different values of  $r_h$  on Fig.1.

For fixed values of  $r_h, r_c$ , the value of  $\ell^2$  varies only a little for  $\Omega \in [0, \Omega_{max}]$  ; e.g. for  $r_h = 1, r_c = 3$  we have  $1/10 \leq 1/\ell^2 \leq 1/11$ . The mass  $E$  and the angular momentum  $J$  of

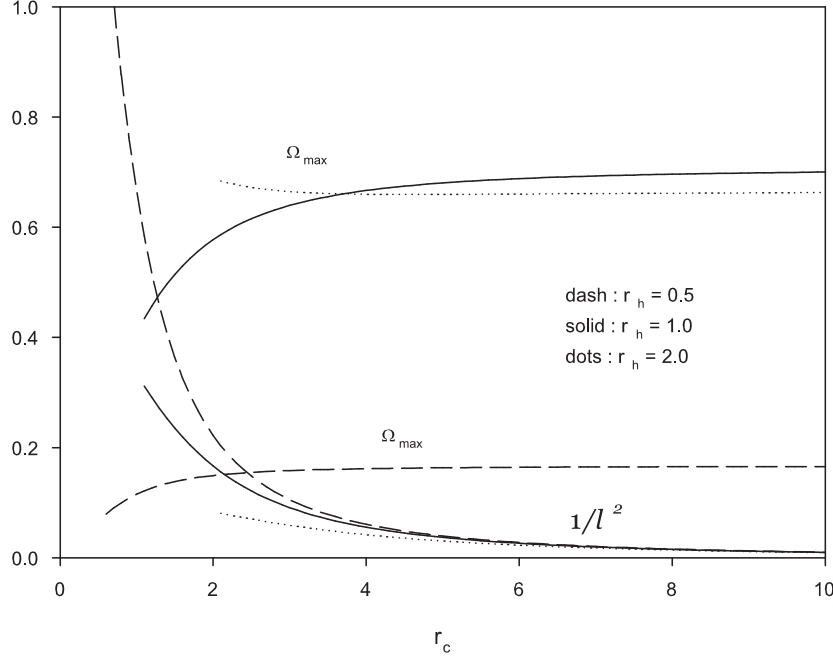


FIG. 1: The value  $\Omega_{max}$  and the parameter  $1/\ell^2$  as functions of  $r_c$  for the MP solution and for  $r_h = 0.5, 1.0, 2.0$

the Myers-Perry solutions are given by [33], [34]

$$E = \frac{V_{d-2}}{4\pi G} M \left( \frac{d-2}{2} - \epsilon \frac{a^2}{2\ell^2} \right) \quad , \quad J = \frac{V_{d-2}}{4\pi G} \frac{d-1}{2} M a \quad (10)$$

where  $V_{d-2}$  denotes the area of the  $d-2$  dimensional sphere. Throughout this paper, we will use the expression  $E$  above to characterize the mass of the solutions although a more elaborated formalism exists [35, 36]. Once applied to the MP solutions [38], it leads to an expression of the form  $\mathcal{M}^{(d)} = E_c^{(d)} - \epsilon E$  where  $E_c^{(d)}$  is interpreted as the Casimir energy of the dual Conformal Field Theory living on the  $d-1$ -dimensional boundary of the space-time [35, 36]. The contribution  $E_c^{(d)}$  appears only for odd values of  $d$  and depends on the space-time dimension and of the cosmological constant [37]; for  $d = 5$ , one finds  $E_c^{(5)} = \frac{V_{d-2}}{4\pi G} \frac{3\ell^2}{16}$

## IV. BOUNDARY CONDITIONS AND PHYSICAL QUANTITIES

### A. Constraints of regularity about the horizon

We are interested in black hole solutions, with an horizon located at  $r = r_h$ , i.e. with  $b(r_h) = f(r_h) = 0$ . Expanding the different functions in the neighbourhood of the horizon

$$\begin{aligned} b(r) &= b_1(r - r_h) + O(r - r_h)^2, & f(r) &= f_1(r - r_h) + O(r - r_h)^2, \\ h(r) &= h_0 + h_1(r - r_h) + O(r - r_h)^2, & w(r) &= w_h + w_1(r - r_h) + O(r - r_h)^2, \\ a_\varphi(r) &= a_0 + a_1(r - r_h) + O(r - r_h)^2, & V(r) &= V_0 + V_1(r - r_h) + O(r - r_h)^2 \end{aligned} \quad (11)$$

and inserting the expansions (11) into the equations (31)-(36) leads to terms proportional to  $1/(r-r_h)$ . For the solutions to be regular at the event horizon  $r_h$  the coefficients of all these singular terms should vanish. Since our numerical analysis will deal mainly with the case  $d=5$ ,  $\epsilon=1$ , we write these rather cumbersome conditions in this case only. The equation for the function  $h$  leads to the condition  $\Gamma_1(r=r_h)=0$  with

$$\begin{aligned}\Gamma_1(r) \equiv & 8b'h^2(12a_\varphi^2+7h) + 4rb'h'h'(12a_\varphi^2+5h) \\ & - 32b'h^2r^2 + 8r^3b'h(f'h-4h') \\ & + 2hr^4\left(\frac{24}{\ell^2}b'h^2 - f'(4(a'_\varphi)^2hw^2 + 8a'_\varphi h w V' + b'h' + 5h^2(w')^2 + 4h(V')^2)\right) \\ & + h'r^5\left(-\frac{24}{\ell^2}b'h^2 - f'(4(a'_\varphi)^2hw^2 + 8a'_\varphi h w V' - b'h' - h^2(w')^2 + 4h(V')^2)\right) \quad (12)\end{aligned}$$

where, for brevity, we dropped the dependence on  $r$  in the various functions. The value  $f'(r_h)$  appearing in the condition  $\Gamma_1(r_h)=0$  can be extracted from the equation for  $f$ , giving

$$f'(r_h) = \frac{4b_1h_0}{r_h^3} \frac{6r_h^4/\ell^2 + 8r_h^2 - 12a_0^2 - 5h_0}{8b_1h_0 + r_h(4h_0(V_1 + a_1w_0)^2 - b_1h_1 - h_0^2w_1^2)} \quad (13)$$

In the same way, the two Maxwell equations lead to a single condition  $\Gamma_2(r=r_h)=0$  with

$$\Gamma_2(r) \equiv 4a_\varphi b'h + r^4 f'(h w' V' + a'_\varphi h w w' - a'_\varphi b') \quad (14)$$

Similar conditions should be imposed at the cosmological horizon  $r_c$  as well :  $\Gamma_1(r_c) = \Gamma_2(r_c) = 0$ .

## B. Physical quantities at the horizon

The Hawking temperature and the event horizon area of these solutions can be obtained in a standard way, leading to

$$T_H = \frac{\sqrt{f_1 b_1}}{4\pi}, \quad A_H = V_{d-2} r_h^{d-2}. \quad (15)$$

The mass and the angular velocity at the horizon are also usefull. They are defined by means of the appropriate Komar integrals over the sphere  $S_H^3$  at the event horizon. It leads to

$$M_H = \frac{V_{d-2}}{8\pi G_d} \sqrt{\frac{f h g^2}{b}} (b' - h w w')|_{r=r_h}, \quad J_H = \frac{V_{d-2}}{8\pi G_d} 2 \sqrt{\frac{f g^2 h^3}{b}} w'|_{r=r_h} \quad (16)$$

These quantities can be evaluated from the numerical solutions. Similar quantities, say  $M_C, J_C$  can be associated with the cosmological horizon. In fact, the masses and angular velocities at the two horizons can be related to each other by formulas of the Smarr-type. Considering a Killing vector  $K_\mu$ , integrating the two sides of the identity

$$\nabla^\mu \nabla_\mu K_\nu = -R^\mu{}_\nu K_\mu = -8\pi (T^\mu{}_\nu - \frac{1}{d-2} T \delta^\mu_\nu) K_\mu \quad (17)$$

over the (truncated) hyper-volume  $\Sigma$  covering the space-like region between the two horizons and using both Einstein equations and Stokes theorem appropriately result into the identities

we are interested in. Choosing first  $K = \partial_\varphi$  one gets

$$J_C - V_{d-2} \sqrt{\frac{f h g^2}{b}} (w a_\varphi + V')|_{r=r_c} = J_H - V_{d-2} \sqrt{\frac{f h g^2}{b}} (w a_\varphi + V')|_{r=r_h} , \quad (18)$$

expressing the conservation of the total angular momentum. The identity associated with the time-translation invariance ( $K = \partial_t$ ) is a bit more involved, it leads to

$$M_C + \Phi_C Q_C - M_H - \Phi_H Q_H = \frac{1}{\pi \ell^2} V_{d-2} \int_{r_h}^{r_c} \sqrt{\frac{f h g^2}{b}} dr \quad (19)$$

where  $Q_H$  (the electric charge at the horizon [39]) and  $\Phi_H$  (the electrostatic potential) are defined respectively by

$$Q_H = \int_0^{\pi/2} d\theta \left( \sqrt{\frac{f h g^2}{b}} \right)|_{r=r_h} , \quad \Phi_H = \left( V + \frac{d-1}{2} \Omega a_\varphi \right)|_{r=r_h} \quad (20)$$

and similarly for  $Q_R, \Phi_R$ . Similarly to [23], the mass at the horizon can further be expressed in terms of the surface gravity  $\kappa$  and angular momentum according to

$$\frac{d-3}{d-2} M_H = \frac{1}{8\pi G_d} \kappa A_H + \frac{d-1}{2} \Omega J_H \quad (21)$$

The various identities above were checked on our numerical solutions.

### C. Asymptotics and global charges

The asymptotic form of the solutions to the Einstein equations can be constructed after some algebraic manipulations, leading to

$$\begin{aligned} b(r) &= -\epsilon \frac{r^2}{\ell^2} + 1 + \frac{\alpha}{r^{d-3}} + O(1/r^{2d-6}), \quad f(r) = -\epsilon \frac{r^2}{\ell^2} + 1 + \frac{\beta}{r^{d-3}} + O(1/r^{d-1}), \\ h(r) &= r^2 \left( 1 + \epsilon \frac{\ell^2 (\beta - \alpha)}{r^{d-1}} + O(1/r^{2d-4}) \right), \quad w(r) = \frac{\hat{J}}{r^{d-1}} + O(1/r^{2d-4}). \end{aligned} \quad (22)$$

They depend on three arbitrary constants  $\alpha$ ,  $\beta$  and  $\hat{J}$ . The choice of the gauge metric  $g = r^2$  in the ansatz (2) and the resulting functions  $f$ ,  $b$ ,  $h$  have the advantage to present a direct connection with the closed form of the MP rotating solution. The asymptotic expression of the gauge potential is similar to the asymptotically flat case

$$V(r) = -\frac{q}{r^{d-3}} + O(1/r^{2d-4}), \quad a_\varphi(r) = \frac{\hat{\mu}}{r^{d-3}} + O(1/r^{2d-4}). \quad (23)$$

Although it is not clear if the thermodynamical properties are well defined in the presence of the positive cosmological horizon, one may still define them in the standard way. In this section, we follow the lines of [25] to present the global charges characterizing the solutions asymptotically. [35, 36]. The mass-energy  $E$  of the solutions and the angular momentum  $J$  associated with an angular direction are respectively given by

$$E = \frac{V_{d-2}}{16\pi G_d} (\beta - (d-1)\alpha), \quad J = \frac{V_{d-2}}{8\pi G_d} \hat{J}. \quad (24)$$



The above relations can be obtained namely by using a background subtraction approach or the counterterm formalism [35, 36]. Here do not take into account the contribution of the Casimir energy of the solution [37, 38].

The electric charge and the magnetic moment of the solutions are given by

$$Q = \frac{(d-3)V_{d-2}}{4\pi G_d} q, \quad \mu = \frac{(d-3)V_{d-2}}{4\pi G_d} \hat{\mu}. \quad (25)$$

## D. Boundary value problem

### 1. Case $\Lambda < 0$

In the case of asymptotically ADS black holes, the formulation of the boundary problem is simple. Fixing  $\Lambda < 0$ , the equations can be integrated in one step on the interval  $r \in [r_h, \infty]$  by imposing the boundary conditions

$$f = 0, \quad b = 0, \quad w = w_h, \quad V = 0, \quad a'_\varphi = a'_h, \quad \Gamma_1 = 0, \quad \Gamma_2 = 0 \quad \text{for } r = r_h, \quad (26)$$

$$b = \frac{r^2}{\ell^2} + 1 + O(1/r^2), \quad h = r^2, \quad (r^2 a_\phi)' = 0, \quad (r^4 w)' = 0 \quad \text{for } r \rightarrow \infty. \quad (27)$$

The arbitrary additive constant of the electric potential has been used to set  $V(r_h) = 0$ . The parameters  $w_h, a'_h$  have to be fixed by hand (alternatively,  $a'_\varphi = a'_h$  can be replaced by  $a_\varphi = a_h$ ). The asymptotic decay of the electric potential  $V$  and of the metric function  $f$  can then be used for crosschecking the numerical solutions.

Extremal black holes are characterized by a double zero of the functions  $f, b$  at  $r = r_h$ . In the case of de Sitter-Reissner-Nordstrom black holes, analytical form follows from (7), the parameters  $M, q$  are then fixed as functions of  $r_h$ . In the case of rotating solutions, the regularity conditions at  $r = r_h$ , leads to a set of rather involved conditions at  $r = r_h$ .

### 2. Case $\Lambda > 0$

The positive cosmological constant leads to the occurrence of a  $\Lambda$ -depending cosmological horizon at  $r = r_c$  with  $r_h < r_c < \infty$  and where  $f(r_c) = b(r_c) = 0$ . This creates several difficulties, namely : (i) the point  $r_c$  constitutes a singular point of the equations, (ii) the point  $r_c$  is not known a priori as a function of  $\Lambda$ , (iii) imposing regularity at both horizons  $r_h$  and  $r_c$  needs twelve boundary conditions, this number exceeds the eleven boundary conditions allowed by orders of the six equations (for instance, one of first order and five of second order).

In order to overcome these difficulties, we supplement the system (31),(36) by the trivial equation  $d\ell^2/dx = 0$ . The new system then needs twelve boundary conditions to be well posed. The strategy consists in solving the equations in two steps. First we choose by hand the event horizon  $r_h$  and the cosmological horizon  $r_c$  and integrate the equations on the interval  $[r_h, r_c]$  by imposing the regularity conditions at the two horizons. It turns out that this needs exactly twelve boundary conditions, namely :

$$f = 0, \quad b = 0, \quad b' = 1, \quad w = w_h, \quad V = 0, \quad a'_\varphi = a'_h, \quad \Gamma_1 = 0, \quad \Gamma_2 = 0 \quad \text{for } r = r_h \quad (28)$$

$$f = 0, \quad b = 0, \quad \Gamma_1 = 0, \quad \Gamma_2 = 0 \quad \text{for } r = r_c \quad (29)$$

The functions  $\Gamma_{1,2}$  are defined above. Here, we choose the arbitrary rescaling of the time variable  $t$  in order to impose  $b'(r_h) = 1$ . The function  $b(r)$  is therefore not normalized according to (22), the appropriate normalisation, say  $b \rightarrow b/\mathcal{N}_b^2$ , can be obtained only after the second step. The constants  $w_h$  and  $a'_h$  are arbitrary and control the total angular momentum and the (electric and magnetic) charges of the black hole. The numerical value of  $\ell^2$  is determined by the first step, together with the values of all the different fields at  $r = r_c$ .

The second step consists in integrating the equations on  $[r_c, \infty]$ . The value of  $\ell^2$  is known from the first step and the numerical values of the fields at  $r = r_c$  can be used as a set of Cauchy data to solve the equations from  $r = r_c$  to  $r \geq r_c$ .

This method presents several features of inconvenience. Namely: (i) a systematic analysis of the solution for a fixed value of the cosmological constant cannot be performed. Fortunately, the numerical value of  $\ell^2$  depends only a little on  $w(r_h)$  and  $a_\varphi(r_h)$  once  $r_h, r_c$  are fixed. (ii) The appropriate normalisation of  $b, w, V$  leading to an asymptotically DeSitter metric is only known a posteriori.

$$b \rightarrow \frac{b}{\mathcal{N}_b^2}, \quad w \rightarrow \frac{w}{\mathcal{N}_b}, \quad V \rightarrow \frac{V}{\mathcal{N}_b}$$

In particular it turns out impossible to study solutions with a fixed value of the angular velocity at the horizon  $\Omega$  (since  $\Omega = w(r_h)/\mathcal{N}_b$ ) and varying the horizon. In the numerical analysis, we put the emphasis on the families of solutions obtained when varying  $\Omega$  for fixed values of the horizons  $r_h, r_c$  and of the parameter  $a'_h$ .

## V. NUMERICAL RESULTS

The system of equations (31)-(36) does not admit, to our knowledge, explicit solutions for generic values of  $w_h$  and  $a'_h$ . We therefore rely on a numerical method to construct solutions. We solved the equations in the case  $d = 5$  and we hope that this case catches the qualitative properties of the pattern of the solutions available for  $d > 5$ . The numerical solver Colsys [40] was used to obtain the results.

### A. Charged solutions for $\Lambda < 0$

Charged 5-dimensional rotating black hole solutions have been presented in details in [25] where the coordinate freedom is fixed by choosing the isotropic coordinate, say  $y$ . The solutions are constructed as function of  $y$  and their pattern is discussed mainly for fixed horizon at  $y = y_h$ . With this definition of the radial coordinate, the pattern of the solutions available for fixed  $y_h$  present several branches. For instance, solutions with different masses but same  $y_h$  and  $\Omega$  can be found. Unfortunately, the solutions can hardly be compared with the explicit form (8) of the Myers-Perry solutions which hold with the Schwarzschild (or the Boyer-Linquist) coordinate. For this reason, and for the purpose of comparison of the solutions between the Anti-DeSitter and DeSitter cases, we present some solutions with  $\Lambda < 0$  in the Schwarzschild coordinate  $g = r^2$  and determine the pattern of solutions available

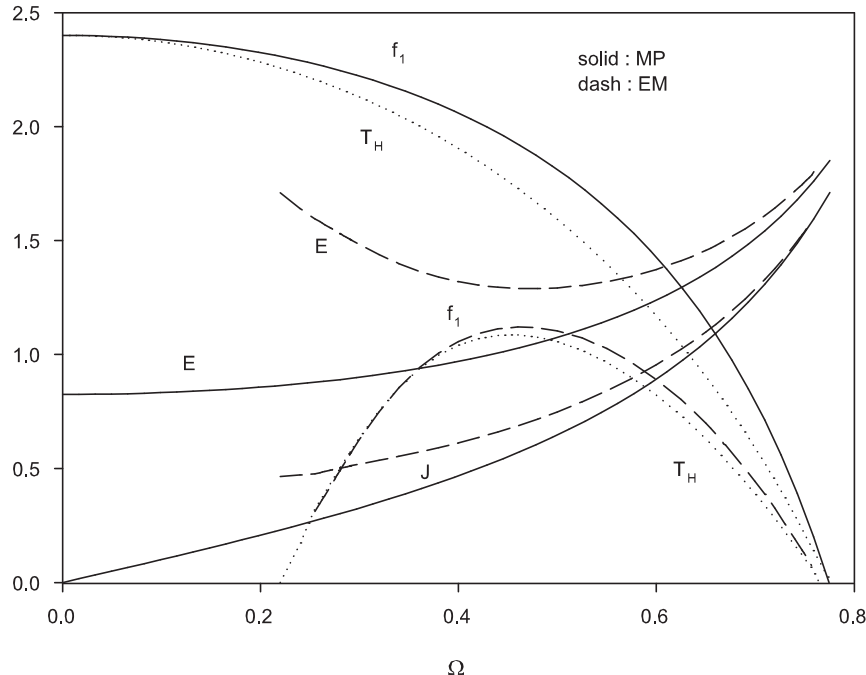


FIG. 2: The Mass  $E$ , the angular momentum  $J$  and the value  $f_1$  as functions of  $\Omega$  for the Myers-Perry solution (solid) and the charged solution with  $A' = 0.5$  (dashed) for  $r_h = 1$  and  $\ell^2 = 0.1$ . The Hawking temperature  $T_H$  is represented by the dotted lines

with  $r_h$  fixed. For all this section we set  $r_h = 1$  and  $\ell^2 = 0.1$ . Several physical parameters characterizing the charged rotating black holes with  $r_h = 1$ ,  $a'_h = 0.5$  are represented on Fig. 2 by the dashed lines. The corresponding parameters associated with the MP solutions are represented on this figure by the solid lines. As already pointed out, vacuum (or MP) solutions exist for positive values of  $\Omega$  up to a maximal value  $\Omega_{max}$ , with the choice of the parameters of the figure, we have  $\Omega_{max} \sim 0.78$ . Our numerical integration of the full equations reveals that charged black holes exist only for a finite interval of the horizon angular velocity :  $\Omega \in [\Omega_a, \Omega_b]$ , with  $\Omega_a \geq 0$  and  $\Omega_b \leq \Omega_{max}$ . The values  $\Omega_{a,b}$  depend on the strength of the electromagnetic fields. In the case of Fig. 2 we find  $\Omega_a \sim 0.22$ ,  $\Omega_b \sim 0.765$ . In the limits  $\Omega \rightarrow \Omega_{a,b}$ , extremal solutions are approached, i.e. corresponding to  $f_1 = b_1 = 0$ . As a consequence, the Hawking temperature  $T_H$  also tends to zero in these extremal limits. It is represented by the dotted lines on Fig. 2. Several parameters characterizing the electromagnetic fields of the solutions have also been computed numerically. The electric charge  $Q$  and the magnetic moment  $\mu$ , vary slowly as functions of the horizon velocity  $\Omega$ , as shown by Fig. 3. Note that that the numerical values of  $Q$  and  $V(\infty)$  are very close to each other and similarly for  $\mu$  and  $A_h$ . Profiles of a typical ADS rotating black hole is presented on Fig. 4. We managed to construct rotating extremal black holes numerically for small values of  $\Omega$ . The profile of such a solution is presented in Fig. 5 corresponding to  $\Omega = 0.09$ . The numerical integration is delicate; a systematic construction of extremal solutions is not aimed in this paper but would require an appropriate reparametrisation of the metric.

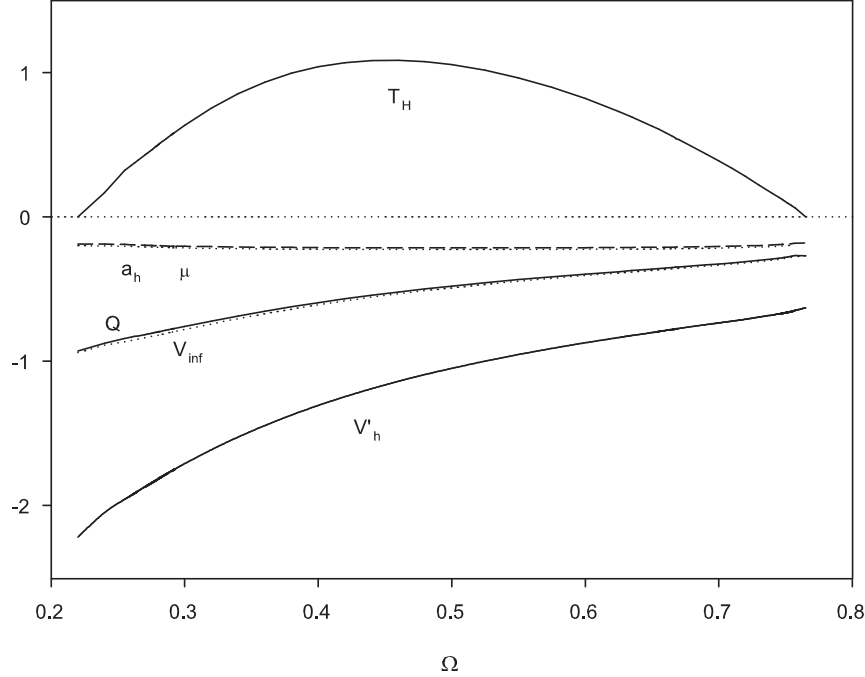


FIG. 3: Parameters of the electromagnetic field of charged ADS-BH as function of  $\Omega$  for  $r_h = 1$ ,  $a'_h = 0.5$ .

### B. Remarks on the Isotropic coordinate

We would like to stress the main differences between the patterns of the solutions once constructed by using the isotropic coordinate and the Schwarzschild coordinate. We make these remarks in the vacuum, asymptotically flat case :  $\Lambda = 0$ . In the present paper, we use the Schwarzschild coordinate  $r$ , denoting the angular velocity at the horizon as  $\Omega \equiv w(r_h)$  fixing the horizon  $r_h$ , we have solutions with a regular horizon for  $\Omega \leq \frac{1}{r_h \sqrt{2}}$ . All physical quantities, the mass in particular are monotonic functions of  $\Omega$ . In contrast, the isotropic coordinate  $y$  is used e.g. in [25], we have

$$y = \exp K(r) \quad , \quad K'(r) = \frac{1}{r\sqrt{f}} \quad (30)$$

While families of solutions (MP) take a rather simple form in the Schwarzschild coordinate it is not the case with the isotropic coordinate. Considering the family of solutions with fixed horizon  $y_h$  in the isotropic coordinate, leads to a rather involved pattern in the  $r_h, \Omega$  plane. Namely it can be that two solutions with equal  $y_h$  and  $\Omega$  can exist but with different masses. This is illustrated on Fig. 6 where two solutions corresponding to  $r_h \approx 1$  and  $r_h \approx 2$  and  $\Omega = 0.33$  have both  $y_h = 0.5$  once expressed in the isotropic coordinate.

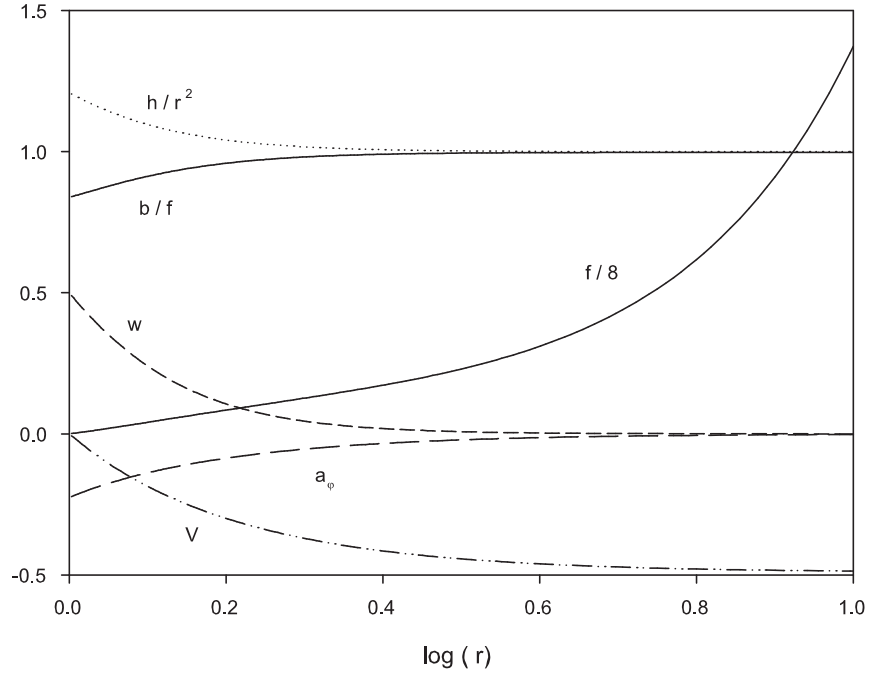


FIG. 4: The profiles of a typical ADS rotating black hole with  $r_h = 1, \Omega = 0.5$  and  $a'_h = 0.5$

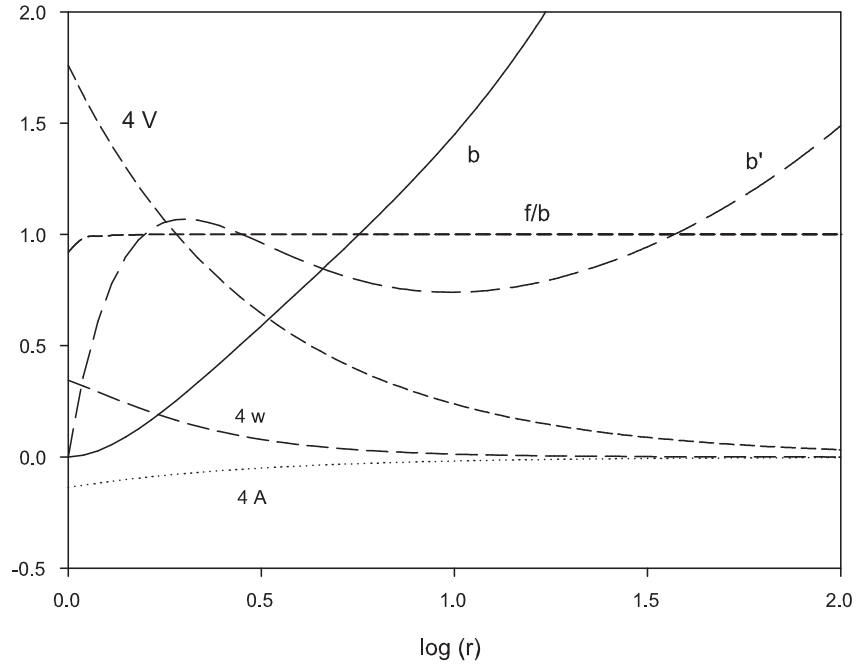


FIG. 5: The profiles of an extremal ADS rotating black hole with  $r_h = 1, \Omega = 0.09$  and  $\Lambda = -0.1$

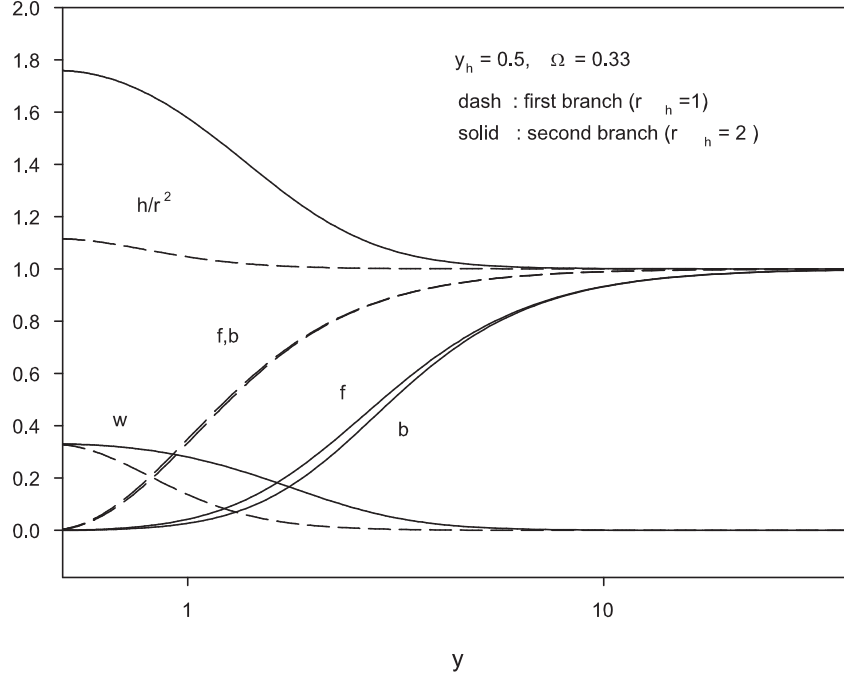


FIG. 6: The profiles of two black hole solutions corresponding to  $y_h = 0.6$ ,  $\Omega = 0.33$

### C. Charged solutions for $\Lambda > 0$

For  $\Lambda > 0$ , the presence of a Maxwell field changes the pattern of solutions corresponding to the vacuum (or MP) solutions. Along with the case  $\Lambda < 0$ , our numerical analysis strongly suggests that charged-rotating black holes exist only for a finite interval of the horizon angular velocity  $\Omega$ , i.e. for  $\Omega \in [\Omega_a, \Omega_b]$  with  $0 < \Omega_a < \Omega_b < \Omega_{max}$ . Here  $\Omega_{max}$  denotes the maximal horizon velocity for the MP solution and the values  $\Omega_{a,b}$  depend of the strength of the electromagnetic field. It must be stressed that the numerical integration of the equations become extremely tedious in the region of the parameters which allow to approach the lower limit  $\Omega \sim \Omega_a$ . The reason of this difficulty comes from the fact that, when the shooting parameter  $w_h$  decreases, the branch of solutions stops as some minimal value, say  $w_{h,min}$ . Then, a second branch of solution emerge for  $w_h \geq w_{h,min}$ . The two branches coincide in the limit  $w_h \rightarrow w_{h,min}$ . Following the second branch shows that it terminate into a critical value of  $w_{h,c}$  and  $f_1 \rightarrow 0$  for  $w_h \rightarrow w_{h,c}$ . This is illustrated on Fig. 7 where a few parameters that are substantially different on the two branches are represented as functions of  $w_h$ . In all figures of this section, the values  $r_h = 1, r_c = 3$  are used corresponding to  $\ell^2 \sim 10$ . In fact the second branch is represented for  $a'_h = 0.5$  (where we find  $w_{h,min} \sim 0.459$  and  $w_{h,c} \sim 0.67$ ). A second branch exist also for the other lines presented on the figure (i.e. corresponding to  $a'_h = 0.1$ ) but it turned out even more difficult to construct it with a good accuracy than for the case  $a'_h = 0.5$  and we refrain to representing it. On the other side on the graphic, solutions seem to exist for arbitrarily large values of the parameter  $w_h$ .

So, once described in terms of  $w_h$ , the pattern of solution looks complicated. However, it is more relevant to plot the different quantities as functions of the physically meaningful parameter  $\Omega$  representing the angular velocity at the horizon. This is obtained by renormal-

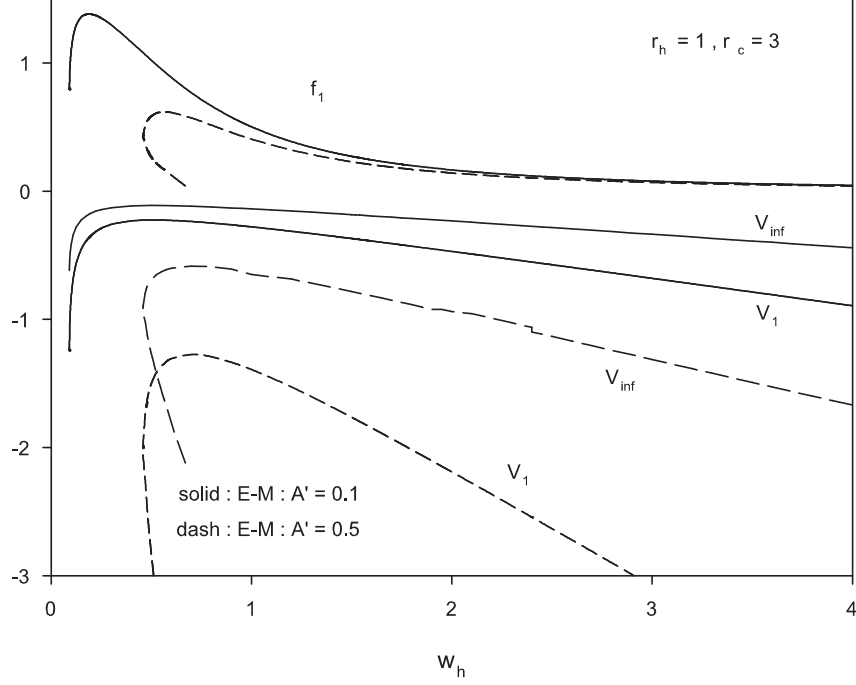


FIG. 7: The values  $f_1$ ,  $V_1$  and  $V_{inf} \equiv V(\infty)$  are reported as functions of  $w_h$  for  $a'_h = 0, 0.1$  and  $a'_h = 0.5$

izing the fields  $b, w, V$  appropriately so that the metric become asymptotically DeSitter.

Several parameters characterizing the metric functions at the horizon are reported on Fig. 8. as function of  $\Omega$ . The results summarized on this graphic suggest that the solutions exist for  $\Omega \in [\Omega_a, \Omega_b]$ , and that in the two limits  $\Omega \rightarrow \Omega_{a,b}$  an extremal black hole is approached at the event horizon  $r_h$ . Physical parameters like the energy  $E$  and the angular momentum  $J$ , which are related to the asymptotic charges, are reported on Fig. 9. The two figures 8, 9 further show that, for  $a'_h$  fixed, the parameters of the charged black hole deviate more substantially from the corresponding Myers-Perry solutions in the region of small values of  $\Omega$ . We note that, for non-vanishing electromagnetic field there is only one solution for given  $\Omega, r_h$ . This contrasts with the description in term the isotropic coordinate where in general two solutions can be associated to one value of  $\Omega$  (see Fig.1 of [25]). Finally several parameters related to the Maxwell fields  $V$  and  $a_\varphi$  are reported of Fig.10. We see that the values of  $\mu, A_0$  which hardly differ for small  $a'_h$  are lifted when the amplitude of the electromagnetic field increases (and similarly for  $Q, V_{inf}$ ).

Profiles of two typical solutions having  $a'_h = 0.5$  are represented on Figs. 11,12 respectively for  $\Omega = 0.62$ , i.e. close to the limit  $\Omega_b \sim 0.64$  and for an intermediate value  $\Omega = 0.44$

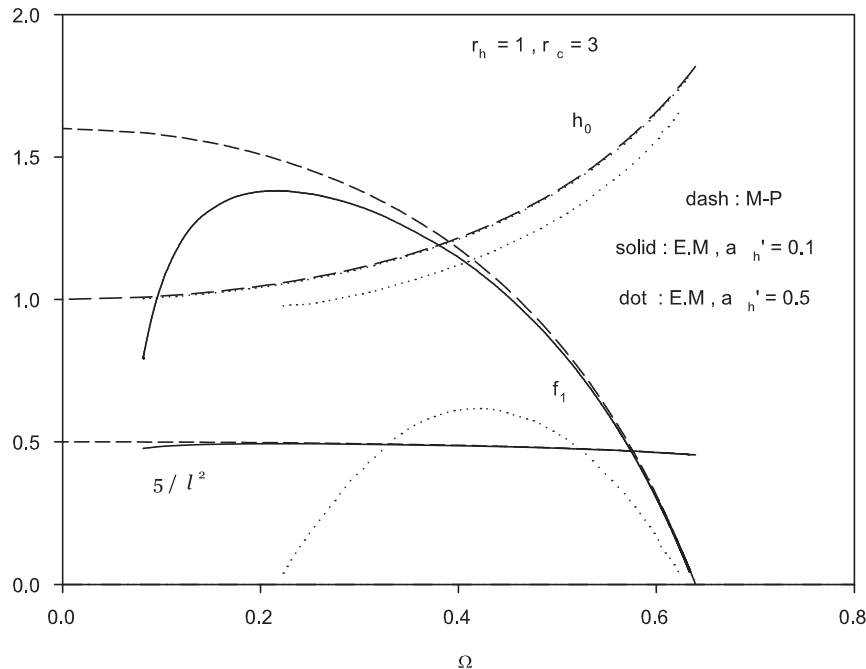


FIG. 8: The value  $f_1$  and  $g_0$  are reported as functions of  $\Omega$  for  $a'_h = 0, 0.1$  and  $a'_h = 0.5$

## VI. CONCLUSIONS

In this paper, numerical arguments are given for the existence of five-dimensional charged and rotating black holes with a positive cosmological constant. Two main features of these solutions are that (i) they are characterized by at least two horizons, an event horizon  $r_h$  and a cosmological horizon  $r_c$ ; for fixed  $r_h, r_c$ , we find a families of black holes characterized by the parameters  $\Omega$  and  $a'_h$  determining respectively the angular velocity and the magnetic fields at the event horizon. We presented several arguments indicating that black hole solutions exist only for  $\Omega$  taking values in a finite interval, say  $[\Omega_a, \Omega_b]$ . The limiting values depend on the strength of the electromagnetic field (in our case this is controlled by parameter  $a'_h$ ) and determine the domain of existence of the black holes in the  $\Omega, a'_h$  plane (with  $r_h, r_c$  fixed). The solutions end up into extremal black holes when the values  $\Omega_{a,b}$  are approached. In the limit  $a'_h \rightarrow 0$ , the MP black holes are recovered, existing for  $\Omega \in [0, \Omega_{max}]$  with  $\Omega_{max}$  known analytically. This feature is common to both signs of the cosmological constant once solution are constructed with a Schwarzschild -like coordinate. Smarr-type formulas have been obtained which relate some physical quantities associated with the event horizon and the cosmological horizon in the case of a positive cosmological constant.



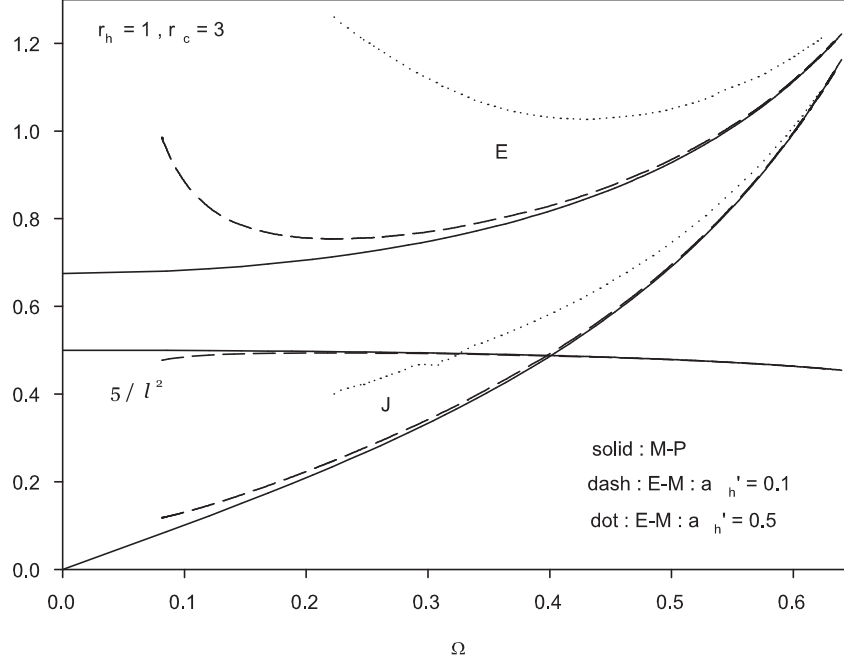


FIG. 9: The Energy of the solutions and the angular momentum  $J$  are reported as functions of  $\Omega$  for  $a'_h = 0; 0.1$  and  $a'_h = 0.5$

## VII. APPENDIX: THE EQUATIONS

In this Appendix, we present the equations associated to the Einstein-Maxwell Lagrangian. First, the equations determining the metric :

$$f' + \frac{f}{(d-2)} \left( -\frac{rh}{2b}w'^2 + \frac{2r}{b}V'^2 + \frac{4rw}{b}a'_\varphi V' + \frac{h'}{h} \left( 1 - \frac{rb'}{2b} \right) - 2r \left( \frac{1}{h} - \frac{w^2}{b} \right) a_\varphi'^2 + \frac{b'}{b} \right) + \frac{(d-1)(d-4)}{r} + \frac{1}{(d-2)r^3} ((3d-5)h + 4(d+1)a_\varphi'^2 - (d-1)^2 r^2) + \frac{(d-1)r}{\ell^2} = 0, \quad (31)$$

$$b'' + \frac{1}{d-2} \left( 4(5-2d)wa'_\varphi V' + \frac{(d-3)}{2h}b'h' - \frac{2(d-3)b}{2rh}h' - 2(2d-5)V'^2 \right) + \frac{1}{2}(3-2d)hw'^2 - 2\left(\frac{b}{h} + (2d-5)w^2\right)a_\varphi'^2 + (d-2) \left( \frac{b'f'}{2f} - \frac{b'^2}{2b} \right) + \frac{(d-3)^2}{r}b' - \frac{(d-3)b}{r^4 f} (12a_\varphi'^2 + h) + \frac{(d-3)b}{r^2} \left( \frac{d-1}{f} + 4-d \right) + \frac{(d-1)(d-2)b}{\ell^2 f} = 0, \quad (32)$$

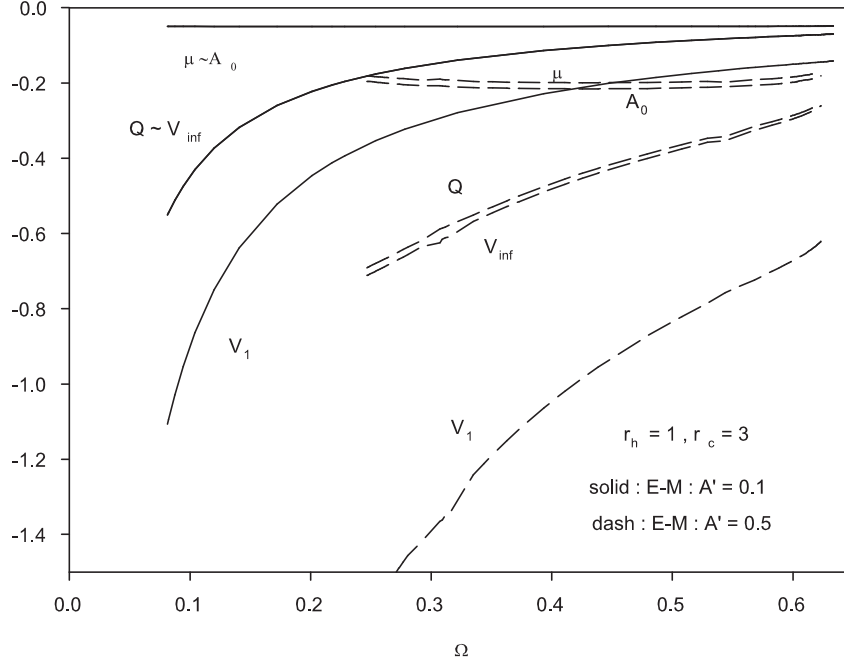


FIG. 10: Electromagnetic parameters as functions of  $\Omega$  for  $a'_h = 0.1$  and  $a'_h = 0.5$

$$h'' + \frac{1}{(d-2)} \left( \frac{(2d-5)h^2}{2b} w'^2 + \frac{2h}{b} V'^2 + \frac{4hw}{b} a'_\varphi V' - \frac{(d-2)h'}{2} \left( \frac{h'}{h} - \frac{f'}{f} \right) \right. \\ \left. + \frac{(d-3)}{2b} b' h' + \frac{(d-3)^2}{r} h' + 2 \left( \frac{hw^2}{b} + 2d-5 \right) a_\varphi'^2 - \frac{(d-3)h}{rb} b' - \frac{(d-3)(2d-3)h^2}{r^4 f} \right. \\ \left. - \frac{12(d-3)a_\varphi'^2 h}{r^4 f} - \frac{(d-3)(d-4)h}{r^2} + \frac{(d-1)h}{f} \left( \frac{d-2}{\ell^2} - \frac{d-3}{r^2} \right) \right) = 0, \quad (33)$$

$$w'' - \frac{4w}{h} a_\varphi'^2 - \frac{4a'_\varphi V'}{h} + \frac{(d-3)w'}{r} + \frac{1}{2} \left( -\frac{b'}{b} + \frac{f'}{f} + \frac{3h'}{h} \right) w' = 0, \quad (34)$$

and, for the Maxwell fields

$$V'' - \frac{w}{b} b' a'_\varphi + \frac{w}{h} a'_\varphi h' + \frac{1}{2} \left( \frac{2(d-3)}{r} - \frac{b'}{b} + \frac{f'}{f} + \frac{h'}{h} \right) V' \\ + \left( 1 + \frac{hw^2}{b} \right) a'_\varphi w' + \frac{hw}{b} V' w' + \frac{2(d-3)a_\varphi h w}{r^4 f} = 0, \quad (35)$$

$$a_\varphi'' + \frac{1}{2} \left( \frac{2(d-3)}{r} + \frac{b'}{b} + \frac{f'}{f} - \frac{h'}{h} \right) a'_\varphi - \frac{h}{b} (w a'_\varphi + V') w' - \frac{2(d-3)a_\varphi h}{r^4 f} = 0. \quad (36)$$

It can easily be seen that the equations of motion have the first integral

$$g^{\frac{(d-3)}{2}} \sqrt{\frac{fh}{b}} (w a'_\varphi + V') = (d-3)q. \quad (37)$$

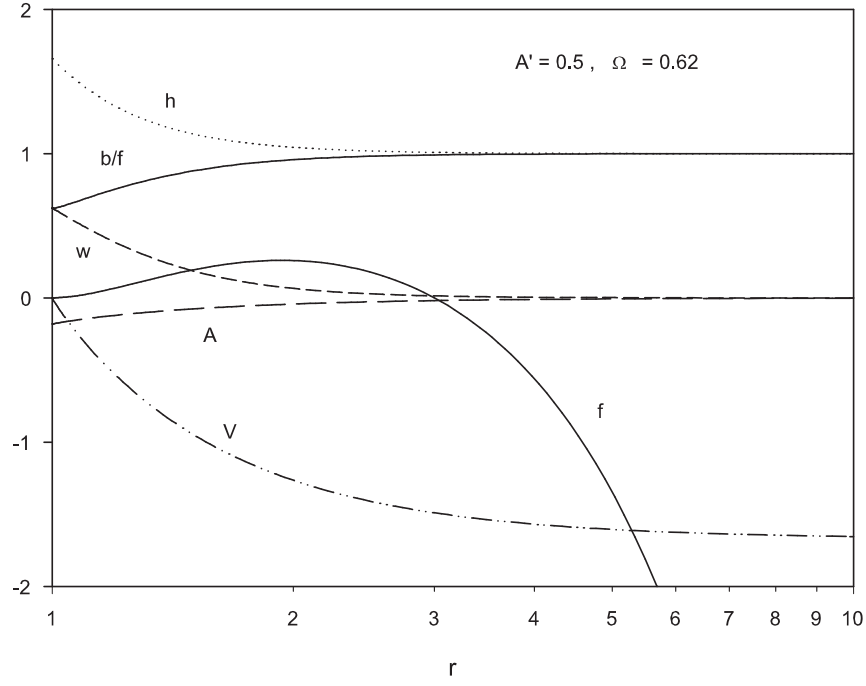


FIG. 11: The profile of the metric and Maxwell function for  $r_c = 3, r_h = 1$  for  $a'_h = 0.5$  and  $\Omega = 0.62$

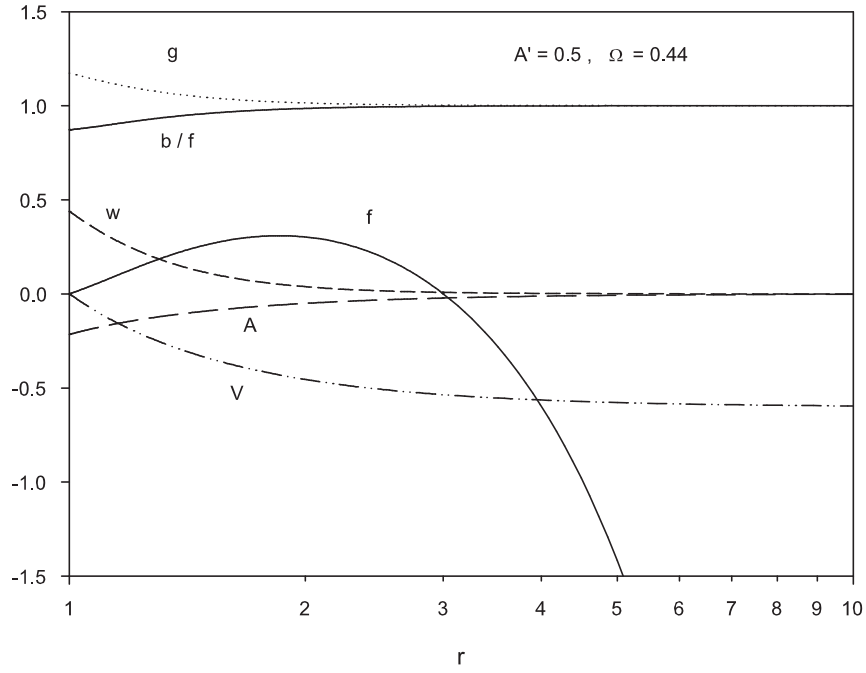


FIG. 12: The profile of the metric and Maxwell function for  $r_c = 3, r_h = 1$  for  $a'_h = 0.5$  and  $\Omega = 0.44$

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